# Efficient multipowers*

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**Abstract**

We show how to minimize the asymptotic variance of multipower estimators to make them efficient, using a linear combination of optimal powers. Taking advantage of the lower variance provided by this technique allows to build superior estimators of integrated volatility powers. In particular, we focus on a new efficient quarticity estimator and we show, using simulated data, that we can drastically reduce the mean square error of traditional estimators. The implementation on US stock prices is in line with the theoretical findings and further shows that using our efficient quarticity estimator reduces drastically the number of detected jumps, and improves the quality of volatility forecasts.

**Keywords:** Efficiency, Realized volatility, Quarticity, Multipowers, Threshold, Jumps

**JEL Classification Codes:** C13, G1.

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# 1 Motivation

A very active literature in financial econometrics focuses on the estimation of integrated volatility powers. The most popular example is realized volatility, which is an estimator of the integrated variance which attracted enormous recent interest. The integral of squared variance, dubbed *quarticity*, is also quite important since is a necessary ingredient in relevant applications, such as estimating the confidence bands for integrated variance estimates, applying jump tests (see, e.g., Dumitru and Urga, 2012 for a review), forecasting volatility (Bollerslev et al., 2015), computing the optimal sampling frequency in the presence of market microstructure noise (see, e.g., Bandi and Russell, 2006). The integrated third volarility power is needed, for example, to build confidence bands in the estimator developed by Reiß (2011) to obtain efficiency in the presence of microstructure noise.

Disentangling price volatility due to continuous movements to that due to discontinuous movements is also of fundamental importance in a number of equally relevant applications. In this field, a primary advancement has been represented by the introduction of multipower estimators, a tool introduced by Barndorff-Nielsen and Shephard (2004) to reduce the bias, due to jumps, in estimating integrated volatility powers. Successful modifications of multipower variation estimators have been proposed in Andersen et al. (2012, 2014), and Corsi et al. (2010), among others.

Grounding on the success of this literature in estimating integrated volatility powers in the presence of price discontinuities, this paper concentrates on *efficiency*, and specifically on how to minimize the variance of multipower estimators. The theory is first developed in the case without jumps. This allows to study the variance minimization problem in the simplified case of continuous semimartingales. We provide a general criterion to find the efficient linear combination of $N$ multipower estimators given the number $m$ of multipowers employed (for example, $m = 2$ for bipower variation, $m = 3$ for tripower variation, and so on), and we present explicit solutions in relevant cases. The (empirically compelling) case with jumps is recovered by applying the same logic to *truncated* returns, in the spirit of Mancini (2009).

The problem of finding the efficient estimator among multipowers is trivial for integrated variance. In this case, standard realized variance (i.e., the sum of squared returns) is the globally efficient estimator, achieving the maximum likelihood bound. However, for other integrated volatility powers the variance minimization problem is not so obvious. Jacod and Rosenbaum (2013) provide an efficient estimator which requires $m \to \infty$, but do not address the case of a fixed $m$. Mykland and Zhang (2009) study the fixed $m$ case and provide a block estimator which is more efficient than popular benchmarks, but do not address the problem of optimization. Our contribution is to propose a methodology to provide efficient estimation when the number of powers $m$ is fixed. In particular, we propose a novel quarticity estimator which has smaller variance, when $m$ is fixed, of both the Mykland and Zhang (2009) and the Jacod and Rosenbaum (2013) estimator, coinciding with the latter (and thus achieving global efficiency) when $m \to \infty$.

Minimization of the variance can be readily exploited to improve the mean square error of the estimate. Indeed, the gain in variance as $m \to \infty$ comes at the cost of a deterioration of the bias. However, these two effects can be traded off to select the value of $m$ which minimizes the mean square error. Thus, by construction, the resulting efficient estimator can only lower the mean square error with respect to traditional counterparts. Thus, our theoretical contribution has the immediate practical application of delivering superior estimates of integrated volatility powers.

Our theoretical results are indeed corroborated by simulated experiments and empirical applications, both focused on estimating quarticity. Using simulations, we show that the efficient multipower estimators perform better than competing alternatives, including standard multipowers, nearest neighbor estimators and the Jacod and Rosenbaum (2013) estimator. This result is robust to the presence of frictions (market microstructure noise and flat pricing) which are not dealt with by our theory but are well known to contaminate the data.

When we apply the proposed quarticity estimator to a set of intraday prices of liquid US stocks, we confirm that estimates obtained with efficient multipowers are less variable than competing estimators, and suffer of less distortion. For example, efficient estimated quarticities are larger than the square of integrated variance (as it should be asymptotically, by Jensen's inequality) only in the 13% of cases, a figure much larger than what obtained with standard multipowers (44% with tripower, 54% with quadpower). Having at disposal superior quarticity estimates has relevant empirical implications. We provide two examples. The first is that the adoption of efficient quarticity estimation for standard jump tests delivers drastically less jumps than what obtained with standard multipowers. This finding supports the recent empirical literature (see, e.g., Christensen et al., 2014) claiming that the number of jumps typically detected in high frequency data is spuriously excessive, suggesting that a significant part of spuriously detected jumps could be due to inefficient quarticity estimation. In the second example, we adopt the specification proposed by Bollerslev et al. (2015) to forecast volatility. In this setting, quarticity is used to correct the coefficients of the HAR model of Corsi (2009). Also in this case, we show that efficient quarticity improves the quality of realized volatility forecasts with respect to standard quarticity measures. Both results clearly emphasize the empirical potential of our contribution.

The paper is structured as follows. In Section 2, we describe the theoretical framework, define the class of estimators of interest and state their asymptotic properties. Section 3 contains the theoretical result on efficient multipower estimators. Section 4 reports the Monte Carlo study, while Section 5 reports the empirical application. Section 6 concludes.

# 2 Efficient multipowers

In a filtered probability space satisfying the usual conditions, denote by $X_t$ (e.g., the log-price of a financial stock or a stock index) an Itō semimartingale of the form

$$dX_t = \mu_t dt + \sigma_t dW_t + dJ_t, \tag{2.1}$$

where $\mu_t$ is predictable, $\sigma_t$ is cádlág, and $J_t$ is a jump process. The estimation target is the integrated volatility power over a finite interval,

$$V_{[0,T]}(R) = \int_0^T (\sigma_s)^R \, ds,$$

for a given $R > 0$. The most important cases in practice are $V_{[0,T]}(2)$ (integrated variance) and $V_{[0,T]}(4)$ (quarticity).

## 2.1 The case $J = 0$

We start with the case without jumps, in which the methodology can be outlined clearly. The empirically compelling case with jumps is treated in the following Section.

Define $\Delta_i X = (X_{iT/n} - X_{(i-1)T/n})$, for $i = 1, \ldots, n$, and write $\Delta_n = T/n$. Multipower estimators (Barndorff-Nielsen and Shephard, 2004, 2006) are defined as follows. Consider an $m-$valued real vector $\mathbf{r} = [r_1, \ldots, r_m]$ with positive coefficients, and let

$$R = \sum_{i=1}^{m} r_j.$$

We define the *multipower variation* estimator $\mathsf{MPV}(\mathbf{r})$ with powers $\mathbf{r}$ as

$$\mathsf{MPV}(\mathbf{r}) = c_{\mathbf{r}} \cdot \sum_{i=1}^{n-m+1} \left( \prod_{j=1}^{m} |\Delta_{i+j-1} X|^{r_j} \right), \tag{2.2}$$

where the constant $c_{\mathbf{r}}$, meant to make the estimator unbiased in small samples under the assumption of constant volatility, is defined by

$$c_{\mathbf{r}} = \left(\frac{n}{T}\right)^{\frac{R}{2}-1} \frac{n}{n-(m-1)} \left( \prod_{j=1}^{m} (\mu_{r_j})^{-1} \right),$$

where $\mu_r = E(|u|^r)$ with $u$ being a standard normal. Important special cases are *bipower* *variation*, used to estimate integrated variance:

$$\mathsf{MPV}([1,1]) = \frac{\pi}{2}\frac{n}{n-1}\sum_{i=1}^{n-1}|\Delta_i X|\,|\Delta_{i+1}X|$$

and tripower and quadpower variation, both used to estimate quarticity:

$$\mathsf{MPV}\left([4/3,4/3,4/3]\right) = \frac{n}{T}\frac{1}{\mu_{\frac{4}{3}}^3}\frac{n}{n-2}\sum_{i=1}^{n-2}|\Delta_i X|^{\frac{4}{3}}\,|\Delta_{i+1}X|^{\frac{4}{3}}\,|\Delta_{i+2}X|^{\frac{4}{3}}\,,$$

$$\mathsf{MPV}\left([1,1,1,1]\right) = \frac{n}{T}\left(\frac{\pi}{2}\right)^2\frac{n}{n-3}\sum_{i=1}^{n-3}|\Delta_i X|\,|\Delta_{i+1}X|\,|\Delta_{i+2}X|\,|\Delta_{i+3}X|\,.$$

When $J_t = 0$, it has been proven, see e.g. Barndorff-Nielsen et al. (2006a), that $\mathsf{MPV}([\mathbf{r}])$ is a consistent and asymptotically normally distributed estimator, as $n \to \infty$, of $\int_0^T (\sigma_s)^R\, ds$. Under mild assumptions on the model (2.1) the following stable central limit theorem holds:

$$\sqrt{\frac{1}{\Delta_n}}\left(\mathsf{MPV}(\mathbf{r}) - \int_0^T (\sigma_s)^R\, ds\right) \underset{n\to\infty}{\Rightarrow} \mathcal{MN}\left(0, V_{\mathbf{r}}\int_0^T (\sigma_s)^{2R}\, ds\right) \tag{2.3}$$

where $\mathcal{MN}$ denotes a standard mixed normal distribution, and

$$V_{\mathbf{r}} = \frac{\displaystyle\prod_{j=1}^{m}\mu_{2r_j} - (2m-1)\prod_{j=1}^{m}\mu_{r_j}^2 + 2\sum_{i=1}^{m-1}\prod_{j=1}^{i}\mu_{r_j}\prod_{j=m-i+1}^{m}\mu_{r_j}\prod_{j=1}^{m-i}\mu_{r_j+r_{j+i}}}{\displaystyle\prod_{j=1}^{m}\mu_{r_j}^2}, \tag{2.4}$$

see Theorem 3 in Barndorff-Nielsen et al. (2006a) for details.

In the large empirical literature that uses multipower estimators, these are invariably implemented with equal powers: bipower variation with $\mathbf{r} = [1,1]$, tripower variation with $\mathbf{r} = [4/3, 4/3, 4/3]$, quadpower variation with $\mathbf{r} = [1,1,1,1]$ and so forth. However, minimizing $V_{\mathbf{r}}$ in formula (2.4) reveals that the variance of the estimator can be reduced considerably by changing powers. For example, while with the standard tripower variation we have $V_{[4/3,4/3,4/3]} = 13.65$, the optimal choice with three powers would be $V_{[3.5455,0.2182,0.2362]} = 9.70$, that is a gain of roughly 30% on the asymptotic variance. This fact has also been remarked in the paper of Mancini and Calvori (2012). As we show next, combining estimators with different powers is even more efficient.[1]

We now define the class of estimators inside which we look for the efficient one. We write a linear combination of $N$ different $\mathsf{MPV}(\mathbf{r})$ estimators which have the maximum number of adjacent returns equal to $m$ as:

$$\mathsf{GMPV}(N, m; \sigma^R) = \sum_{j=1}^{N} w_j \, \mathsf{MPV}(\mathbf{r}^{(j)}) \tag{2.5}$$

where the weights $w_j, j = 1, \ldots, N$ are such that

$$\sum_{j=1}^{N} w_j = 1$$

and for each $j = 1, \ldots, N$, $\mathbf{r}^{(j)}$ is a vector in $\mathbb{R}^m$ with nonnegative components such that $\sum_{i=1}^{m} r_i^{(j)} = R$. By construction, $\mathsf{GMPV}(N, m; \sigma^R)$ is a consistent estimator of $V_{[0,T]}(R)$, that is

$$\mathsf{GMPV}(N, m; \sigma^R) \xrightarrow{p} \int_0^T (\sigma_s)^R \, ds$$

as $n \to \infty$ with fixed $N$ and $m$. Our problem is to find the efficient estimator in this class, that is the problem of finding the optimal weights $w_j$ and power vectors $\mathbf{r}^{(j)}$ delivering the minimum asymptotic variance of $\mathsf{GMPV}(N, m; \sigma^R)$. We denote by $\mathsf{GMPV}^*(N, m; \sigma^R)$ the *efficient* estimator in this class for fixed $N, m, R$. The following proposition provides the desired result.

**Proposition 2.1.** *The efficient estimator in the class of linear combinations of multi-power estimators is given by:*

$$\mathsf{GMPV}^*(N, m; \sigma^R) = \sum_{j=1}^{N} w_j^* \, \mathsf{MPV}(\mathbf{r}^{*(j)}), \tag{2.6}$$

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[1] The concept of efficiency used here corresponds to that used when estimating the parameter $\sigma^R$ in the model $X_t = \sigma W_t$ with constant $\sigma$, the lower bound for $V_{\mathbf{r}}$ being fixed by maximum likelihood at the value $R^2/2$. The same concept of efficiency has been used by Jacod and Rosenbaum (2013).

*where the efficient $\mathbb{R}^m$-valued power vectors $\mathbf{r}^{*(1)}, \ldots, \mathbf{r}^{*(N)}$ minimize the quantity*

$$\widetilde{V}(N, m; \sigma^R) = \frac{1}{\sum_{i=1}^{N} \sum_{j=1}^{N} \left(C^{-1}\right)_{ij}},$$

*$C^{-1}$ is the inverse of the $N \times N$ symmetric matrix $C$ defined as:*

$$\begin{aligned} C_{ij} = &\left(\prod_{k=1}^{m} \mu_{r_k^{(i)}} \mu_{r_k^{(j)}}\right)^{-1} \left(\prod_{k=1}^{m} \mu_{r_k^{(i)} + r_k^{(j)}} + \sum_{k=1}^{m-1} \prod_{\ell=1}^{k} \mu_{r_\ell^{(i)}} \prod_{\ell=1}^{m-k} \mu_{r_{k+\ell}^{(i)} + r_\ell^{(j)}} \prod_{\ell=m-k+1}^{m} \mu_{r_\ell^{(i)}} \right. \\ &\left. + \sum_{k=1}^{m-1} \prod_{\ell=1}^{k} \mu_{r_\ell^{(j)}} \prod_{\ell=1}^{m-k} \mu_{r_{k+\ell}^{(j)} + r_\ell^{(i)}} \prod_{\ell=m-k+1}^{m} \mu_{r_\ell^{(j)}} - (2m-1) \prod_{k=1}^{m} \mu_{r_k^{(i)}} \mu_{r_k^{(j)}} \right), \end{aligned} \tag{2.7}$$

*for $i, j = 1, \ldots, N$; and the efficient weights coefficient $w_j^\star$, $j = 1, \ldots, N$, are given by:*

$$w_j^\star = \widetilde{V}(N, m; \sigma^R) \sum_{i=1}^{N} \left(C^{-1}\right)_{ij}. \tag{2.8}$$

*For the efficient estimator it holds:*

$$\sqrt{\frac{1}{\Delta_n}} \left( \mathsf{GMPV}^*(N, m; \sigma^R) - \int_0^T (\sigma_s)^R ds \right) \underset{n \to \infty}{\Rightarrow} \mathcal{MN}\left(0, \widetilde{V}(N, m; \sigma^R) \int_0^T \sigma_s^{2R} ds \right), \tag{2.9}$$

*where the above convergence is stable in law.*

*Proof.* See Appendix A. □

The above proposition provides a procedure that allows to find, by numerical optimization, the efficient combination with given $N$, $m$ and $R$. For example, when $R = 3$ the efficient estimator with $N = 1$, $m = 2$ is found to be $\mathsf{GTMPV}^*(1, 2; \sigma^3) = \mathsf{TMPV}([0.1358, 2.8642])$, with $\widetilde{V}(1, 2; \sigma^3) = 4.7947$, the lower bound imposed by maximum likelihood being 4.5 in this case. Adding more estimators in a linear combination would improve efficiency. So, when $N = 2$ the efficient estimator is found to be

$$\mathsf{GTMPV}^*(2, 2; \sigma^3) = 0.2634\, \mathsf{TMPV}([1.5, 1.5]) + 0.7366\, \mathsf{TMPV}([3, 0]), \tag{2.10}$$

Table 1: Reports the values of $\widetilde{V}(m, N; \sigma^4)$ for several values of $m$ and $N$. The globally efficient value is 8.

| | $N$ | | | | |
|---|---|---|---|---|---|
| $m$ | 1 | 2 | 3 | 4 | 5 |
| 1 | 10.666 | | | | |
| 2 | 10.050 | 9.600 | | | |
| 3 | 9.701 | 9.314 | 9.143 | | |
| 4 | 9.478 | 9.119 | 8.989 | 8.965 | |
| 5 | 9.323 | 9.019 | 8.856 | 8.850 | 8.801 |

with $\widetilde{V}(2, 2; \sigma^3) = 4.7324$; while when $N = 3$ the efficient estimator is

$$\begin{aligned} \mathsf{GTMPV}^*(3, 2; \sigma^3) = & -0.1345\, \mathsf{TMPV}\left([0.6868, 2.3132]\right) + 0.3639\, \mathsf{TMPV}\left([1.3020, 1.6980]\right) \\ & + 0.7706\, \mathsf{TMPV}\left([0.0016, 2.9984]\right), \end{aligned} \tag{2.11}$$

with $\widetilde{V}(3, 2; \sigma^3) = 4.7320$. It is interesting to look, always in the case $R = 3$, at the efficient estimator with $N = 3, m = 3$, which is found to be

$$\mathsf{GTMPV}^*(3, 3; \sigma^3) = 0.1865\, \mathsf{TMPV}\left([0, 1.5, 1.5]\right) + 0.1865\, \mathsf{TMPV}\left([1.5, 0, 1.5]\right) \tag{2.12}$$

$$+ 0.6270\, \mathsf{TMPV}\left([0, 3, 0]\right), \tag{2.13}$$

with $\widetilde{V}(3, 3; 3) = 4.6666$. We can conjecture the emergence a clear structure for the case $N = m$. We will exploit this structure in the definition of suitable quarticity estimator.

Tables 1 and 2 report the efficient values found when fixing $m$ and $N$ in the cases $R = 3$ and $R = 4$ respectively. We can see that, as $m$ and $N$ increase, there is convergence of the proposed estimator to the globally efficient value $R^2/2$.

## 2.2 The case $J \neq 0$

The case with $J \neq 0$ makes the analysis of the efficiency of multipower estimators more complicated. Consistency of multipower estimators is indeed lost, in the presence of

Table 2: Reports the values of $\widetilde{V}(m, N; \sigma^3)$ for several values of $m$ and $N$. The globally efficient value is 4.5.

| | $N$ | | | | |
|---|---|---|---|---|---|
| $m$ | 1 | 2 | 3 | 4 | 5 |
| 1 | 4.891 | | | | |
| 2 | 4.795 | 4.732 | | | |
| 3 | 4.745 | 4.693 | 4.667 | | |
| 4 | 4.714 | 4.667 | 4.666 | 4.641 | |
| 5 | 4.693 | 4.653 | 4.643 | 4.629 | 4.620 |

jumps, if $\max\{r_1, \ldots, r_m\} \geq 2$, and the asymptotic variance of the multipower estimator also contains a part influenced by jumps if $\max\{r_1, \ldots, r_m\} \geq 1$, see Barndorff-Nielsen et al. (2006a,b); Veraart (2010); Vetter (2010); Woerner (2006). Thus, the central limit Theorem in Eq. (2.3) only holds with these two constraints.

However, a simple solution at hand is provided by applying multipower estimators to truncated returns. Consider a positive stochastic process $\vartheta_t$, which we call a *threshold* (Mancini, 2009). Write $\vartheta_t = \xi_t \Theta(\Delta)$, with $\Theta(\Delta)$ being a real function satisfying

$$\lim_{\Delta \to 0} \Theta(\Delta) = 0, \qquad \lim_{\Delta \to 0} \frac{\Delta \log \frac{1}{\Delta}}{\Theta(\Delta)} = 0, \tag{2.14}$$

and $\xi_t$ being a stochastic process on $[0, T]$ which is a.s. bounded and with a strictly positive lower bound. Using the threshold, we define *truncated* equally-spaced returns, observed over the interval $[0, T]$ as

$$\not{\Delta}_i X = (\Delta_i X)\, I_{\{|\Delta_i X| \leq \vartheta_{(i-1)T/n}\}}, \; i = 1, \ldots, n \tag{2.15}$$

where $I_{\{A\}}$ is the indicator function of the set $A$. The truncation is meant to annihilate returns larger than a given threshold, while leaving all the remaining returns unchanged. We define the *threshold multipower variation* estimator $\mathsf{TMPV}(\mathbf{r})$ (Corsi et al., 2010) as

$$\mathsf{TMPV}(\mathbf{r}) = c'_{\mathbf{r}} \cdot \sum_{i=1}^{n-m+1} \left( \prod_{j=1}^{m} |\not{\Delta}_{i+j-1} X|^{r_j} \right). \tag{2.16}$$

where the constant $c'_{\mathbf{r}}$, meant to make the estimator unbiased in small sample under the assumption of constant volatility, is now defined by

$$c'_{\mathbf{r}} = \left(\frac{n}{T}\right)^{\frac{R}{2}-1} \frac{n}{n-(m-1)-n_J} \left(\prod_{j=1}^{m} (\mu_{r_j})^{-1}\right), \tag{2.17}$$

where $n_J$ is the number of terms vanishing in the sum in Eq. (2.16) because of the indicator function. Then, under mild conditions for the process (2.1), the following stable central limit continues to hold as $n \to \infty$:

$$\sqrt{\frac{1}{\Delta_n}} \left(TMPV(\mathbf{r}) - \int_0^T (\sigma_s)^R \, ds\right) \Rightarrow \mathcal{MN}\left(0, V_{\mathbf{r}} \int_0^T (\sigma_s)^{2R} \, ds\right), \tag{2.18}$$

see Theorem 2.3 in Corsi et al. (2010) for the case with finite activity jumps, and Theorem 13.2.1 in Jacod and Protter (2012), for the general case allowing for infinite activity jumps.

Using truncated returns, the results in Section 2.1 can be readily recovered. The disadvantage of the truncation is the cost of estimating an additional parameter, that is the threshold $\vartheta$. However, using a reasonably high threshold (in this paper, we always use 5 "local" standard deviations) will leave out very big jumps only, whose impact would be the largest, while remaining small jumps are dealt with the multipower technique. This double-sword feature smoothes the hurdle of having to select an additional threshold and makes the proposed estimators virtually immune from the bias due do the presence of the jumps, so that we can concentrate on variance reduction.

An alternative way to avoid threshold selection has been proposed by Andersen et al. (2012) and Andersen et al. (2014), consisting in a "comparison" method based on the nearest neighbors. We also consider this kind of estimators in the Monte Carlo study and empirical application, namely the minRQ estimator, defined as

$$\mathsf{minRQ} = \frac{\pi}{3\pi - 8} \frac{n^2}{n-1} \sum_{i=1}^{n-1} \min\left(|\Delta_i X|, |\Delta_{i+1} X|\right)^4, \tag{2.19}$$

and the $\mathsf{medRQ}$ estimators, defined as

$$\mathsf{medRQ} = \frac{3\pi}{9\pi + 72 - 52\sqrt{3}} \frac{n^2}{n-2} \sum_{i=1}^{n-2} \text{med}\left(|\Delta_i X|, |\Delta_{i+1} X|, |\Delta_{i+2} X|\right)^4. \tag{2.20}$$

Both estimators are consistent for quarticity in the presence of jumps, allow for a central limit theorem in the same form of Eq. (2.18) with $V_{\mathbf{r}}$ replaced by 18.54 for the $\mathsf{minRQ}$ estimator, and by 14.16 for the $\mathsf{medRQ}$ estimator. Thus, the nonlinear structure of these estimators implies a quite large asymptotic variance. Changing powers might help also here, but the asymptotic variance appears to be more difficult to compute theoretically.

In what follows, we cast the theory in the more realistic case in which $J_t \neq 0$ using truncated returns and threshold multipower estimators.

# 3 Estimating quarticity

In practice, two cases appear to be more relevant: the case $R = 2$ (integrated volatility), and the case $R = 4$ (integrated quarticity).

In the case $R = 2$ the efficient estimator, for every value of $m$ and $N$, is simply $\mathsf{TMPV}([2])$, that is the threshold realized variance proposed by Mancini (2009). Indeed, this coincides (asymptotically) with the maximum likelihood estimator of the parameter $\sigma^2$ in the model $X_t = \sigma W_t$. The case with $R = 2$ is the unique value of $R$ for which the efficient estimator is the same for all $m$. Indeed, inspection of the proof of Proposition 2.1 reveals that the efficiency of $\mathsf{GTMPV}(N, m; \sigma^R)$ can be formally related to the efficiency of the GMM estimator with a number $N$ of moment restrictions, where $N$ could diverge to infinity. The problem of efficient GMM estimator with a continuum of moment restrictions (when $N \to \infty$) is discussed in Carrasco and Florens (2014), whose Proposition 4.1 shows that efficiency can be reached if the moment restrictions span the score of the likelihood (namely, with respect to the parameter $\sigma^R$ for the model $X_t = \sigma W_t$). Since the score (the derivative of the log-likelihood with respect to the parameter of interest) takes the form

$$s_i = 1 - (\Delta_i X)^2 (\sigma^R)^{-1/2}, \tag{3.1}$$

for $i = 1, \ldots, n$, it is clear that efficiency can be reached only in the case $R = 2$, since for different values of $R$ the score cannot be spanned, even when we allow $N$ to diverge.

## 3.1 Efficient quarticity estimation

The case $R = 4$ (quarticity) is more intriguing and particularly important in financial applications: it is used, for example, in determining the optimal sampling frequency in the presence of market microstructure noise (Bandi and Russell, 2006); in computing jump tests; in determining the confidence intervals of realized variance; in forecasting volatility; see, for example, the discussion in Balter (2014), who provides an estimator based on the observation of the whole price path.

Based on the above analysis, we propose to use the quarticity estimator which we find to be efficient when $N = m$, that is[2]:

$$\mathsf{GTMPV}^{**}(m; \sigma^4) = \frac{3}{2m+1}\,\mathsf{TMPV}([4]) + \frac{2}{2m+1}\sum_{j=0}^{m-2}\mathsf{TMPV}([2, \underbrace{0, \ldots, 0}_{j\ terms}, 2]), \tag{3.2}$$

The weights in Eq. (3.2) sum up to 1, so that $\mathsf{GTMPV}^{**}(m; \sigma^4)$ is a consistent estimator, as $n \to \infty$, of $V_{[0,T]}(4)$ for every fixed value of $m$. The next proposition provides the asymptotic distribution of $\mathsf{GTMPV}^{**}(m; \sigma^4)$ for a fixed $m$,

**Proposition 3.1.** *As $n \to \infty$, if $m$ is fixed,*

$$\sqrt{\frac{1}{\Delta_n}}\left(\mathsf{GTMPV}^{**}(m; \sigma^4) - \int_0^T \sigma_s^4 ds\right) \Rightarrow \mathcal{M}N\left(0, V^{**}(m, \sigma^4)\int_0^T \sigma_s^8 ds\right), \tag{3.3}$$

*where*

$$V^{**}(m, \sigma^4) = 8 + \frac{8}{2m+1}. \tag{3.4}$$

[2]The global efficiency of the estimator in Eq. (3.2) for fixed $m$ is left as a theoretical conjecture.

*and the above convergence is stable in law.*

*Proof.* See Appendix A. □

The estimator (3.2) is asymptotically equivalent to that of Jacod and Rosenbaum (2013), since its variance converges to $8\int_0^T \sigma_s^8 ds$ when $m \to \infty$. In small samples, the Jacod and Rosenbaum (2013) can also be written, ignoring end-effects, as a linear combination of multipower estimators (see the proof of Proposition 3.2, stated below). The difference consists in the weights of the linear combination. For the estimator 3.2, the weights are purposedly designed to be efficient for fixed $m$. Thus, the estimator 3.2 has smaller variance than the Jacod and Rosenbaum (2013) estimator implemented with the same window. As we discuss below, this fact is very important in practice. Indeed, larger $m$ also implies larger bias. When the objective is the minimization of a loss function which depends on both variance and bias, as it is customary, an intermediate value of $m$ is optimal. Thus, minimizing the variance in the fixed $m$ case is important.

The problem of improving efficiency when $m$ is fixed has also been studied by Mykland and Zhang (2009), who propose a block estimator for integrated volatility powers which is UMVU in each block.[3] This clearly improves the asymptotic variance, but does not explore (as we do here) the possibility of interaction among blocks. For this reason, our estimator is more efficient. The asymptotic variance of the Mykland and Zhang (2009) estimator when $R = 4$ is indeed given, from Eq. (58) in their paper, by:

$$V^{MZ}(m, \sigma^4) = 8 + 8\frac{m^2 + 2m}{m^2 - 1} \tag{3.5}$$

so that the relative efficiency compares favorably for $\mathsf{GTMPV}^{**}(m; \sigma^4)$ since

$$\frac{V^{MZ}(m, \sigma^4)}{V^{**}(m, \sigma^4)} = \frac{1}{2}\frac{m(2m+1)(m+2)}{(m+1)^2(m-1)} > 1 \tag{3.6}$$

Figure 1 shows the Asymptotic Relative Efficiency (ARE) of both estimators as a func-

[3]The setting of Mykland and Zhang (2009) is without jumps. It can however be reconciled with our framework by applying their estimator to truncated returns, as we do here, instead of the original ones.

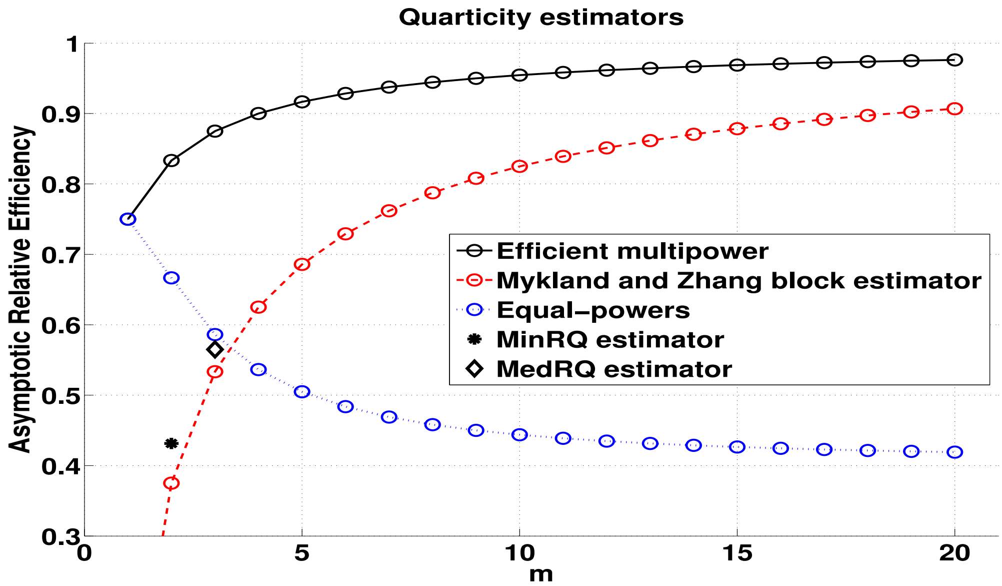


Figure 1: Asymptotic Relative Efficiency (with respect to the case $m \to \infty$) for quarticity estimation. Several estimators are compared: the $\mathsf{GTMPV}^{**}$ efficient multipower estimator in Eq. (3.2), the Mykland and Zhang (2009) block estimator, the standard multipower estimator with all equal powers, and the minRQ and medRQ estimator.

tion of $m$, showing that the advantage in using the efficient multipowers estimator can be quit large for small values of $m$. The figure also shows the ARE of the standard multipower estimator $\mathsf{TMPV}(\underbrace{[4/m, \ldots, 4/m]}_{m \ terms})$ with all equal powers (for $m = 1$ it trivially coincides with the efficient estimator; for $m = 2$ this is bipower variation, for $m = 3$ tripower variation, for $m = 4$ quadpower variation, and so on), which shows that using equal powers for large $m$ is definitively not the best option; and the ARE of the minRQ (with $m = 2$) and the medRQ estimator (with $m = 3$), which shows that the min-med estimators have lower relative efficiency than traditional multipowers. For efficient multipowers and the block estimator, the asymptotic relative efficiency increases with $m$, converging to 1 when $m \to \infty$. The convergence is faster for the efficient multipower estimator.

## 3.2 The choice of the optimal $m$: bias considerations

Achieving efficiency is clearly not the end of the story. Typically, the objective is the minimization of a loss function which depends also on the bias, such as the mean square error. While the variance tends to decrease with $m$, the bias tends to increase with $m$, thus originating the usual bias-variance tradeoff.

A convenient expression of the asymptotic bias is provided by the next proposition, in which $m$ is allowed to diverge at a suitable rate. The result borrows from the work of Jacod and Rosenbaum (2013).

**Proposition 3.2.** *Assume that volatility is driven by the process*

$$d\sigma_t^2 = \mu_t^\sigma dt + \Lambda_t dW_t + dJ_t^\sigma,$$

*where $\mu_t$ is a predictable process, $\Lambda_t$ is cádlág and $J_t^\sigma$ is a jump process. If $n, m \to \infty$ in such a way that $m^2/n \to \theta$ and $m^3/n \to \infty$, we have*

$$\sqrt{\frac{1}{\Delta_n}}\left(\mathsf{GTMPV}^{**}(m;\sigma^4) - \int_0^T \sigma_s^4 ds\right) \Rightarrow B_1 + B_2 + B_3 + \mathcal{MN}\left(0, 8\int_0^T \sigma_s^8 ds\right), \quad (3.7)$$

*where*

$$B_1 = -\frac{\theta}{2}\left(\sigma_0^4 + \sigma_T^4\right), \quad (3.8)$$

$$B_2 = -\frac{\theta}{6}\int_0^T \Lambda_s^2 ds, \quad (3.9)$$

$$B_3 = -\frac{\theta}{6}\sum\left(\Delta\sigma_s^2\right)^2, \quad (3.10)$$

*and $\Delta\sigma_s^2 = \sigma_s^2 - \sigma_{s-}^2$ are the jumps in the variance process.*

The (asymptotic) bias thus consists of three negative terms,[4] the first due to the border-effect, and the second and the third due to the variability of volatility.

We can take advantage of the small sample approximation of the bias provided by

[4] Note, that in finite samples there is another source of negative bias, due to the truncation of the largest observations. However, it is of smaller order with respect to the other two biases, hence it is not considered here.

Table 3: Reports the values of $m^{opt}$ that optimize the mean square error (3.11) for different values of $n$ and $\overline{\Lambda} = \left(\int_0^T \Lambda_s^2 ds + \sum (\Delta\sigma_s^2)^2\right)^{1/2}$, in the case $\int_0^T \sigma_t^8 dt = \sigma_0 = \sigma_T = 1$.

| | $\overline{\Lambda} = 0.1$ | $\overline{\Lambda} = 0.5$ | $\overline{\Lambda} = 1$ | $\overline{\Lambda} = 2$ | $\overline{\Lambda} = 3$ | $\overline{\Lambda} = 5$ |
|---|---|---|---|---|---|---|
| | | | optimal $m$ | | | |
| $n = 40$ | 56 | 15 | 9 | 5 | 4 | 2 |
| $n = 80$ | 73 | 20 | 12 | 7 | 5 | 3 |
| $n = 400$ | 109 | 38 | 22 | 13 | 9 | 6 |

Proposition 3.2 with the small sample approximation of the variance provided by Proposition 3.1 to get the following approximation of the mean square error:

$$MSE \approx \Delta_n \left(8 + \frac{8}{2m+1}\right) Q^2 + \Delta_n^3 m^4 B^2 \tag{3.11}$$

where $Q^2 = \int_0^T \sigma_s^8 ds$ and $B = \frac{1}{2}(\sigma_0^4 + \sigma_T^4) + \frac{1}{6}\left(\int_0^T \Lambda_s^2 ds + \sum (\Delta\sigma_s^2)^2\right)$.

In principle, this MSE could be estimated from the data as a function of $m$, which could then be optimized. The main problem would be to estimate the volatility of volatility term $\int_0^T \Lambda_s^2 ds + \sum (\Delta\sigma_s^2)^2$ with sufficiently low error. To gain feeling of what we can get, we set $\int_0^T \sigma_t dt = 1$, we ignore the end-effect term $\frac{1}{2}(\sigma_0^4 + \sigma_T^4)$ since we use the constant $c_{\mathbf{r}}$ in Eq. (2.17) to compensate for it, and optimize the MSE for various values of $\overline{\Lambda} = \left(\int_0^T \Lambda_s^2 ds + \sum (\Delta\sigma_s^2)^2\right)^{1/2}$. Table 3 reports the optimal $m$ we found for different choices of $n$, and shows that the optimal $m$ would strongly depend on the volatility-of-volatility estimate. Given the notorious difficulty in estimating this parameter, we propose the following alternative approach.

We assume to start from a quarticity estimator $\widehat{Q}$, which might not be efficient, but which is assumed to be unbiased (for example, the traditional multipower estimator). Under this assumption, we can always improve (in the mean square error sense, or in the sense of any alternative loss function combining bias and variance) with respect to the estimator $\widehat{Q}$ by choosing $m^*$ that minimizes the MSE:

$$\widetilde{MSE} = \left[\mathsf{GTMPV}^{**}(m;\sigma^4) - \widehat{Q}\right]^2 + Var(\mathsf{GTMPV}^{**}(m;\sigma^4)) \tag{3.12}$$

where the variance of the efficient estimator can be estimated, using the asymptotic expression (3.3), by

$$Var(\mathsf{GTMPV}^{**}(m;\sigma^4)) \approx \left(8 + \frac{8}{2m+1}\right)\mathsf{TMPV}([8/3, 8/3, 8/3]),$$

or replacing $\mathsf{TMPV}([8/3, 8/3, 8/3])$ by another consistent estimator of $\int_0^T \sigma_s^8 ds$.[5] By construction, the efficient estimator $\mathsf{GTMPV}^{**}(m^*;\sigma^4)$ will have a smaller mean square error than the original estimator $\widehat{Q}$. This is the technique we use in the empirical applications on jump testing and volatility forecasting, using $\widehat{Q} = \mathsf{TMPV}([4])$.

# 4 Monte Carlo simulations

In this Section, we perform a series of Monte Carlo experiments focusing on quarticity estimation at different frequencies. In particular, on a simulated typical trading day in the US market of 6.5 hours, we consider $n = 40, 80, 400$ which roughly corresponds to 10, 5 and 1-minutes returns respectively.

In order to generate a realistic price dynamics, we simulate the jump-diffusion model:

$$\begin{aligned} d\log p_t &= \mu\, dt + \gamma_t \sigma_t dW_{p,t} + dJ_t \\ d\log \sigma_t^2 &= (\alpha - \beta \log \sigma_t^2)\, dt + \eta dW_{\sigma,t}, \end{aligned} \tag{4.1}$$

where $W_p$ and $W_\sigma$ are standard Brownian motions with $\text{corr}\,(dW_p, dW_\sigma) = \rho$, $\sigma_t$ is a stochastic volatility factor and $\gamma_t$ is an *intraday effect*. We use the model parameters estimated by Andersen et al. (2002) on S&P500 prices: $\mu = 0.0304$, $\alpha = -0.012$, $\beta =$

[5]Alternatively, the variance of $\mathsf{GTMPV}^{**}(m;\sigma^4)$ could be estimated, in small samples, using wild bootstrap. Our numerical experiments indicate that the two ways of estimating the variance are equivalent, so that we suggest to use the handy asymptotic approximation.

0.0145, $\eta = 0.1153$, $\rho = -0.6127$, where the parameters are expressed in daily units and returns are in percentage, and we use:

$$\gamma_{t,\tau} = \frac{1}{0.1033}(0.1271\tau^2 - 0.1260\tau + 0.1239),$$

as estimated by us on S&P500 intraday returns. We discretize model (4.1) in the interval $[0,1]$ with the Euler scheme, using a discretization step of $\sqrt{1/n}$.

Instead of specifying the jump process as a compound Poisson process with random jump sizes we restrict its realizations to a fixed number of jumps of a known size. In particular, we consider the case of absence of jumps and the case of a single jump with deterministic size equal to $3\sqrt{1/n}$ (small jump, notice that in the simulations $\sigma_t \simeq 1$) and $10\sqrt{1/n}$ (big jump).

In order to make the Monte Carlo simulations more realistic, we further simulate additional frictions which are not considered in the theoretical framework of this paper, but are known to be present in the data. The frictions we consider are microstructure noise, in the form of distortions to the price process, and flat prices, that is prices that do not change due, for example, to liquidity reasons.[6] We thus consider three possible scenarios:

1. The price process is observed without frictions.
2. The price process is contaminated by microstructure noise.
3. The price process is contaminated by flat prices.

In order to save space, since microstructure noise and flat pricing have the largest impact at high frequency, we study these frictions only at $n = 400$.

To estimate quarticity we implement standard (non truncated) quadpower variations $\mathsf{MPV}([1,1,1,1])$, threshold tripower and quadpower variations, that is $\mathsf{TMPV}([\frac{4}{3},\frac{4}{3},\frac{4}{3}])$

[6] Multipower estimators could be robustified against the presence of market microstructure noise using pre averaging techniques, see e.g. Hautsch and Podolskij (2013). On flat pricing, a theoretical analysis is offered by Phillips and Yu (2009) and Bandi et al. (2014).

and $\mathsf{TMPV}([1,1,1,1])$ respectively; the single power $\mathsf{TMPV}([4])$ estimator; and multi-power estimators $\mathsf{GTMPV}^{**}(m)$ for different values of $m$ (for simplicity, we omit the $\sigma^4$ in the notation). We also implement the nearest neighbor minRQ and medRQ estimators defined in Eq. (2.19) and Eq. (2.20) Finally, we consider the efficient estimator $\mathsf{QVeff}(k_n)$ of Jacod and Rosenbaum (2013), see Eq. (A.4), implemented with the bias correction proposed in Jacod and Rosenbaum (2015) for different values of $k_n$.

In order to set up the threshold for truncated multipowers, we use

$$\vartheta_t = c_\vartheta \cdot \widehat{\sigma}_t^n \tag{4.2}$$

with $c_\vartheta = 5$ and and $\widehat{\sigma}_t^n$ is an estimator of local standard deviation (that is over the interval $\sqrt{1/n}$) obtained as in Corsi et al. (2010). The choice of $c_\vartheta = 5$ is meant to truncate only returns that are extremely large with respect to the estimated local standard deviation.

On each replication we compute the generated quarticity value $IQ_k$ and the estimated quarticity value $\widehat{IQ}_k$ according to different estimators ($k = 1, \ldots, M$). We report the relative bias,

$$\text{Bias} = \frac{1}{M} \sum_{k=1}^{M} \frac{\widehat{IQ}_k - IQ_k}{IQ_k},$$

the relative standard deviation,

$$\text{Std} = \sqrt{\frac{1}{M} \sum_{k=1}^{M} \left( \frac{\widehat{IQ}_k - IQ_k}{IQ_k} - bias \right)^2},$$

and the relative Root Mean Square Error:

$$\text{RMSE} = \sqrt{\text{Std}^2 + \text{Bias}^2}.$$

The figures are computed with $M = 10,000$ replications.

## 4.1 Estimation without frictions

Tables 4, 5, 6 report the results in the case $n = 40$ (10 minutes), $n = 80$ (5 minutes), $n = 400$ (1 minute) respectively for the competing estimators, and in three cases: absence of jumps, presence of a single small jump and presence of a single large jump.

Generally speaking, the performance of truncated estimators is largely better than non-truncated ones. In the case with a big jump, $\mathsf{MPV}$ is literally shattered: the RMSE decreases with increasing frequency, but it is still roughly +400% at the one-minute frequency. Basically, non-truncated estimators are severely biased. Standard threshold multipower estimators reduce the bias considerably. For example, in the 5-minute case in the presence of a small jump, the bias goes from the +151% of the standard $\mathsf{MPV}([1,1,1,1])$ estimator to the −6% of the truncated $\mathsf{TMPV}([1,1,1,1])$ estimator. However, as we discuss below, the RMSE of $\mathsf{TMPV}([1,1,1,1])$ (and $\mathsf{TMPV}([\frac{4}{3},\frac{4}{3},\frac{4}{3}])$) is higher than that of the $\mathsf{GTMPV}^{**}$ estimator because of the efficiency loss.

Efficient multipowers also perform better than the neighbor truncation min-med estimators. At the frequency of 5 minutes (the most typical in applications) the RMSE of min-med estimators is more than double than that of $\mathsf{GTMPV}^{**}$ estimators, both in the case in which there are small or large jumps, and still roughly 50% higher on paths without jumps, which again reflects the efficiency loss of these estimators.

Regarding the $\mathsf{GTMPV}^{**}(m)$ estimators, we can see that they generally display a small negative bias, which is consistent with the theory. Using the efficient estimator, we obtain a gain in terms of relative RMSE, with respect to the standard multipower estimators $\mathsf{TMPV}([1,1,1,1])$ and $\mathsf{TMPV}([\frac{4}{3},\frac{4}{3},\frac{4}{3}])$, of roughly 20% at 10 minutes (from 60% to 40%), of 13% at 5 minutes (from 43% to 30%) and of 5% at 1 minute (from 20% to 15%). Thus, the gain is substantial and is entirely due to the smaller variance of the efficient estimators, which more than offsets the loss in bias due to the higher value of $m$ employed.

The performance of the efficient estimators of Jacod and Rosenbaum (2013) reflects its asymptotic nature: it improves with large $k_n$. In order to compare it with the

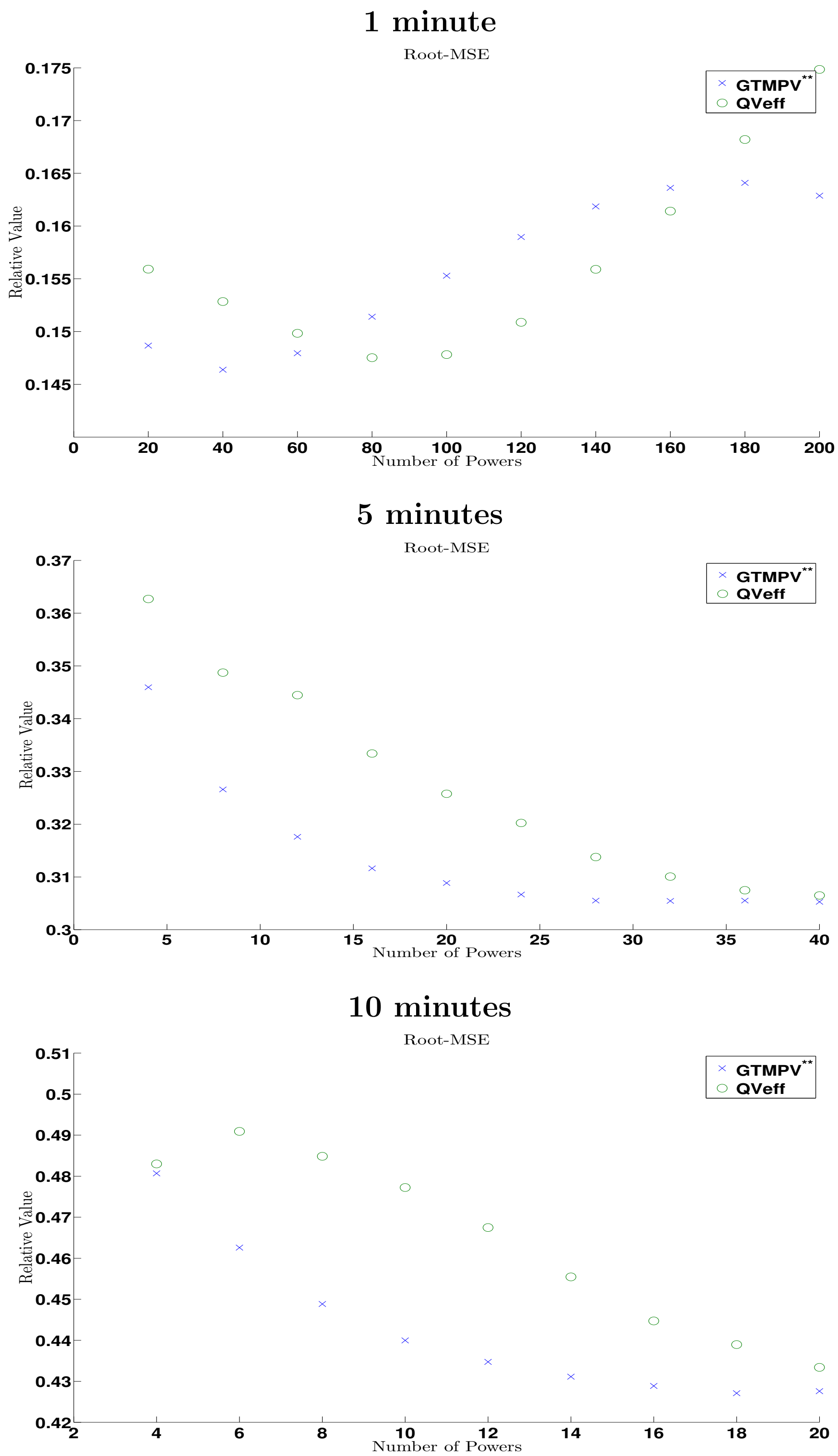


Figure 2: Relative RMSE obtained on simulations for the $\mathsf{GTMPV}**(m;4)$ and $\mathsf{QVeff}$, the Jacod and Rosenbaum estimator with $k_n = m+1$ for different values of $m$, at three different sampling frequencies.

$\mathsf{GTMPV}^{**}(m;4)$, we fix $k_n = m+1$ and compute the two estimators for various $m$. The relative RMSE for the three different frequencies used in the simulation study in the case without jumps is shown in Figure 2. It is clear that using the estimator $\mathsf{GTMPV}^{**}(m;4)$, which has a lower variance than the Jacod and Rosenbaum estimator for fixed $m$, can obtain a better result with respect to an estimator which is designed to have the lowest possible variance when $m$ (that is, $k_n - 1$) diverges to $+\infty$. Again, this is important in practice, since the number of multipowers (or the window used for preliminary estimates of spot variance) is actually fixed. The performance of the two estimators tend to be similar when $m$ is large, since the two estimators coincide (excluding finite sample bias corrections) for large $m$. The Jacod and Rosenbaum estimator is also better when $n$ is large (at the 1-minute frequency) for intermediate values of $m$.

Summarizing, the Monte Carlo experiments in this section show that i) truncating returns is essential to get reasonable quarticity estimates in the presence of jumps; ii) standard multipower estimators, including the min-med estimator, suffer substantial efficiency loss with respect to the efficient multipower estimators, which results in a deteriorated estimate in terms of the mean square error; iii) an efficient estimator designed for fixed $m$ can be beneficial, in terms of mean square error, with respect to an efficient estimator designed for diverging $m$.

## 4.2 Estimation in the presence of microstructure noise

The first type of friction we introduce in simulated experiments is microstructure noise in the form of autocorrelated price distortion. The observed prices, $X_j$, are generated as follows:

$$X_j = X_j^* + \epsilon_j, \tag{4.3}$$

where $X_j^*$ is simulated as in the previous section, and

$$\epsilon_j = \rho_\epsilon \epsilon_{j-1} + \epsilon_j^*, \quad \epsilon_j^* \sim \mathcal{N}(0, \sigma_\epsilon^2). \tag{4.4}$$

Table 4: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $\mathbf{n} = \mathbf{40}$ intraday returns generated by model (4.1). The threshold is set as in Eq. (4.2) with $c_\vartheta = 5$.

| | **no jumps** | | | **small jump** | | | **big jump** | | |
|---|---|---|---|---|---|---|---|---|---|
| | RMSE | Std | Bias | RMSE | Std | Bias | RMSE | Std | Bias |
| MPV([1, 1, 1, 1]) | 0.6036 | 0.5998 | -0.0674 | 3.6144 | 3.0225 | 1.9820 | 11.9077 | 9.6054 | 7.0378 |
| TMPV([4]) | 0.5353 | 0.5353 | -0.0007 | 0.5286 | 0.5277 | -0.0306 | 0.5388 | 0.5379 | -0.0320 |
| TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$) | 0.5914 | 0.5898 | -0.0441 | 0.5652 | 0.5520 | -0.1218 | 0.5864 | 0.5747 | -0.1167 |
| TMPV([1, 1, 1, 1]) | 0.6041 | 0.6002 | -0.0686 | 0.5758 | 0.5519 | -0.1641 | 0.5946 | 0.5738 | -0.1558 |
| minRQ | 0.7094 | 0.7092 | -0.0166 | 1.1396 | 1.1123 | 0.2478 | 1.0556 | 1.0307 | 0.2278 |
| medRQ | 0.6053 | 0.6042 | -0.0359 | 0.9878 | 0.9566 | 0.2461 | 0.9228 | 0.8931 | 0.2326 |
| GTMPV$^{**}$(4) | 0.4751 | 0.4743 | -0.0272 | 0.4719 | 0.4695 | -0.0478 | 0.4878 | 0.4855 | -0.0480 |
| GTMPV$^{**}$(8) | 0.4432 | 0.4392 | -0.0594 | 0.4405 | 0.4330 | -0.0810 | 0.4546 | 0.4475 | -0.0801 |
| GTMPV$^{**}$(12) | 0.4278 | 0.4202 | -0.0803 | 0.4299 | 0.4178 | -0.1010 | 0.4385 | 0.4265 | -0.1021 |
| GTMPV$^{**}$(16) | 0.4220 | 0.4121 | -0.0912 | 0.4253 | 0.4106 | -0.1109 | 0.4326 | 0.4180 | -0.1115 |
| GTMPV$^{**}$(20) | 0.4198 | 0.4095 | -0.0924 | 0.4249 | 0.4105 | -0.1094 | 0.4328 | 0.4187 | -0.1099 |
| GTMPV$^{**}$(24) | 0.4213 | 0.4132 | -0.0820 | 0.4259 | 0.4149 | -0.0961 | 0.4344 | 0.4236 | -0.0963 |
| QVeff(4) | 0.4785 | 0.4708 | -0.0857 | 0.4727 | 0.4556 | -0.1261 | 0.4894 | 0.4732 | -0.1250 |
| QVeff(8) | 0.4777 | 0.4773 | -0.0192 | 0.4636 | 0.4592 | -0.0636 | 0.4799 | 0.4756 | -0.0635 |
| QVeff(12) | 0.4557 | 0.4546 | -0.0325 | 0.4460 | 0.4389 | -0.0790 | 0.4566 | 0.4495 | -0.0802 |
| QVeff(16) | 0.4376 | 0.4339 | -0.0567 | 0.4310 | 0.4180 | -0.1052 | 0.4387 | 0.4258 | -0.1056 |
| QVeff(20) | 0.4268 | 0.4188 | -0.0821 | 0.4256 | 0.4061 | -0.1275 | 0.4319 | 0.4126 | -0.1278 |
| QVeff(24) | 0.4266 | 0.4157 | -0.0957 | 0.4267 | 0.4024 | -0.1418 | 0.4325 | 0.4087 | -0.1412 |

We consider a persistent noise process ($\rho_\epsilon = 0.5$). The microstructure noise is virtually not present at moderate frequencies, hence we do not consider the 5- and 10-minute frequencies in the present subsection. For $n = 400$ (1-minute), as in the simulation design of Podolskij and Vetter (2009), we set $\sigma_\epsilon^2 = 0.0005 IV_t$, with $IV_t$ denoting the daily integrated variance.

Results, shown in Table 7, show that microstructure noise induces a strong distortion in all estimates, in form of an upper bias of roughly +50% which is very similar across all considered estimators. Still, the observed variance of efficient multipowers is the lowest among competitors.

Even if the magnitude of the bias due to microstructure noise could vary given a different data generating process for the price dynamics and the noise itself, these results suggest that the impact of the noise is anyway translated into an upper bias which affects the competing estimators very similarly, suggesting that it would still be beneficial to concentrate on variance reduction even in the presence of this kind of friction.

Table 5: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $\mathbf{n} = \mathbf{80}$ intraday returns generated by model (4.1). The threshold is set as in Eq. (4.2) with $c_\vartheta = 5$.

| | **no jumps** | | | **small jump** | | | **big jump** | | |
|---|---|---|---|---|---|---|---|---|---|
| | RMSE | Std | Bias | RMSE | Std | Bias | RMSE | Std | Bias |
| MPV([1,1,1,1]) | 0.4376 | 0.4362 | -0.0356 | 2.7541 | 2.3016 | 1.5124 | 9.1583 | 7.4072 | 5.3860 |
| TMPV([4]) | 0.3761 | 0.3759 | -0.0130 | 0.3806 | 0.3802 | -0.0174 | 0.3805 | 0.3801 | -0.0161 |
| TMPV($[\frac{4}{3},\frac{4}{3},\frac{4}{3}]$) | 0.4248 | 0.4240 | -0.0263 | 0.4263 | 0.4209 | -0.0676 | 0.4247 | 0.4202 | -0.0620 |
| TMPV([1,1,1,1]) | 0.4380 | 0.4364 | -0.0366 | 0.4413 | 0.4319 | -0.0905 | 0.4342 | 0.4257 | -0.0854 |
| minRQ | 0.5051 | 0.5049 | -0.0153 | 0.6991 | 0.6877 | 0.1255 | 0.6497 | 0.6376 | 0.1246 |
| medRQ | 0.4362 | 0.4354 | -0.0259 | 0.5935 | 0.5799 | 0.1265 | 0.5781 | 0.5629 | 0.1317 |
| GTMPV**(5) | 0.3361 | 0.3350 | -0.0278 | 0.3406 | 0.3389 | -0.0342 | 0.3391 | 0.3377 | -0.0306 |
| GTMPV**(10) | 0.3208 | 0.3167 | -0.0510 | 0.3221 | 0.3167 | -0.0591 | 0.3217 | 0.3169 | -0.0551 |
| GTMPV**(15) | 0.3138 | 0.3060 | -0.0696 | 0.3136 | 0.3037 | -0.0781 | 0.3134 | 0.3043 | -0.0748 |
| GTMPV**(20) | 0.3091 | 0.2973 | -0.0845 | 0.3100 | 0.2958 | -0.0930 | 0.3090 | 0.2957 | -0.0898 |
| GTMPV**(30) | 0.3065 | 0.2890 | -0.1020 | 0.3081 | 0.2879 | -0.1098 | 0.3070 | 0.2878 | -0.1068 |
| GTMPV**(40) | 0.3072 | 0.2894 | -0.1030 | 0.3078 | 0.2877 | -0.1094 | 0.3070 | 0.2878 | -0.1069 |
| QVeff(5) | 0.3427 | 0.3338 | -0.0776 | 0.3461 | 0.3332 | -0.0935 | 0.3460 | 0.3343 | -0.0889 |
| QVeff(10) | 0.3419 | 0.3414 | -0.0192 | 0.3411 | 0.3389 | -0.0389 | 0.3414 | 0.3397 | -0.0346 |
| QVeff(15) | 0.3349 | 0.3341 | -0.0231 | 0.3315 | 0.3286 | -0.0438 | 0.3320 | 0.3296 | -0.0399 |
| QVeff(20) | 0.3261 | 0.3237 | -0.0388 | 0.3215 | 0.3159 | -0.0599 | 0.3219 | 0.3168 | -0.0569 |
| QVeff(30) | 0.3117 | 0.3023 | -0.0762 | 0.3126 | 0.2971 | -0.0973 | 0.3111 | 0.2962 | -0.0951 |
| QVeff(40) | 0.3095 | 0.2900 | -0.1081 | 0.3118 | 0.2838 | -0.1291 | 0.3102 | 0.2827 | -0.1276 |

Table 6: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $\mathbf{n} = \mathbf{400}$ intraday returns generated by model (4.1). The threshold is set as in Eq. (4.2) with $c_\vartheta = 5$.

| | **no jumps** | | | **small jump** | | | **big jump** | | |
|---|---|---|---|---|---|---|---|---|---|
| | RMSE | Std | Bias | RMSE | Std | Bias | RMSE | Std | Bias |
| MPV([1,1,1,1]) | 0.2094 | 0.2092 | -0.0096 | 1.2485 | 1.0169 | 0.7245 | 4.1396 | 3.3419 | 2.4431 |
| TMPV([4]) | 0.1772 | 0.1772 | -0.0019 | 0.1748 | 0.1746 | -0.0068 | 0.1749 | 0.1748 | -0.0062 |
| TMPV($[\frac{4}{3},\frac{4}{3},\frac{4}{3}]$) | 0.2017 | 0.2016 | -0.0069 | 0.1953 | 0.1948 | -0.0148 | 0.1986 | 0.1981 | -0.0141 |
| TMPV([1,1,1,1]) | 0.2094 | 0.2092 | -0.0099 | 0.2034 | 0.2024 | -0.0199 | 0.2069 | 0.2060 | -0.0188 |
| minRQ | 0.2373 | 0.2373 | -0.0015 | 0.2449 | 0.2437 | 0.0243 | 0.2501 | 0.2490 | 0.0236 |
| medRQ | 0.2046 | 0.2046 | -0.0049 | 0.2158 | 0.2144 | 0.0243 | 0.2198 | 0.2185 | 0.0238 |
| GTMPV**(20) | 0.1516 | 0.1499 | -0.0229 | 0.1505 | 0.1482 | -0.0260 | 0.1506 | 0.1483 | -0.0261 |
| GTMPV**(30) | 0.1501 | 0.1462 | -0.0337 | 0.1493 | 0.1447 | -0.0367 | 0.1493 | 0.1447 | -0.0369 |
| GTMPV**(40) | 0.1497 | 0.1432 | -0.0436 | 0.1491 | 0.1417 | -0.0463 | 0.1493 | 0.1418 | -0.0468 |
| GTMPV**(50) | 0.1504 | 0.1409 | -0.0527 | 0.1499 | 0.1393 | -0.0554 | 0.1499 | 0.1392 | -0.0558 |
| QVeff(20) | 0.1582 | 0.1581 | -0.0044 | 0.1558 | 0.1554 | -0.0104 | 0.1568 | 0.1565 | -0.0099 |
| QVeff(30) | 0.1569 | 0.1569 | -0.0042 | 0.1547 | 0.1544 | -0.0105 | 0.1549 | 0.1546 | -0.0099 |
| QVeff(40) | 0.1549 | 0.1547 | -0.0083 | 0.1534 | 0.1528 | -0.0141 | 0.1537 | 0.1530 | -0.0141 |
| QVeff(50) | 0.1534 | 0.1528 | -0.0139 | 0.1522 | 0.1510 | -0.0192 | 0.1521 | 0.1509 | -0.0198 |

## 4.3 Estimation in the presence of flat prices

We finally consider a form of friction which we document to be present at the highest frequencies, that is flat pricing. Flat pricing consists in the observation of spurious zero returns, which might be due to lack of liquidity in the market or asymmetric information (see the discussion in Bandi et al., 2014). In our simulation setting, we assume that the generated returns are given by:

$$\Delta_j X = \Delta_j X^* \cdot \psi_j, \tag{4.5}$$

where $X^*$ is the process generated without frictions, and

$$\psi_j = \begin{cases} 0, & \text{with probability} \quad p_\psi \\ 1, & \text{with probability} \quad (1 - p_\psi) \end{cases} \tag{4.6}$$

We set $p_\psi = 0.3$ and, as before, we consider only the case $n = 400$ (1-minute data). This choice is motivated by data analysis, since the phenomenon of flat prices tends to fade away, for the stocks we consider in the empirical application, at the 5-minutes frequency. Results are show in Table 8.

Even if the probability of flat trading we use in the simulated experiments is quite high with respect to the observed values (reported in Table 9), the impact on final estimates is quite small. For efficient multipowers, the relative RMSE increases from about 15% to 17% only. This is basically due to increased estimator variance, in line with the theoretical predictions of Phillips and Yu (2009) for realized variance. In this sense, microstructure noise and flat trading have an impact which is completely different on estimators, the first kind of friction mostly affecting the estimator bias, whereas the second kind of friction mostly affecting the estimator variance.

Interestingly, efficient multipowers are much more robust than competing estimators to this form of friction. Indeed, the relative RMSE standard TMPV estimators increases from roughly 20% to 45% for tripower and 60% for quadpower; for minRQ it increases from 23% to 30%; and for medRQ it increases from 20% to 23%. The gap between

Table 7: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $n = 400$ intraday returns generated by model (4.3), which includes **microstructure noise**. The threshold is set as in Eq. (4.2) with $c_\vartheta = 5$. The autocorrelation of the noise process is $\rho_\epsilon = 0.5$.

| | **no jumps** | | | **small jump** | | | **big jump** | | |
|---|---|---|---|---|---|---|---|---|---|
| | RMSE | Std | Bias | RMSE | Std | Bias | RMSE | Std | Bias |
| MPV([1, 1, 1, 1]) | 0.6615 | 0.3249 | 0.5762 | 2.1188 | 1.3929 | 1.5966 | 5.9511 | 4.4924 | 3.9030 |
| TMPV([4]) | 0.6352 | 0.2703 | 0.5748 | 0.6358 | 0.2646 | 0.5781 | 0.6251 | 0.2588 | 0.5690 |
| TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$) | 0.6546 | 0.3116 | 0.5756 | 0.6529 | 0.3144 | 0.5722 | 0.6366 | 0.3009 | 0.5610 |
| TMPV([1, 1, 1, 1]) | 0.6612 | 0.3253 | 0.5756 | 0.6555 | 0.3329 | 0.5647 | 0.6381 | 0.3204 | 0.5518 |
| minRQ | 0.6912 | 0.3640 | 0.5876 | 0.7336 | 0.3839 | 0.6251 | 0.7388 | 0.3802 | 0.6335 |
| medRQ | 0.6705 | 0.3225 | 0.5878 | 0.7060 | 0.3354 | 0.6212 | 0.7040 | 0.3257 | 0.6242 |
| GTMPV**(30) | 0.5889 | 0.2298 | 0.5422 | 0.5812 | 0.2252 | 0.5358 | 0.5718 | 0.2201 | 0.5277 |
| GTMPV**(40) | 0.5765 | 0.2262 | 0.5303 | 0.5688 | 0.2207 | 0.5243 | 0.5596 | 0.2173 | 0.5157 |
| GTMPV**(50) | 0.5654 | 0.2245 | 0.5190 | 0.5567 | 0.2166 | 0.5128 | 0.5493 | 0.2152 | 0.5054 |
| QVeff(30) | 0.6287 | 0.2430 | 0.5798 | 0.6155 | 0.2388 | 0.5673 | 0.6076 | 0.2320 | 0.5616 |
| QVeff(40) | 0.6233 | 0.2393 | 0.5755 | 0.6113 | 0.2349 | 0.5644 | 0.6036 | 0.2302 | 0.5579 |
| QVeff(50) | 0.6141 | 0.2339 | 0.5678 | 0.6056 | 0.2342 | 0.5585 | 0.5960 | 0.2282 | 0.5506 |

Table 8: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $n = 400$ intraday returns generated by model (4.5), which includes **flat pricing**. The threshold is set as in Eq. (4.2) with $c_\vartheta = 5$. The probability of observing a zero return is $p_\epsilon = 0.3$.

| | **no jumps** | | | **small jump** | | | **big jump** | | |
|---|---|---|---|---|---|---|---|---|---|
| | RMSE | Std | Bias | RMSE | Std | Bias | RMSE | Std | Bias |
| MPV([1, 1, 1, 1]) | 0.5742 | 0.1347 | -0.5582 | 0.7554 | 0.6872 | -0.3135 | 2.1987 | 2.1738 | 0.3301 |
| TMPV([4]) | 0.3895 | 0.2752 | 0.2757 | 0.4212 | 0.2868 | 0.3085 | 0.4084 | 0.2875 | 0.2900 |
| TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$) | 0.4301 | 0.1555 | -0.4010 | 0.4916 | 0.1465 | -0.4692 | 0.4672 | 0.1470 | -0.4435 |
| TMPV([1, 1, 1, 1]) | 0.5760 | 0.1347 | -0.5600 | 0.6283 | 0.1265 | -0.6154 | 0.6465 | 0.1142 | -0.6363 |
| minRQ | 0.3009 | 0.2286 | -0.1957 | 0.3318 | 0.2575 | -0.2093 | 0.3139 | 0.2627 | -0.1719 |
| medRQ | 0.2337 | 0.2183 | -0.0834 | 0.2596 | 0.2531 | -0.0577 | 0.2575 | 0.2527 | -0.0495 |
| GTMPV**(30) | 0.1737 | 0.1694 | -0.0384 | 0.1793 | 0.1736 | -0.0449 | 0.1780 | 0.1725 | -0.0442 |
| GTMPV**(40) | 0.1731 | 0.1652 | -0.0518 | 0.1789 | 0.1694 | -0.0574 | 0.1777 | 0.1682 | -0.0573 |
| GTMPV**(50) | 0.1736 | 0.1622 | -0.0618 | 0.1795 | 0.1654 | -0.0696 | 0.1781 | 0.1648 | -0.0677 |
| QVeff(30) | 0.1833 | 0.1833 | -0.0018 | 0.1880 | 0.1879 | -0.0050 | 0.1862 | 0.1860 | -0.0096 |
| QVeff(40) | 0.1801 | 0.1799 | -0.0092 | 0.1866 | 0.1861 | -0.0148 | 0.1834 | 0.1825 | -0.0178 |
| QVeff(50) | 0.1790 | 0.1781 | -0.0181 | 0.1835 | 0.1815 | -0.0271 | 0.1817 | 0.1792 | -0.0301 |

efficient multipowers and competing estimator becomes then wider in the presence of flat trading.

Table 9: Reports the list of the sixteen blue chip stocks used in the empirical application, their ticker and the percentage of zero-returns at one and five minutes.

| Company | Ticker | 1-minute zero returns (%) | 5-minute zero returns (%) |
|---|---|---|---|
| Bank of America | BAC | 9.46 | 4.71 |
| Citigroup Inc. | C | 12.51 | 7.77 |
| JPMorgan Chase & Co. | JPM | 6.24 | 2.76 |
| Wells Fargo & Company | WFC | 6.24 | 2.76 |
| Boeing | BA | 5.60 | 2.39 |
| Caterpillar Inc. | CAT | 4.45 | 1.90 |
| FedEx Corporation | FDX | 4.93 | 2.05 |
| Honeywell International Inc. | HON | 7.96 | 3.41 |
| Hewlett-Packard Company | HPQ | 8.65 | 3.75 |
| International Business Machines Corp. | IBM | 4.02 | 1.83 |
| AT&T Inc. | T | 9.55 | 4.44 |
| Texas Instruments Incorporated | TXN | 12.94 | 6.09 |
| Kraft Foods Inc. | KFT | 13.33 | 6.32 |
| PepsiCo, Inc. | PEP | 8.19 | 3.61 |
| The Procter & Gamble Company | PG | 7.94 | 3.54 |
| Time Warner Inc. | TWX | 10.72 | 4.70 |

# 5 Empirical Application

The purpose of the empirical application is to apply the efficient multipower estimator of quarticity to real data from the financial market, to compare its performance with respect to competing estimators, and to evaluate the advantage in using efficient estimators in relevant applications, namely jump testing and volatility forecasting. As in the Monte Carlo section, we restrict our attention to the $\mathsf{GTMPV}(m)$ estimator in Eq. (3.2) (we omit the $\sigma^4$ in the notation for simplicity). By $m^*$ we indicate the value that minimizes the MSE in Eq. (3.12), using $\widehat{Q} = \mathsf{TMPV}([4])$.

The data set we use is the collection of sixteen blue chip stocks quoted on the New York Stock Exchange. The stocks are all very liquid and they are listed, together with their corresponding ticker, in Table 9. One-minute prices were recovered from the TickData One Minute Equity Data (OMED) dataset, from 3 January 2007 to 29 June 2012, for a total of $1,385$ trading days. Our sample then lies in the middle of the credit crunch crisis, characterized by very high volatility levels and a supposedly high number of jumps. The data went trough a standard filtering procedure. TickData one-minute equity data are adjusted for corporate actions such as mergers and acquisitions or symbol changes. Moreover, the underlying tick data used to build 1-minute time series are first controlled for cancelled trades, or records not temporally aligned with previous/subsequent data; then filtered to identify bad ticks which are corrected using validation with third-party sources. All the measures reported here are for daily units and percentage returns.

Table 9 also shows the percentage of zero returns at 1 minute and 5 minutes. We can see that the impact of flat trading at 1 minute is substantial in all stocks, while it is much less impactful at 5 minutes. On each day in the sample, we compute the same quarticity estimators used in the Monte Carlo section. The 1-minute frequency corresponds to $n = 390$; the 5-minutes and 10-minutes frequencies are just defined on the corresponding grids, and since there are many possible grids we subsample the estimators by averaging them over all possible grids.

## 5.1 Quarticity: efficient versus non-efficient estimation

Table 10 reports summary statistics on pooled daily quarticity estimates ($\widehat{IQ}$) and the ratio $\sqrt{\widehat{IQ}}/\widehat{IV}$, where we use $\mathsf{TRV}([2])$ as an estimator $\widehat{IV}$ of integrated volatility. Asymptotically, we expect this ratio to be always greater than 1, by Jensen inequality. However, since we are using estimated quantities, the ratio could be less than one. Using the same normalization for all measures allows the comparison of different days and different stocks. We apply the estimators at three sampling frequencies: 1, 5 and 10 minutes. In Table 10, we exclude the Flash Crash day (May 6, 2010) since the difference between truncated and non-truncated estimators in this specific day is many orders of magnitudes away than what observed in the other days.

We observe a large difference between mean and median quarticity estimates at all the considered frequencies, indicating a very skewed distribution. The difference is less pronounced for the ratios. We start the discussion with the five-minute frequency. Here, we

Table 10: Summary statistics of pooled quarticity estimators $\widehat{IQ}$ and the ratios $\sqrt{\widehat{IQ}}/\widehat{IV}$, where $\widehat{IV}$ is threshold realized variance (TRV([2])). The Flash Crash day is excluded.

| | $\widehat{IQ}$ | | | $\sqrt{\widehat{IQ}}/\widehat{IV}$ | | | |
|---|---|---|---|---|---|---|---|
| **1-minute frequency** | | | | | | | |
| | **mean** | **median** | **std** | **mean** | **median** | **std** | **kurtosis** |
| MPV($[\frac{4}{3},\frac{4}{3},\frac{4}{3}]$) | 316.7 | 3.5 | 9314.0 | 1.578 | 1.366 | 1.005 | 570.7 |
| MPV($[1,1,1,1]$) | 343.5 | 4.0 | 9852.4 | 1.721 | 1.463 | 1.227 | 526.5 |
| minRQ | 512.3 | 4.4 | 19736.3 | 1.895 | 1.530 | 3.477 | 12383.1 |
| medRQ | 464.7 | 4.4 | 17007.9 | 1.838 | 1.522 | 2.467 | 8600.2 |
| TMPV($[4]$) | 337.7 | 4.6 | 9290.8 | 1.769 | 1.582 | 0.810 | 54.3 |
| TMPV($[\frac{4}{3},\frac{4}{3},\frac{4}{3}]$) | 310.3 | 3.6 | 8677.6 | 1.576 | 1.394 | 0.782 | 81.0 |
| TMPV($[1,1,1,1]$) | 293.7 | 3.1 | 8601.8 | 1.480 | 1.310 | 0.744 | 99.0 |
| GTMPV$^{**}(m^*)$ | 331.3 | 4.5 | 9219.9 | 1.750 | 1.564 | 0.798 | 54.9 |
| $(m^*)$ | (5.7) | (3.0) | (7.2) | | | | |
| **5-minute frequency** | | | | | | | |
| | **mean** | **median** | **std** | **mean** | **median** | **std** | **kurtosis** |
| MPV($[\frac{4}{3},\frac{4}{3},\frac{4}{3}]$) | 207.3 | 1.9 | 7595.9 | 1.104 | 1.018 | 0.502 | 427.7 |
| MPV($[1,1,1,1]$) | 238.6 | 2.2 | 7893.6 | 1.203 | 1.081 | 0.765 | 2801.0 |
| minRQ | 315.9 | 2.3 | 10433.0 | 1.275 | 1.114 | 1.119 | 2102.6 |
| medRQ | 249.0 | 2.3 | 7799.7 | 1.236 | 1.103 | 0.957 | 2771.3 |
| TMPV($[4]$) | 209.8 | 2.5 | 6418.3 | 1.219 | 1.179 | 0.204 | 4.3 |
| TMPV($[\frac{4}{3},\frac{4}{3},\frac{4}{3}]$) | 194.2 | 1.9 | 7490.8 | 1.075 | 1.033 | 0.265 | 5.7 |
| TMPV($[1,1,1,1]$) | 177.9 | 1.7 | 7377.2 | 1.015 | 0.975 | 0.269 | 5.8 |
| GTMPV$^{**}(m^*)$ | 203.2 | 2.5 | 6296.7 | 1.203 | 1.160 | 0.201 | 4.6 |
| $(m^*)$ | (8.1) | (5.0) | (8.2) | | | | |
| GTMPV$^{**}(5)$ | 189.9 | 2.3 | 6228.4 | 1.144 | 1.109 | 0.150 | 5.0 |
| GTMPV$^{**}(10)$ | 165.1 | 2.0 | 5530.0 | 1.081 | 1.054 | 0.112 | 4.6 |
| GTMPV$^{**}(20)$ | 141.1 | 1.7 | 4621.9 | 1.005 | 0.990 | 0.070 | 5.5 |
| GTMPV$^{**}(30)$ | 125.0 | 1.6 | 3868.2 | 0.962 | 0.957 | 0.054 | 1.7 |
| GTMPV$^{**}(35)$ | 120.0 | 1.5 | 3634.4 | 0.948 | 0.947 | 0.052 | 7.0 |
| QVeff(5) | 191.0 | 2.3 | 6364.7 | 1.149 | 1.108 | 0.179 | 5.2 |
| QVeff(10) | 181.4 | 2.2 | 6236.2 | 1.129 | 1.096 | 0.144 | 5.0 |
| QVeff(20) | 162.3 | 1.9 | 5759.2 | 1.049 | 1.030 | 0.096 | 4.5 |
| QVeff(30) | 137.1 | 1.7 | 4539.8 | 0.982 | 0.973 | 0.069 | 5.0 |
| QVeff(35) | 128.6 | 1.6 | 4125.7 | 0.955 | 0.948 | 0.063 | 3.1 |
| **10-minute frequency** | | | | | | | |
| | **mean** | **median** | **std** | **mean** | **median** | **std** | **kurtosis** |
| MPV($[\frac{4}{3},\frac{4}{3},\frac{4}{3}]$) | 134.0 | 1.2 | 5062.8 | 0.914 | 0.836 | 0.484 | 1305.1 |
| MPV($[1,1,1,1]$) | 154.4 | 1.5 | 5477.1 | 1.009 | 0.906 | 0.601 | 836.2 |
| minRQ | 180.6 | 1.6 | 5316.7 | 1.075 | 0.929 | 0.788 | 756.2 |
| medRQ | 159.6 | 1.6 | 5372.0 | 1.043 | 0.936 | 0.706 | 1545.6 |
| TMPV($[4]$) | 165.4 | 1.7 | 7657.7 | 1.043 | 0.991 | 0.377 | 9.0 |
| TMPV($[\frac{4}{3},\frac{4}{3},\frac{4}{3}]$) | 123.7 | 1.2 | 4565.5 | 0.891 | 0.839 | 0.363 | 7.3 |
| TMPV($[1,1,1,1]$) | 111.8 | 1.0 | 4473.8 | 0.830 | 0.783 | 0.348 | 6.7 |
| GTMPV$^{**}(m^*)$ | 158.3 | 1.7 | 6959.0 | 1.036 | 0.985 | 0.369 | 9.3 |
| $(m^*)$ | (5.2) | (5.0) | (3.4) | | | | |

can see that the efficient multipower estimator $\mathsf{GTMPV}^{**}(m^*)$ has the lowest in-sample standard deviation, both in terms of average estimates and ratios. The average (data-driven) $m^*$ used in these computation is 8.1. To gain insight, we also report efficient multipowers at several values of $m$, and compare them with the Jacod-Rosembaum estimator $\mathsf{QVeff}(m)$. We can see that, as $m$ increases, the in-sample standard deviation and average of both estimators decreases, with the efficient multipower being less variable at all fixed frequencies. The choice of $m^*$ balances the bias and the variance. All these results are in line with the theory.

Figure 3 shows the estimated probability density functions of the ratio $\sqrt{\widehat{IQ}}/\widehat{IV}$ for pooled daily quarticity estimates, for different quarticity estimators: $\mathsf{TMPV}([\frac{4}{3}, \frac{4}{3}, \frac{4}{3}])$ (labelled threshold tripower), $\mathsf{TMPV}([1, 1, 1, 1])$ (threshold quadpower), minRV, medRV, $\mathsf{GTMPV}^{**}(m^*)$. We use 5-minute returns. The Figure shows clearly the empirical potential of the theory. The efficient quarticity estimator delivers a much more concentrated ratio, with largely thinner tails. The inefficient estimators have sparser ratios. The estimated ratio is spuriously less then one for 54.17% of estimates using threshold quadpower, 43.92% using threshold tripower, 35.55% using minRV and 33.37% using medRV; the violation is instead observed only in 12.95% of cases with the efficient estimator.[7]

At 1 minute and 10 minutes, results are qualitatively the same. At 1 minute, where the impact of market microstructure noise and flat trading is likely to be higher, standard multipowers display a smaller variability and a smaller average than efficient multipowers. At 10 minutes, where the bias is relatively higher since the number of observations is smaller, we observe the same phenomenon. Again, the efficient multipower estimator is implemented here to balance bias and variance in an optimal way. Finally, notice that, at all frequencies, the difference in the kurtosis between non-truncated and truncated ratios is particularly pronounced, indicating that truncated estimator have much thinner tails.

[7] We are taking advantage here also of the fact that $\mathsf{TMPV}[4]$ delivers, by construction, a ratio greater than one, and is also used to compute the bias when optimizing $m^*$.

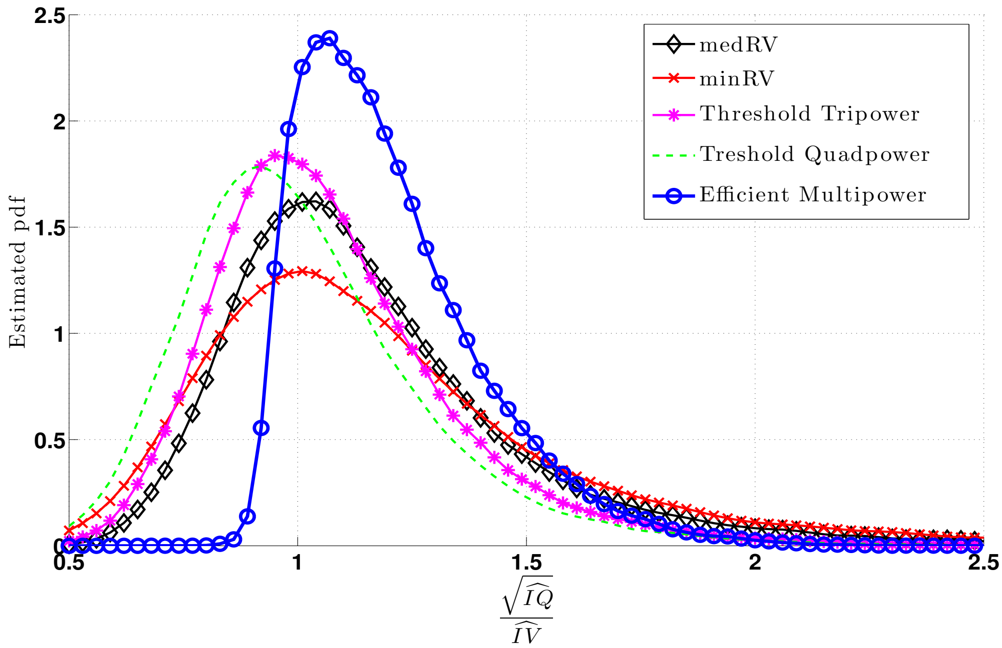


Figure 3: Estimated probability density functions of pooled daily ratios $\sqrt{\widehat{IQ}}/\widehat{IV}$, where $\widehat{IQ}$ are different quarticity estimators and $\widehat{IV}$ is threshold realized variance (TRV([2])). Estimates are obtained at the 5-minute frequency and the Flash Crash day is excluded.

## 5.2 Impact on jump testing

A popular way for testing for jumps is to take the difference between realized variance (which contains jumps) and bipower variation (which does not), and standardize this with the standard deviation of the difference. The technique has been basically laid out in Barndorff-Nielsen and Shephard (2006), see also Huang and Tauchen (2005), and is largely used in empirical work. We follow this empirical strategy using the version of the test proposed by Corsi et al. (2010), namely:

$$CTz(QV) = \frac{1}{n^{1/2}} \frac{\frac{RV - C\,\mathsf{TMPV}([1,1])}{RV}}{\sqrt{\theta \max\left(\frac{QV}{(C\,\mathsf{TMPV}([1,1])^2}, 1\right)}}, \tag{5.1}$$

where $\theta = (\pi^2/4 + \pi - 5)$. In the definition (5.1), $QV$ is a consistent quarticity estimator. Consistently with their proposed estimators, Corsi et al. (2010) suggest to use standard multipowers $\mathsf{TMPV}([4/3, 4/3, 4/3])$ or $\mathsf{TMPV}([1, 1, 1, 1])$ to estimate quarticity. In what follows, we study the sensitivity of the test to different quarticity estimators. In total we have $2392 \times 16 \approx 40\ 000$ tests in our sample.

Table 11 reports the number of detected jumps when standard (threshold tripower, and threshold quadpower) and efficient ($\mathsf{GTMPV}^{**}(m^*)$) multipower estimators are used. At all confidence intervals, at all sampling frequencies, the number of detected jumps is significantly larger than what predicted by the confidence interval, as exhaustively reported by the empirical literature. However, when using efficient multipower, the percentage of detected jumps reduces drastically, reducing the number of detection of roughly a half.

This empirical finding might help in explaining why, in the literature, it appears that "too many" jumps are detected, as documented in Christensen et al. (2014) using ultra-high frequency data. We suggest, indeed, that most of the jumps are spurious artifact of inefficient quarticity measurements. Our results show also that the impact of such spurious detections can be substantially reduced by employing an efficient quarticity estimator.

### 5.3 Impact on volatility forecasting

As suggested by Bollerslev et al. (2015), quarticity estimation can assist volatility forecasting. Denote by $\widehat{IV}_t$ at day $t$ an integrated variance estimator, and define $\widehat{IV}_{t-j|t-h} = \frac{1}{h}\sum_{i=j}^{h} \widehat{IV}_{t-j}$. One of the most popular model for forecasting daily integrated variance is the Heterogeneous Autoregression (HAR) model of Corsi (2009):

$$\widehat{IV}_t = \beta_0 + \beta_1 \widehat{IV}_1 + \beta_2 \widehat{IV}_{t-1|t-5} + \beta_3 \widehat{IV}_{t-1|t-22} + u_t, \tag{5.2}$$

where $u_t$ is a stationary error process. It is well known that the $\beta$ coefficients estimated with the HAR model are affected by measurement errors in the realized volatilities. In order to account for the presence of the measurement errors, Bollerslev et al. (2015)

Table 11: Reports the percentage of detected jumps in our sample three confidence intervals (99%,99.9%,99.99%); three different quarticity estimators (the standard threshold tripower and quadpower variation, and the efficient multipower variation); three different sampling frequencies (1, 5, 10 minutes). Results indicate a drastic reduction of detected jumps when the efficient multipower estimator is used.

| c.i. | standard quarticity estimation | | efficient quarticity estimation |
|---|---|---|---|
| | $\mathsf{TMPV}([\frac{4}{3}, \frac{4}{3}, \frac{4}{3}])$ | $\mathsf{TMPV}([1,1,1,1])$ | $\mathsf{GTMPV}^{**}(m^*)$ |
| | 1 minute frequency - percentage of detected jumps | | |
| 99% | 45.12 | 48.11 | 35.22 |
| 99.9% | 31.29 | 34.66 | 20.31 |
| 99.99% | 22.77 | 26.22 | 13.68 |
| | 5 minutes frequency - percentage of detected jumps | | |
| 99% | 20.56 | 21.65 | 12.30 |
| 99.9% | 12.84 | 13.56 | 8.23 |
| 99.99% | 8.69 | 9.14 | 5.68 |
| | 10 minutes frequency - percentage of detected jumps | | |
| 99% | 18.31 | 18.96 | 11.92 |
| 99.9% | 10.38 | 10.72 | 6.99 |
| 99.99% | 6.06 | 6.26 | 3.92 |

introduce the HARQ model, whose dynamics is specified by:

$$\widehat{IV}_t = \beta_0 + \left(\beta_1 + \beta_{1,Q}\widehat{IQ}_{t-1}^{1/2}\right)\widehat{IV}_1 + \beta_2\widehat{IV}_{t-1|t-5} + \beta_3\widehat{IV}_{t-1|t-22} + u_t, \tag{5.3}$$

where $\widehat{IQ}_{t-1}$ is an integrated quarticity estimator at day $t$. We then study the sensitivity of the results obtained with model (5.3) to different quarticity estimator.

The variable to be forecasted in our exercise is $\widehat{IV} = \mathsf{TRV}$, that is threshold realized variance. The different quarticity estimators are those examined so far. Our exercise is fully out-of-sample. We obtain the forecast of $\widehat{IV}_t$ at time $t$ forecast using estimation of

Table 12: Reports the out-of-sample forecast relative RMSE of $HARQ$ model for TRV with different quarticity estimators, standardized by the relative RMSE of the HAR model, so that a value less than 1 indicates superiority with respect to the HAR model. In bold, we indicate the estimator with the best performance for the given stock.

| | Quarticity estimators | | | | | |
|---|---|---|---|---|---|---|
| Ticker | minRQ | medRQ | TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$) | TMPV($[1, 1, 1, 1]$) | TMPV($[4]$) | GTMPV$^{**}(m^*)$ |
| BAC | 0.8584 | 0.8504 | 0.8704 | 0.8767 | 0.7439 | **0.7425** |
| C | 0.7600 | **0.7584** | 0.7842 | 0.7853 | 0.7958 | 0.7948 |
| JPM | 0.9008 | 0.9156 | 0.9125 | 0.9433 | **0.8610** | 0.8619 |
| WFC | 0.9565 | 0.9517 | 0.9539 | 0.9651 | 0.9332 | **0.9329** |
| BA | 0.9654 | 0.9602 | **0.9519** | 0.9570 | 0.9630 | 0.9641 |
| CAT | 0.9348 | 0.9405 | 0.9385 | 0.9342 | 0.9328 | **0.9318** |
| FDX | 0.9754 | 0.9607 | 0.9608 | 0.9696 | **0.9458** | 0.9460 |
| HON | 0.8769 | 0.8721 | 0.8707 | 0.8785 | **0.8672** | 0.8681 |
| IBM | 0.9267 | 0.9194 | 0.9182 | 0.9167 | 0.9131 | **0.9100** |
| HPQ | **0.8918** | 0.8932 | 0.9139 | 0.9345 | 0.9263 | 0.9273 |
| TXN | 0.9903 | 0.9812 | 0.9686 | **0.9574** | 0.9642 | 0.9640 |
| T | 0.9978 | 0.9153 | 0.9245 | 0.9311 | 0.9023 | **0.9020** |
| KFT | 1.0094 | 1.0078 | 0.9888 | 0.9858 | 0.9799 | **0.9798** |
| PEP | 0.9646 | 0.9653 | **0.9497** | 0.9549 | 0.9541 | 0.9541 |
| PG | 0.8581 | **0.8559** | 0.8610 | 0.8628 | 0.8620 | 0.8627 |
| TWX | 0.9663 | 0.9533 | 0.9545 | **0.9503** | 0.9614 | 0.9607 |
| mean | 0.9271 | 0.9188 | 0.9201 | 0.9252 | 0.9066 | **0.9064** |

model (5.3) on the past year till day $t-1$, so that estimates are performed on a rolling window. As the loss function, we use the traditional Root Mean Square (relative) Error. Table 12 shows the RMSE of the HARQ model corresponding to different quarticity estimates, standardized by RMSE obtained with the HAR model.

We confirm the empirical evidence of Bollerslev et al. (2015): The HARQ model provides superior forecasts with respect to the HAR model, since the ratio is generally less than one, with rare exceptions, for all stocks and all quarticity estimators. Among different quarticity, efficient multipower achieve the best value of the loss function in most of the time, and is also the best estimator on average, indicating a clear advantage in using our efficient estimator. The second-best estimator is TMPV([4]), which is also an efficient estimator (with $m = 1$), and which has been indicated by Bollerslev et al. (2015) as their best performing quarticity estimator. Again, this is an indication that an estimator built to be more precise can yield substantial improvement also in terms of volatility forecasting.

# 6 Conclusions

We studied the efficiency of multipower estimator, and provided methods to find efficient estimators in a given the class of linear combinations of multipower estimators. In particular, we propose a specific quarticity estimator which is more efficient than estimators in the literature in the case in which the numbers of multipowers employed is fixed.

Based on these results, we show, on simulated data, that efficient multipowers outperform benchmark estimators in terms of mean square error. With respect to the Jacod and Rosenbaum (2013) quarticity estimator, we improve in the fact that the estimator we propose has a smaller variance when $m$, the number of multipowers employed, is fixed; while the Jacod and Rosenbaum estimator is the globally efficient when $m \to \infty$.

In practice, having an efficient estimator for fixed $m$ is more convenient. We indeed use this result to propose a data-driven selection of $m$ which, by construction, improves the loss function of any unbiased estimator.

Our empirical application confirms that estimation methods based on truncation provide quite different estimates (typically less variable) than what provided by traditional multipowers. Moreover, they help in delivering substantially less jumps than what previously found with high-frequency data, and in improving the quality of realized volatility forecasts. We thus conclude that efficient multipower estimators should replace existing alternatives for empirical work.

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# A Proofs

*Proof of Proposition 2.1.* First notice we can consider the case $X_t = \sigma W_t$ with constant $\sigma$, and set $\vartheta_t = +\infty, T = 1$, without loss of generality, see Theorem 11.2.1 in Jacod and Protter (2012). In this case, the parameter to be estimated is $\sigma^R$. Define:

$$h^j(\Delta_1 X, \ldots, \Delta_m X; \sigma^R) = c_{\mathbf{r}} \frac{n-m+1}{n} \cdot \prod_{k=1}^{m} |\Delta_k X|^{r_k^{(j)}} - \sigma^R \tag{A.1}$$

for $j = 1, \ldots, N$. By construction, we have $\mathbf{E}\left[h^j(\Delta_1 X, \ldots, \Delta_m X; \sigma^R)\right] = 0$, which can be interpreted as a set of $N$ moment restrictions in a GMM setting. The empirical moment vector is $\mathbf{h} = \left[\frac{1}{n-m+1}\sum_{i=1}^{n-m+1} h^j\left(\Delta_i X, \ldots, \Delta_{i+m-1} X; \sigma^R\right)\right]_{j=1,\ldots,N}$, which is proportional to the vector $\mathsf{TMPV}(\mathbf{r}^{(j)}) - \sigma^R$ for $j = 1, \ldots, N$. This shows that our estimation problem is equivalent to GMM. The GMM estimator of $\sigma^R$ is then given by

$$\widehat{\sigma^R} = \arg\min_{\sigma^R} \mathbf{h}' W^{-1} \mathbf{h},$$

for a suitable weighting matrix $W$. It is well known that the minimum variance is achieved for $W = C$, where $C$ is the covariance matrix of the vector of moment restrictions $h^j$ as $n \to \infty$. Direct calculations, as in Barndorff-Nielsen et al. (2006a), show that $C$ is given by formula (2.7).

The efficient estimator corresponds to the minimum of the quadratic form $\mathbf{h}'C^{-1}\mathbf{h}$. The first order condition is:

$$C^{-1}\mathbf{h} = C^{-1}(\mathbf{h} + i_N \widehat{\sigma^R}) - C^{-1} i_N \widehat{\sigma^R} = \mathbf{0}, \tag{A.2}$$

where $i_N$ is the $N \times 1$ vector of ones, $\mathbf{0}$ is a vector of zeros, and the vector $\mathbf{T} = \left(\mathbf{h} + i_N \widehat{\sigma^R}\right)$, which has coefficients $\mathsf{TMPV}(\mathbf{r}^{(j)})$ for $j = 1, \ldots, N$, does not depend on $\widehat{\sigma^R}$. Multiplying Eq. (A.2) by $i_N'$ from the right we deduce that the efficient GMM estimator of $\sigma^R$ takes the form:

$$\widehat{\sigma^R} = \frac{i_N' C^{-1} \mathbf{T}}{\left(i_N' C^{-1} i_N\right)^{-1}}.$$

In other words, the efficient estimator of $\sigma^R$ is the weighted average of $\mathsf{TMPV}(\mathbf{r}^{(j)})$ with the optimal weights given Eq. (2.8). Straightforward computations show that, for a fixed set of powers, the minimum variance is indeed given by

$$\widetilde{V} = \frac{1}{i'_N C^{-1} i_N}.$$

□

*Proof of Proposition 3.1.* Without loss of generality, we can restrict to the model $X_t = \sigma W_t$ with constant $\sigma$, and set $\vartheta_t = +\infty, T = 1$. The proof of Eq. (3.3) and (3.4) is straightforward.

□

*Proof of Proposition 3.2.* Consider the corrected estimator of Jacod and Rosenbaum (2013) given by:

$$\begin{aligned}\Delta_n \left(1 - \frac{3}{k_n}\right) \sum_{i=1}^{n-k_n+1} (\widehat{c}_i^n)^2 + \frac{\Delta_n}{4} \sum_{i=1}^{t/\Delta_n - 2k_n+1} \left(\widehat{c}_{i+k_n}^n - \widehat{c}_i^n\right)^2 + \frac{(k_n-1)\,\Delta_n}{2} \left((\widehat{c}_1^n)^2 + \left(\widehat{c}_{t/\Delta_n - k_n+1}^n\right)^2\right) \\ = \Delta_n \left(1 - \frac{2}{k_n}\right) \sum_{i=1}^{n-k_n+1} (\widehat{c}_i^n)^2 + \frac{\Delta_n}{4} \sum_{i=1}^{t/\Delta_n - 2k_n+1} \left(\widehat{c}_{i+k_n}^n - \widehat{c}_i^n\right)^2 \\ - \frac{\Delta_n}{k_n} \sum_{i=1}^{n-k_n+1} (\widehat{c}_i^n)^2 + \frac{(k_n-1)\,\Delta_n}{2} \left((\widehat{c}_1^n)^2 + \left(\widehat{c}_{t/\Delta_n - k_n+1}^n\right)^2\right),\end{aligned} \tag{A.3}$$

where

$$\widehat{c}_i^n = \frac{n}{T}\frac{1}{k_n} \sum_{j=0}^{k_n-1} (\Delta_{i+j} X)^2 .$$

The Jacod and Rosenbaum (2013) estimator is equal to the first term of the equation (A.3) and it takes the form:

$$\mathsf{QVeff}(k_n) = \frac{T}{n}\left(1 - \frac{2}{k_n}\right) \sum_{i=1}^{n-k_n+1} \left(\widehat{c}_i^n\right)^2 . \tag{A.4}$$

We now show that the asymptotic behavior of $\mathsf{GTMPV}^{**}(m;\sigma^4)$ and $\mathsf{QVeff}(k_n)$ are the same. Write:

$$\mathsf{QVeff}(k_n) = \frac{T}{n}\left(1 - \frac{2}{k_n}\right) \sum_{i=1}^{n-k_n+1} \left(\frac{n}{T}\frac{1}{k_n} \sum_{j=0}^{k_n-1} (\Delta_{i+j} X)^2\right)^2$$

$$=\frac{n}{T}\left(1-\frac{2}{k_n}\right)\frac{1}{k_n^2}\sum_{i=1}^{n-k_n+1}\left(\sum_{j=0}^{k_n-1}(\Delta_{i+j}X)^2\right)^2$$

$$=\frac{n}{T}\left(1-\frac{2}{k_n}\right)\frac{1}{k_n^2}\left(\sum_{j=0}^{k_n-1}\sum_{i=1}^{n-k_n+1}(\Delta_{i+j}X)^4+2\sum_{j_1=0}^{k_n-1}\sum_{j_2=j_1+1}^{k_n-1}\sum_{i=1}^{n-k_n+1}(\Delta_{i+j_1}X)^2(\Delta_{i+j_2}X)^2\right)$$

Now, for all $j=0,\ldots,k_n-1$ we have:

$$\frac{1}{3}\frac{n}{T}\sum_{i=1}^{n-k_n+1}(\Delta_{i+j}X)^4=\mathsf{TMPV}([4])+O_p(k_n/n),$$

and, for all $j_1=0,\ldots,k_n-1$ and $j_2=j_1+1,\ldots,k_n-1$ we have

$$\frac{n}{T}\sum_{i=1}^{n-k_n+1}(\Delta_{i+j_1}X)^2(\Delta_{i+j_2}X)^2=\mathsf{TMPV}([2,\underbrace{0,\ldots,0}_{j_2-j_1-1\text{ terms}},2])+O_p(k_n/n)$$

This proves that the $\mathsf{QVeff}$ estimator can be written in a generalized form of the $\mathsf{GTMPV}(k_n-1,k_n-1;4)$ estimator, since it is a linear combination of the $k_n-1$ estimators $\mathsf{TMPV}([4])$, $\mathsf{TMPV}([2,2])$, $\mathsf{TMPV}([2,0,2])$, $\mathsf{TMPV}([2,0,0,2])$, $\ldots$, $\mathsf{TMPV}([2,\underbrace{0,\ldots,0}_{k_n-2\ terms},2])$ plus end-effects which are of order $k_n^2/n$ (since there are $k_n-1$ terms which are $O_p(k_n/n)$). The generalization is in the fact that the sum of the weights in the linear combination is not necessarily 1; but it has to converge to 1, as $n,k_n\to\infty$, since the two quantities converge to the same object.

When $k_n^2/n\to\theta$, equations (2.14), (2.16) and (2.17) of Jacod and Rosenbaum (2015) imply that

$$\sqrt{\frac{1}{\Delta_n}}\left(\frac{(k_n-1)\,\Delta_n}{2}\left((\widehat{c}_1^n)^2+\left(\widehat{c}^n_{t/\Delta_n-k_n+1}\right)^2\right)\right)\xrightarrow{P}B_1, \tag{A.5}$$

and

$$\sqrt{\frac{1}{\Delta_n}}\left(\frac{\Delta_n}{4}\sum_{i=1}^{t/\Delta_n-2k_n+1}\left(\widehat{c}^n_{i+k_n}-\widehat{c}^n_i\right)^2-\frac{\Delta_n}{k_n}\sum_{i=1}^{n-k_n+1}(\widehat{c}^n_i)^2\right)\xrightarrow{P}B_2, \tag{A.6}$$

while Theorem 2.6 in Jacod and Rosenbaum (2015) guarantees convergence of the jump part to the term $B_3$. □