

Truncated Realized Covariance when prices have infinite variation jumps¹

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Abstract

The speed of convergence of the Truncated Realized Covariance (TRC) to the Integrated Covariation between the Brownian parts of two semimartingales is heavily influenced by the presence of infinite activity jumps with infinite variation (iV), through both the degree of dependence and the jump activity indices of the two small jumps processes. To show this, marginal stable-like small jumps with a parametric dependence structure are considered. The estimator is efficient in only some cases of iV jumps.

The result of this paper is relevant in financial economics, since by the TRC it is possible to separately estimate the common jumps among two assets, which has important implications in risk management and contagion modeling.

Keywords. Brownian correlation coefficient, co-jump, integrated covariation, Lévy copula, threshold estimator

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1 Introduction

We consider two state variables evolving as follows

$$\begin{aligned} dX_t^{(1)} &= a_t^{(1)} dt + \sigma_t^{(1)} dW_t^{(1)} + dZ_t^{(1)}, \\ dX_t^{(2)} &= a_t^{(2)} dt + \sigma_t^{(2)} dW_t^{(2)} + dZ_t^{(2)}, \quad t \in [0, T] \end{aligned} \quad (1.1)$$

with T fixed, where $W^{(1)}$ and $W^{(2)}$ are dependent Wiener processes with instantaneous correlation coefficient ρ_t for any $t \in [0, T]$, and $Z^{(1)}$ and $Z^{(2)}$ are correlated pure jump semimartingales (SMs). Given discrete equally spaced observations $X_{t_i}^{(1)}, X_{t_i}^{(2)}$, $i = 1..n$, in the interval $[0, T]$, with $t_i = ih, h = \frac{T}{n}$, we are interested in the identification of the *Integrated Covariation* $IC := \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} dt$. It is well known that, as the observation step h tends to 0, the *Realized Covariance* $\sum_{i=1}^n \Delta_i X^{(1)} \Delta_i X^{(2)}$, where $\Delta_i X^{(m)} := X_{t_i}^{(m)} - X_{t_{i-1}}^{(m)}$, converges to the global quadratic covariation $[X^{(1)}, X^{(2)}]_T = \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} dt + \sum_{0 \leq t \leq T} \Delta Z_t^{(1)} \Delta Z_t^{(2)}$, where $\Delta Z_t^{(m)} = Z_t^{(m)} - Z_{t-}^{(m)}$, containing also the *co-jumps* $\Delta Z_t^{(1)} \Delta Z_t^{(2)}$. It is also well known that the *Threshold Realized Covariance*, or *Truncated Realized Covariance*,

$$\hat{IC} = \sum_{i=1}^n \Delta_i X^{(1)} I_{\{(\Delta_i X^{(1)})^2 \leq r_h\}} \Delta_i X^{(2)} I_{\{(\Delta_i X^{(2)})^2 \leq r_h\}},$$

with a properly chosen deterministic function r_h , e.g. $r_h = h^{2u}$ with $u \in (0, 1/2)$, is consistent to IC ([16], [8]). Further, a CLT for \hat{IC} has been established when the jump activity of the processes is relatively moderate, namely, either when (as in [16]) the jumps have *finite activity* (FA), i.e. only finitely many jumps can occur, along each path, in each finite time interval, or when (as in [9], Thm 7.4) the jumps have infinite activity (IA) but *finite*

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variation (FV), i.e. $\sum_{s \leq T} |\Delta X_s^{(m)}| < \infty$ a.s., for both $m = 1, 2$. In such cases the estimation error is asymptotically mixed Gaussian and converges at rate (or speed) \sqrt{n} .

Many features of \hat{IC} have already been studied. In [16] the estimator has been compared in efficiency with two alternative estimators of IC ; it has been used to estimate the sum of the cojumps of $X^{(1)}$ and $X^{(2)}$ as well as each single cojump; it has been studied in the presence of irregular sampling and non synchronous data. In [4] and in the web appendix of [16] the finite sample performance of \hat{IC} has been evaluated on simulated data. Finally, similarly as in [15], \hat{IC} tends to zero in the presence of microstructure noises in the data.

In the univariate case \hat{IC} becomes the Truncated Realized Variance (TRV) \hat{IV} which has been much used to estimate the *Integrated Variance* $IV := \int_0^T \sigma_t^2 dt$ of X and also adapted to estimate the spot variance σ_t^2 (see the references in [17]). The rate of convergence of \hat{IV} is known even in the presence of infinite variation (iV) jumps (see [14]).

Here we are interested in investigating the rate of convergence of \hat{IC} in the case where at least one jump component has iV. This was not known up to now. Similarly as in the univariate case, the small jumps play a crucial role. In order to tackle the problem, the small jumps of each semimartingale are assumed to be the small jumps of a Lévy stable process, where we denote by α_1, α_2 the stability indices and assume $\alpha_1 \leq \alpha_2$ and $\alpha_2 \geq 1$. Further, a simple dependence structure is imposed to the two stable processes: the dependence degree is measured by a parameter, $\gamma \in [0, 1]$, and can range from complete dependence ($\gamma = 0$) to independence ($\gamma = 1$). We find that the rate is determined not only by α_1, α_2 , but also by the dependence degree γ . In fact, as it is detailed in section 3, we have that when the small jumps are dependent ($\gamma \in [0, 1)$), then the estimation error $\hat{IC} - IC$ tends to zero as: \sqrt{h} , when α_1 is small and α_2 is close to 1; $(1 - \gamma)r_h^{1+\alpha_2/\alpha_1-\alpha_2}$, if either α_1 is a bit larger but still α_2 is close to 1 or they are both large and very close; $hr_h^{-\alpha_2}$, when α_2 is large and strictly larger than α_1 . When the small jumps are independent then the behavior is only determined by α_2 . In the univariate case what is found here reduces to the behavior described in [14].

The optimal rate in estimating IC is \sqrt{n} if the jumps have finite variation, but it is not exactly known when the jumps have infinite variation. For the univariate case, in [11] an estimator reaching \sqrt{n} is found, when the model belongs to a given class, that we call \mathcal{S}_{Stab}^{loc} , of Ito semimartingales X having in particular α stable-like small jumps and semimartingale volatility σ . Note that, the univariate version of our model not necessarily belongs to \mathcal{S}_{Stab}^{loc} , because we have a general càdlàg process σ . In [10] it is shown that the quantity $\rho_n = (n \log n)^{(1-r/2)}$ is the smallest possible rate to be a uniform upper bound for any estimator of IV when the model falls in a class \mathcal{S}_A^r of Ito semimartingales X where the sizes Γ of the small jumps satisfy $\sup_{s \leq T} \int (|\Gamma(\omega, x, s)|^r \wedge 1) \nu(dx) \leq A$, with $r \in (1, 2]$ and $A \in \mathbb{R}$, ν being the jumps Lévy measure. Note that, for the univariate version of our model, by properly choosing the function r_h , the threshold estimator reaches the rate ρ_n . More precisely, when the model has α stable small jumps (as in [14]) then (if A is sufficiently large) it belongs to \mathcal{S}_A^r for any $r > \alpha$ but not to \mathcal{S}_A^α . Fixed an $r > \alpha$, by taking threshold function $r_h = (h/|\log h|)^{2u}$, with $2u = (1 - r/2)/(1 - \alpha/2)$ then the threshold estimator reaches rate $(n \log n)^{2u(1-\alpha/2)} = \rho_n$.

Now, given an estimator \tilde{IV} , a possible estimator of IC is given by $\tilde{IV}(X^{(1)} + X^{(1)})/2 - \tilde{IV}(X^{(1)})/2 - \tilde{IV}(X^{(1)})/2$, thus the best convergence rate of an estimator of IC is bounded by ρ_n if the model falls within \mathcal{S}_A^r and is \sqrt{n} if the model falls within \mathcal{S}_{Stab}^{loc} . It follows that the rate of \hat{IC} is optimal and better than ρ_n , when the estimation error tends to zero as \sqrt{h} . Further, in this last case the asymptotic variance of $(\hat{IV} - IV)/\sqrt{h}$ is the optimal $2 \int_0^T \sigma_s^4 ds$, as in the case of symmetric jumps in [11], but is better than in the case of not symmetric jumps in [11]. When $\hat{IC} - IC$ tends to 0 as $h\epsilon^{-\alpha_2}$ then the rate is worse than \sqrt{n} but is better than ρ_n ; while when $\hat{IC} - IC$ tends to 0 as $(1 - \gamma)r_h^{1+\alpha_2/\alpha_1-\alpha_2}$ then the rate is worse than both \sqrt{n} and ρ_n .

Estimation of IC is of strong interest both in financial econometrics (see e.g. [3]) and for portfolio risk and hedge funds management ([6]), in particular $[X^{(1)}, X^{(2)}] - \hat{IC}$ gives a tool for measuring the propagation among assets of effects due to important negative

or positive economic events. Knowledge of the convergence rate helps in assessing the reliability of the estimator on finite samples.

An outline of the paper is as follows. In section 2 we illustrate the framework and our assumptions, in section 3 we establish the exact convergence speed when both the processes $Z^{(m)}$ have IA and at least one has iV, and in section 4 we conclude. All the proofs are contained in Appendix, together to the needed auxiliary results and comments.

2 The framework

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ supporting two independent standard Brownian motions $W^{(1)} = (W_t^{(1)})_{t \in [0, T]}$, $W^{(3)} = (W_t^{(3)})_{t \in [0, T]}$, and two Poisson random measures $\mu^{(j)}$, $j = 1, 2$ on $\mathbb{R} \times [0, T]$, let $X^{(1)} = (X_t^{(1)})_{t \in [0, T]}$, $X^{(2)} = (X_t^{(2)})_{t \in [0, T]}$ be two real processes defined by (1.1), where $X_0 = (0, 0)$, and $W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1 - \rho_t^2} W_t^{(3)}$,

Assumption A1. *the random coefficients $\sigma^{(m)} = (\sigma_t^{(m)})_{t \in [0, T]}$, $a^{(m)} = (a_t^{(m)})_{t \in [0, T]}$, $m = 1, 2$, and $\rho = (\rho_t)_{t \in [0, T]}$ are adapted càdlàg processes,*

Assumption A2. *for $m = 1, 2$, $Z^{(m)} = J^{(m)} + M^{(m)}$ are jump Ito SMs, with*

$$J^{(m)} := \int_0^\cdot \int_{\{|\gamma^{(m)}| > 1\}} \gamma^{(m)} d\mu^{(m)}, \quad M^{(m)} := \int_0^\cdot \int_{\{|\gamma^{(m)}| \leq 1\}} \gamma^{(m)} d\tilde{\mu}^{(m)},$$

where, for each $m = 1, 2$, $d\mu^{(m)} = \mu^{(m)}(\omega, dx, ds)$ counts the jumps of $Z^{(m)}(\omega)$, $d\tilde{\mu}^{(m)} := d\mu^{(m)} - \nu^{(m)}(dx)ds$ is its compensated measure and $\gamma^{(m)} = \gamma^{(m)}(\omega, x, s)$ is a predictable function on $\Omega \times \mathbb{R} \times [0, T]$ (see [9]).

It turns out that $J^{(m)}$ are FA jump processes; they account for the rare and large (with size bigger in absolute value than 1) jumps of $X^{(m)}$. On the contrary, $M^{(m)}$ have generally IA jumps (the path ω of $M^{(m)}$ jumps infinitely many times on $[0, t]$ iff $\int_0^t \int_{\{|\gamma^{(m)}| \leq 1\}} d\mu^{(m)} = \infty$); $M^{(m)}$ are compensated sums of very frequent and small jumps.

For each $n \in \mathbb{N}$ we observe $X^{(1)}, X^{(2)}$ discretely and synchronously at times $t_i = ih$. Since $h = T/n$, then $h \rightarrow 0$ iff $n \rightarrow \infty$.

Assumption A3. *We choose a deterministic function r_h of h , called threshold, s.t.*

$$\lim_{h \rightarrow 0} r_h = 0, \quad \lim_{h \rightarrow 0} \frac{h \log \frac{1}{h}}{r_h} = 0.$$

Denote, for each $m = 1, 2$, $D_t^{(m)} = \int_0^t a_s^{(m)} ds + \int_0^t \sigma_s^{(m)} dW_s^{(m)}$ and $Y_t^{(m)} = D_t^{(m)} + J_t^{(m)}$ respectively the Brownian semimartingale part (BSM) of $X^{(m)}$ and the BSM part plus the FA jump component.

TRC is able to separately capture IC because it excludes from $\sum_{i=1}^n \Delta_i X^{(1)}$ $\Delta_i X^{(2)}$ those increments where jumps bigger than the threshold occurred, and when $h \rightarrow 0$ all the jumps are excluded (see point iii) in the proof of Theorem 1 in [13], then Lemma A.2 in [5] and Lemma 1 in [13]). However, the remaining jumps, that in absolute value are below $\sqrt{r_h}$, determine, in some cases, the speed of convergence of \hat{IC} .

We find here the speed of convergence of $\hat{IC} - IC$ to 0 when both $M^{(m)} \neq 0$ and at least one of them has iV. We specialize our analysis to the case where the small jumps of each $X^{(m)}$ are stable, i.e. $M_t^{(m)} = L_t^{(m)} - z^{(m)}_t - \sum_{s \leq t} \Delta L_s^{(m)} I_{\{|\Delta L_s^{(m)}| > 1\}}$, where $L^{(m)}$ are α_m -stable Lévy processes with characteristic triplets $(z^{(m)}, 0, \nu^{(m)}(dx))$, with $\nu^{(m)}$ given below. Further, we assume that the occurrence of the joint jumps of $L^{(1)}$ and $L^{(2)}$ is characterized by a Lévy copula C ranging in a given class. We have $\alpha_m \in]0, 2[$ for each $m = 1, 2$ and assume without loss of generality (wlg) $\alpha_1 \leq \alpha_2$. Since we are interested in the case where

at least one $\alpha_m \geq 1$, we assume $\alpha_2 \geq 1$. Further, for simplicity, but wlg, we develop our proofs for the case where the Lévy measure of each $L^{(m)}$ is one sided, i.e. $L^{(m)}$ only makes jumps with positive sizes.

Assumption A4. Take $\alpha_2 \geq 1$, and $\alpha_1 \in (0, \alpha_2]$. With $c_m > 0$, $m = 1, 2$, the jumps of each $L^{(m)}$ have Lévy measure

$$\mathbf{v}^{(m)}(dx_m) = c_m x_m^{-1-\alpha_m} I_{\{x_m > 0\}} dx_m.$$

We denote, for each $m = 1, 2$, by

$$U_m(x_m) := \mathbf{v}^{(m)}([x_m, +\infty[) = c_m \frac{x_m^{-\alpha_m}}{\alpha_m}, \quad x_m > 0 \quad (2.1)$$

the tail integral of the marginal Lévy measure $\mathbf{v}^{(m)}$ of the jumps of $L^{(m)}$. Note that α_m is the *Blumenthal-Gettoor index* of $L^{(m)}$, of $M^{(m)}$ and of $X^{(m)}$.

In order to describe the joint jumps, we make use of Lévy copulas, because, due to the stationarity of the Lévy processes increments, the Lévy copulas allow to separate the time component in the law of a bivariate pure jump Lévy process \mathbf{L} from the jump sizes component and allow to describe the dependence between $L^{(1)}$ and $L^{(2)}$ through only the dependence of their jump sizes. Lévy copulas were introduced by Peter Tankov in his Phd thesis, further studied in [12] and their properties are well summarized in [6].

Assumption A5. For any t the joint jumps occurrence of $(L_t^{(1)}, L_t^{(2)})$ is described by the following tail integrals

$$U(x_1, x_2) = \mathbf{v}_\gamma([x_1, +\infty) \times [x_2, +\infty)) = C_\gamma(U_1(x_1), U_2(x_2))$$

where $C_\gamma(u, v)$ is a Lévy copula of the form

$$C_\gamma(u, v) = \gamma C_\perp(u, v) + (1 - \gamma) C_\parallel(u, v),$$

where $C_\perp(u, v) = u I_{\{v=\infty\}} + v I_{\{u=\infty\}}$ is the independence copula, $C_\parallel(u, v) = u \wedge v$ is the total positive dependence copula, and γ ranges in $[0, 1]$.

A5 means that, at any t , $\mathbf{L} := (L^{(1)}, L^{(2)})$ can only have two basically different classes of jumps: i) the disjoint ones, meaning that \mathbf{L}_t jumps with size either $(0, x_2)$ or $(x_1, 0)$. This type of jumps is regulated only by C_\perp ; ii) the joint ones, meaning that \mathbf{L}_t jumps with size falling into a point (x_1, x_2) with both $x_m \neq 0$. This type of jumps is regulated only by C_\parallel , which characterizes a bivariate jump Lévy process $\bar{\mathbf{L}}$ whose marginals $L^{(m)}$ are Lévy and only make joint jumps which are completely positively monotonic, i.e. there exists a strictly increasing, strictly positive function $f: \forall s > 0, \Delta \bar{L}_s^{(2)} = f(\Delta \bar{L}_s^{(1)})$. In fact the sizes (x_1, x_2) realized by the jumps of $\bar{\mathbf{L}}_s$ turn out to be supported by the graph of $f(x_1) = U_2^{-1}(U_1(x_1))$, which in our case of one sided α -stable marginals is given by $f(x_1) = ((c_1 \alpha_2) / (\alpha_1 c_2))^{-1/\alpha_2} x_1^{\alpha_1/\alpha_2}$.

Our assumption that \mathbf{L} has Lévy measure \mathbf{v}_γ means that its jumps are supported by the union of the graph of f and the positive sides of the Cartesian axes. Each marginal $\mu^{(m)}$ counts the projection on axis x_m of all the realized jumps of \mathbf{L} . However when a realized jump x_1 is so that there exists a realized x_2 such that $x_2 = f(x_1)$ then x_1 is interpreted as the first component of a joint jump. Any other types of jump of $L^{(1)}$ are interpreted as being associated to a zero complementary component, i.e. as being the projection of a disjoint jump (and analogously for $L^{(2)}$). By changing γ we keep the same marginals $L^{(m)}$ and the same joint or disjoint jumps, but we change the weight given to the different classes of jumps by the underlying probability measure. Process $\bar{\mathbf{L}}$ has joint Lévy measure $\mathbf{v}_\parallel([x_1, +\infty) \times [x_2, +\infty)) = I_{\{x_1 \neq 0, x_2 \neq 0\}} \mathbf{v}^{(1)}([x_1 \vee f^{-1}(x_2), +\infty))$, so the \mathbf{v}_γ defined by A5 is

equivalently writable as

$$\begin{aligned} v_\gamma([x_1, +\infty) \times [x_2, +\infty)) &= \gamma I_{\{x_2=0\}} v^{(1)}([x_1, +\infty)) + \\ &+ \gamma I_{\{x_1=0\}} v^{(2)}([x_2, +\infty)) + (1 - \gamma) I_{\{x_1 \neq 0, x_2 \neq 0\}} v^{(1)}([x_1 \vee f^{-1}(x_2), +\infty)). \end{aligned} \quad (2.2)$$

Remarks. i) A5 is equivalently expressed by:

$$L^{(m)} = L'^{(m)} + \bar{L}^{(m)}, \quad m = 1, 2,$$

where $L'^{(m)}$ has triplet $(z'^{(m)}, 0, \gamma v^{(m)}(dx))$, $m = 1, 2$, $\bar{L}^{(1)}$ has $(\bar{z}^{(1)}, 0, (1 - \gamma)v^{(1)}(dx))$, $(L'^{(1)}, L'^{(2)}, \bar{L}^{(1)})$ are independent while, as said, $\Delta \bar{L}_s^{(2)} = f(\Delta \bar{L}_s^{(1)})$. In particular A5 is satisfied when the bivariate jumps \mathbf{Z} follow a *factor model*

$$Z^{(1)} = V^{(1)}, Z^{(2)} = aV^{(2)} + bV^{(1)},$$

with $V^{(1)}, V^{(2)}$ independent pure jump Lévy processes, and $a, b \in \mathbb{R}$: in this case $\bar{\mathbf{L}} = (V^{(1)}, bV^{(1)})$ and $f(x) = bx$.

ii) Note that in our framework the two components of $\bar{\mathbf{L}}$ have the *same* number of jumps, however they can have different jump indices α_m . In a model with $\Delta_t \bar{L}^{(2)} = f(\Delta_t \bar{L}^{(1)})$ but $f(x) \neq bx$, $\bar{L}^{(1)}$ could make jumps much smaller than $\bar{L}^{(2)}$, implying $\bar{\alpha}_1 < \bar{\alpha}_2$. When instead $f(x) = bx$ then the two $\bar{L}^{(m)}$ have the same jump activity index. \square

The processes we chose to deal with are quite representative since in fact many commonly used models in finance (Variance Gamma model, CGMY model, NIG model, etc.) have Lévy measures related to the ones in assumption A4, in the sense that they are tempered stable processes where the order of magnitude of the tail integrals as $x_m \rightarrow 0$ is as in (2.1). Moreover C_γ allows to range from a framework of independent jumps components to a framework where the components are completely positively monotonic.

The speed of convergence of $\hat{IC} - IC$ is strictly related to the speed of convergence to zero of the sum of the small co-increments $\Delta_i M^{(1)} I_{|\Delta_i M^{(1)}| \leq \sqrt{r_h}} \Delta_i M^{(2)} I_{|\Delta_i M^{(2)}| \leq \sqrt{r_h}}$ (as it happened in [14] for the univariate case), which substantially behaves like the sum of the small co-jumps $\sum_{s \leq T} \Delta M_s^{(1)} I_{|\Delta M_s^{(1)}| \leq \sqrt{r_h}} \Delta M_s^{(2)} I_{|\Delta M_s^{(2)}| \leq \sqrt{r_h}}$ (see [2], Lemma 5), whose expectation is $T \int_{0 \leq x, y \leq \sqrt{r_h}} xy v_\gamma(dx, dy)$. Note that, as soon as $\varepsilon < 1$, in restriction to the set of jump sizes $(0, \varepsilon] \times (0, \varepsilon]$, the jumps of the bivariate processes \mathbf{M} and \mathbf{L} coincide. We need assumption A5 in order to control the speed of convergence to zero of integrals like $\int_{0 \leq x, y \leq \varepsilon} x^k y^m v_\gamma(dx, dy)$, for $\varepsilon > 0$ and integers k, m .

Further notations. Given two (possibly random) sequences U_n, V_n , we say that $U_n = O_P(V_n)$ if for any $\varepsilon > 0$ there exists a constant $\eta > 0$ and an \bar{n} such that for all $n \geq \bar{n}$, $P(|U_n| > \eta |V_n|) < \varepsilon$. We write $U_n \sim V_n$ when as $n \rightarrow \infty$ we have both $U_n = O_P(V_n)$ and $V_n = O_P(U_n)$. When for all n we have $V_n \neq 0$ a.s., $U_n = O_P(V_n)$ means that, for sufficiently large n , the sequence U_n/V_n is bounded in probability (i.e. tight); with a a constant, $U_n \approx aV_n$ means that $U_n/V_n \xrightarrow{P} a$, with \xrightarrow{P} denoting convergence in probability; $U_n \ll V_n$ means that $U_n/V_n \xrightarrow{P} 0$; $U_n \gg V_n$ means that $U_n/V_n \xrightarrow{P} +\infty$. K is a mute name for any positive constants: it keeps the same name passing from one side to the other of an inequality/equality, even when the constant changes.

3 Main results

In our main Theorem (Theorem 3.2) we are going to show that

$$\hat{IC} - IC \sim \sqrt{h} U_h + \sum_{i=1}^n \xi_i + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta M_s^{(2)}| > \sqrt{r_h}\}} \geq 1\}}, \quad (3.1)$$

where

$$\xi_i = \xi_i^\varepsilon := \Delta_i M'^{(1)} \Delta_i M'^{(2)},$$

for $m = 1, 2$

$$M_t^{(m)} := M_t^{(m)} - \sum_{s \leq t} \Delta M_s^{(m)} I_{\{|\Delta M_s^{(m)}| > \varepsilon\}} = \int_0^t \int_{\{0 < x \leq \varepsilon\}} x d\bar{\mu}^{(m)} - t \int_{\{\varepsilon < x \leq 1\}} x v^{(m)}(dx),$$

and U_h is a sequence of rvs converging stably in law to a mixed Gaussian rv. So we preliminarily state the following crucial result, which deals with the asymptotic behavior of $\sum_{i=1}^n \xi_i$.

Theorem 3.1. *Assume $0 < \alpha_1 \leq \alpha_2 < 2$, $\alpha_2 \geq 1$, $0 < c_1 \leq c_2$, $\varepsilon = \sqrt{r_h} = h^u$, $u \in (\frac{1}{4}, \frac{1}{2})$. As $h \rightarrow 0$ we have*

i) if $\gamma \in [0, 1)$, then for any choices of α_1, α_2 and u as in the assumptions:

$$\sum_i \xi_i \approx nE[\xi_1] \approx T(1 - \gamma)C(1, 1)\varepsilon^{1 + \frac{\alpha_2}{\alpha_1} - \alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} + Thc_{A_1} c_{A_2} F_0(\varepsilon),$$

where $F_0(\varepsilon) = -\varepsilon^{1 - \alpha_2} I_{\{\alpha_1 \leq \alpha_2 u, \alpha_2 > 1\}} + \log \frac{1}{\varepsilon} I_{\{\alpha_1 \leq \alpha_2 u, \alpha_2 = 1\}}$, for $m = 1, 2$, $c_{A_m} := \frac{c_m}{1 - \alpha_m}$.

$\cdot I_{\alpha_m \neq 1} + c_m I_{\alpha_m = 1}$ and, for $k, m \geq 0$, $C(k, m) := \left(\frac{\alpha_2 c_1}{\alpha_1 c_2}\right)^{\frac{k}{\alpha_1}} \frac{c_2}{m + \frac{\alpha_2}{\alpha_1} k - \alpha_2} > 0$;

ii) if $\gamma = 1$ but $(\alpha_1, \alpha_2) \neq (1, 1)$, and: if we are in $\{\alpha_1 < 1, \alpha_2 \geq 1\} \cup \{\alpha_1 = 1 < \alpha_2\}$ then we take $u \in (\frac{1}{2 + \alpha_2 - \alpha_1}, \frac{1}{2})$; while if we are in $\{1 < \alpha_1 \leq \alpha_2\}$ then we take $u \in (\frac{1}{\alpha_1 + \alpha_2}, \frac{1}{2})$; then we have

$$\sum_i \xi_i \approx nE[\xi_1] \approx Thc_{A_1} c_{A_2} F_1(\varepsilon),$$

where $F_1(\varepsilon) = -\varepsilon^{1 - \alpha_2} I_{\{\alpha_1 < 1 < \alpha_2\}} + \log \frac{1}{\varepsilon} I_{\{\alpha_1 < 1 = \alpha_2\}} - \varepsilon^{1 - \alpha_2} \log \frac{1}{\varepsilon} I_{\{\alpha_1 = 1 < \alpha_2\}} + \log^2 \frac{1}{\varepsilon} \cdot I_{\{\alpha_1 = \alpha_2 = 1\}} + \varepsilon^{2 - \alpha_1 - \alpha_2} I_{\{1 < \alpha_1 \leq \alpha_2\}}$;

iii) if $\gamma = \alpha_1 = \alpha_2 = 1$, for any $u \in (0, \frac{1}{2})$: with $C_m(k) = \frac{c_m}{k - \alpha_m}$, for $k, m = 1, 2$, we have

$$\sum_i \xi_i \approx \sqrt{n \text{Var}(\xi_1)} U_h \approx \sqrt{h} \varepsilon \sqrt{TC_1(2)C_2(2)} U_h.$$

Remarks. i) Since $c_{A_m} > 0$ for $\alpha_m \leq 1$ while $c_{A_m} < 0$ for $\alpha_m > 1$ and within F_0 we only have $\alpha_1 < 1$, then we always have $c_{A_1} c_{A_2} F_i(\varepsilon) > 0$, $i = 0, 1$.

ii) As for ii) above, if either $\alpha_1 < 1$ or $\alpha_1 = 1 < \alpha_2$ then we have $\alpha_1 < \alpha_2$ and requiring that $u > 1/(2 + \alpha_2 - \alpha_1)$ is possible because $1/(2 + \alpha_2 - \alpha_1) < 1/2$. On the contrary, the set $\{1 < \alpha_1 \leq \alpha_2\}$ contains the case $\alpha_1 = \alpha_2$ in which $u > 1/(2 + \alpha_2 - \alpha_1) = 1/2$ is not admissible. Note that condition $u > 1/(2 + \alpha_2 - \alpha_1)$ implies $u > 1/(\alpha_1 + \alpha_2)$ when $\alpha_2 > \alpha_1 > 1$.

iii) The speed of convergence of $\sum_i \xi_i$ is determined not only by each α_1, α_2 but also by the degree γ of dependence of the two small jumps components of \mathbf{Z} .

iv) We have that $\sum_{i=1}^n \xi_i$ tends to zero much faster when $\gamma = 1$ than when $\gamma \in [0, 1)$ (we obtain that by using Proposition 5.4 below and comparing $nE[\xi_1]$ in (5.1) with $nE[\xi_1]$ or $\sqrt{n \text{Var}(\xi_1)}$ in (5.2), while matching all the sets of (α_1, α_2)). In other words, the co-increments ξ_i tend to zero much faster when $M^{(1)}, M^{(2)}$ are independent, in fact ξ_i is led by the small co-jumps and in the independent case the sum of the small co-jumps is zero (rather than being small).

v) Comparing the asymptotic behavior of $\sum_i \xi_i$ with \sqrt{h} , we reach that $\sum_i \xi_i \ll \sqrt{h}$ substantially when α_1 is sufficiently small (and still $\alpha_2 \geq 1$). In this case the co-increments of $M^{(1)}, M^{(2)}$ are negligible with respect to (wrt) the Brownian co-increments. More precisely, using Proposition 5.4 and Theorem 3.1, defined

$$\alpha_1^* := \frac{\alpha_2 u}{\alpha_2 u - u + 1/2} \in (2u, 1), \quad \alpha_1^{**} := \frac{1 + 2u(2 - \alpha_2)}{2u} > \frac{1}{2u} > 1,$$

we reach (see the proof in Appendix) that for $u > 1/4$:

$$\begin{cases} \text{if } \gamma \in [0, 1): & \sum_i \xi_i \ll \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^*; \\ \text{if } \gamma = 1: & \sum_i \xi_i \ll \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^{**}. \end{cases} \quad (3.2)$$

Since $\alpha_1^* < 1 < \alpha_1^{**}$, the above result means that when the two small jumps components $M^{(m)}$ are independent, then the impact of their co-increments on the convergence speed of $\hat{I}\hat{C} - IC$ is negligible, wrt the impact \sqrt{h} of the Brownian co-increments, for a wider range of values α_1 . \square

Here is the main result of our paper.

Theorem 3.2. *Let us assume $\rho \neq 0$ and, for all the possible entries $\eta = \rho, \sigma^{(1)}, \sigma^{(2)}$, let η satisfy, when $h \rightarrow 0$,*

$$\forall s \geq t: s - t \leq h, \text{ then } E[|\eta_s - \eta_t|^2] \leq K(s - t), \quad (3.3)$$

assume $0 < \alpha_1 \leq \alpha_2 < 2$, $\alpha_2 \geq 1$, $0 < c_1 \leq c_2$, $\varepsilon = \sqrt{r_h} = h^u$, and $u > 0$ such that

$$1/2 > u > \begin{cases} \frac{1}{2+\alpha_2-\alpha_1} \vee \frac{1}{3-\frac{\alpha_2}{2}} & \text{if } \alpha_1 < \alpha_2 \\ \frac{1}{3-\frac{\alpha_2}{2}} & \text{if } \alpha_1 = \alpha_2. \end{cases} \quad (3.4)$$

Then, as $h \rightarrow 0$, we have

$$\hat{I}\hat{C} - IC \sim \sqrt{h}U_h + \sum_{i=1}^n \xi_i + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\sum_{s \in [t_{i-1}, t_i]} I_{\{|\Delta M_s^{(2)}| > \sqrt{r_h}\}} \geq 1\}} \quad (3.5)$$

$$\sim \sqrt{h} + (1 - \gamma)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} + h\varepsilon^{-\alpha_2} \quad (3.6)$$

$$\sim \sqrt{h} I_{\{\alpha_2 \in [1, \frac{1}{2u}]\}} \left[I_{\{\gamma=1\}} + I_{\{\gamma \in [0, 1), \alpha_1 \leq \alpha_1^*\}} \right] \quad (3.7)$$

$$+ (1 - \gamma)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\gamma \in [0, 1)\}} \left[I_{\{\alpha_1^* < \alpha_1 \leq \alpha_2 \in [1, \frac{1}{2u}]\}} + I_{\{\alpha_2 \geq \frac{1}{2u}\}} I_{\{\alpha_1 \leq \alpha_2 < \alpha_1(\frac{1}{u}-1)\}} \right] \quad (3.8)$$

$$+ h\varepsilon^{-\alpha_2} I_{\{\alpha_2 \geq \frac{1}{2u}\}} \left[I_{\{\gamma=1\}} + I_{\{\gamma \in [0, 1)\}} I_{\{\alpha_2 \geq \alpha_1(\frac{1}{u}-1)\}} \right]. \quad (3.9)$$

Remarks.

i) Condition $\alpha_2 < \alpha_1(1/u - 1)$ is equivalent to $u < \alpha_1/(\alpha_2 + \alpha_1)$ and we did not include it among the ones in (3.4) because such conditions are required for the convergence of some terms of I_4 (defined within the proof of the Theorem) in $\hat{I}\hat{C} - IC$, while $\alpha_1/(\alpha_2 + \alpha_1)$ is only a separator to establish whether the leading term is $\varepsilon^{1+\alpha_2/\alpha_1-\alpha_2}$ or $h\varepsilon^{-\alpha_2}$. There is another proof for the convergence of some of the cited terms of I_4 , which avoids conditions (3.4), but it is much longer than the one given in Appendix.

ii) Note that $\alpha_1(1/u - 1) \leq \alpha_2$ implies $\alpha_1 < \alpha_2$; $\alpha_1(1/u - 1) > \alpha_2$ implies $\alpha_2 u < \alpha_1$. If $\alpha_1 < \alpha_2$, (3.4) implies that $u > 1/4$. Note however that we always have to take $u > 1/(3 - \alpha_2/2)$. Such a lower bound is less than $1/2$, but it increases and tends to $1/2$ as α_2 increases and tends to 2.

iii) Similarly as for $\sum_{i=1}^n \xi_i$, the convergence speed of $\hat{I}\hat{C} - IC$ depends both on the jump activity indices α_1, α_2 and on the dependence degree γ of the small jumps. This implies that $\hat{I}\hat{C}$ contains information that we could exploit to estimate such a dependence degree.

Note that when the dependence degree increases (γ decreases) then the leading term of $\sum_{i=1}^n \xi_i$ also increases ($\sum_i E[\xi_i]$ increases and $\sqrt{n \text{Var}(\xi_1)} \ll \sum_i E[\xi_i]$), and the estimation error $\hat{I}\hat{C} - IC$ increases. An higher leading term of $\sum_i \xi_i$ means that the average weight of the small jumps is higher so that the disturbing noise when estimating the Brownian feature IC is higher. That is: the higher the dependence degree, the higher the disturbing noise.

iv) Basically, when u is close to $1/2$ (i.e. satisfying conditions (3.4)), if the small jumps are dependent ($\gamma \in [0, 1)$), the estimation error $\hat{I}\hat{C} - IC$ tends to zero as: \sqrt{h} , when α_1

is small ($\alpha_1 < \alpha_1^*$) and α_2 is close to 1 ($1 \leq \alpha_2 < 1/(2u)$); $(1-\gamma)r_h^{1+\alpha_2/\alpha_1-\alpha_2}$, if either α_1 is a bit larger ($\alpha_1^* < \alpha_1$) but still α_2 is close to 1 or they are both large and very close ($\alpha_1 = \alpha_2 \geq 1/(2u)$ or $\alpha_2 \geq \frac{1}{2u}$, $\alpha_2 \in (\alpha_1, \alpha_1(1/u-1))$); $h\varepsilon^{-\alpha_2}$, when α_2 is large and strictly larger than α_1 ($\alpha_2 \geq \alpha_1(1/u-1) \vee 1/(2u)$)).

If the small jumps are independent ($\gamma = 1$), then the speed is: \sqrt{h} if $\alpha_2 < 1/(2u)$; $h\varepsilon^{-\alpha_2}$ if $\alpha_2 \geq 1/(2u)$.

v) For $\gamma = 0$ or $\gamma \in (0, 1)$ we have the same cases: in the presence of the parallel component, the independent component does not modify the speed of convergence. On the contrary, in the presence of the independent component, the parallel component does worsen the speed of convergence. This is due to the fact that the independent component contribution is smaller than the one of the parallel component.

vi) When the leading term of $\sum_{i=1}^n \xi_i$ is $\sqrt{n\text{Var}(\xi_1)} \sim \sqrt{h}\varepsilon^{2-\alpha_1/2-\alpha_2/2}$ (e.g. in the case $\gamma = \alpha_1 = \alpha_2 = 1$, by Proposition 5.4; or in the case $\gamma = 1$ and $1 < \alpha_1 \leq \alpha_2 < 1/(2u)$, since then $u < 1/(2\alpha_2) \leq 1/(\alpha_1 + \alpha_2)$) it holds that $\sqrt{n\text{Var}(\xi_1)}/\sqrt{h} \rightarrow 0$, so $\sum_{i=1}^n \xi_i$ is dominated by \sqrt{h} and the term $\sqrt{h}\varepsilon^{2-\alpha_1/2-\alpha_2/2}$ never appears.

vii) The speed is \sqrt{h} even in some cases with $\alpha_2 \geq 1$ (but $\alpha_2 < 1/(2u)$): any α_1 is (still with $\alpha_1 \leq \alpha_2$), if $\gamma = 1$; for α_1 sufficiently small ($\alpha_1 \leq \alpha_1^*$) if the parallel component is present. In this case we also have a CLT (see below).

viii) When $\alpha_1 = \alpha_2 := \alpha \geq 1$ but the two jump components are not necessarily completely monotonic, then $\hat{I}C - IC$ converges to zero as: $(1-\gamma)\varepsilon^{2-\alpha}$ if $\gamma \in [0, 1)$; \sqrt{h} if $\gamma = 1$ and $\alpha < 1/(2u)$ (note that $\alpha < \alpha_1^* < 1 \leq \alpha_2$ is not in our assumption $\alpha_1 = \alpha_2$); $h\varepsilon^{-\alpha}$ if $\gamma = 1$ and $\alpha \geq 1/(2u)$.

ix) The univariate case is when $\alpha_1 = \alpha_2$ and $\gamma = 0$, in this case $\hat{I}C - IC$ converges to zero as $\varepsilon^{2-\alpha} = r_h^{1-\alpha/2}$, for any $\alpha \geq 1$, consistently with [14], where, when $\alpha \geq 1$, the estimation error $\hat{I}V - IV$ is led by the IA and iV jump part.

x) For fixed h , the asymptotic behavior of $\hat{I}C - IC$, described in (3.7, 3.8, 3.9), is a function $s(\gamma, \alpha_1, \alpha_2, u)$ of our parameters. Such a function is smooth most of the times, however it has some singularities (as is evident in Figure 1; see the details in Appendix).

xi) The slowest convergence of $\hat{I}C - IC$ is obtained when $\gamma \in [0, 1)$. Since we have the same speed for $\gamma \in (0, 1)$ or $\gamma = 0$, let us take $\gamma = 0$. For fixed h and u , define R the region identified by the initial assumptions on α_1, α_2 and by (3.4) and A, B, C the subregions identified respectively in (3.7), (3.8) and (3.9):

$$R = \{(\alpha_1, \alpha_2) : \alpha_1 \in (0, 2), \alpha_2 \in [1, 2), \alpha_1 \leq \alpha_2\} \cap$$

$$\left(\left\{ \alpha_1 < \alpha_2, \frac{1}{2 + \alpha_2 - \alpha_1} \vee \frac{1}{3 - \frac{\alpha_2}{2}} < u \right\} \cup \left\{ \alpha_1 = \alpha_2, \frac{1}{3 - \frac{\alpha_2}{2}} < u \right\} \right),$$

$$A = R \cap \left\{ \alpha_1 \leq \alpha_1^*, \alpha_2 < \frac{1}{2u} \right\}, B = B_1 \cup B_2 \cup B_3, B_1 = R \cap \left\{ \alpha_1 > \alpha_1^*, \alpha_2 < \frac{1}{2u} \right\},$$

$$B_2 = R \cap \left\{ \alpha_1 = \alpha_2, \alpha_2 \geq \frac{1}{2u} \right\}, B_3 = R \cap \left\{ \alpha_2 \geq \frac{1}{2u}, \alpha_1 < \alpha_2 < \alpha_1 \left(\frac{1}{u} - 1 \right) \right\},$$

$$C = R \cap \left\{ \alpha_2 \geq \frac{1}{2u} \vee \alpha_1 \left(\frac{1}{u} - 1 \right) \right\}.$$

Then, noting that $1/(3 - \alpha_2/2) < u$ iff $\alpha_2 < 2(3 - 1/u)$, the slowest convergence is approached by

$$\sup_{(\alpha_1, \alpha_2) \in R} s(0, \alpha_1, \alpha_2, u) = \sup_{B_1 \cup B_2} s = \sup_{\alpha_1 = \alpha_2, 1 \leq \alpha_2 < 2(3 - 1/u)} \varepsilon^{1 + \frac{\alpha_1}{\alpha_2} - \alpha_2} = \varepsilon^{2(\frac{1}{u} - 2)} = h^{2-4u} :$$

note that, for vales $u > 0.4$, $h^{2-4u} \gg \sqrt{h}$, and the closer is u to $1/2$ the slower is the convergence of $\hat{I}C$.

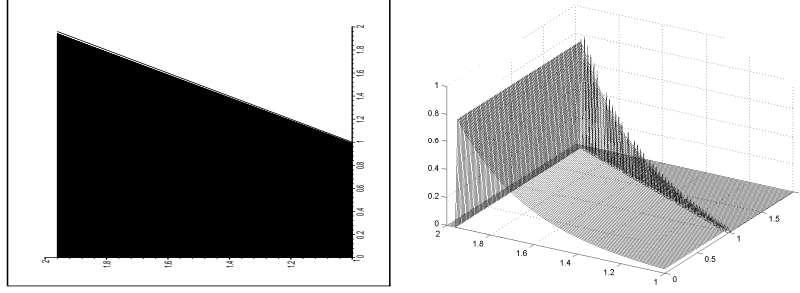


Figure 1: Left: Region R (black); Right: plot of $s(0, \alpha_1, \alpha_2, u)$ (outside R it is set equal to 0), for fixed $h = 1/1000$ and $u = 0.495$. α_2 varies within the axes having range $[1, 2]$, α_1 varies within the one having range $[0, 2]$

Remark. When $\alpha_2 < 1/(2u)$ and either $\gamma = 1$ or both $\gamma \in [0, 1)$ and $\{\alpha_1 < \alpha_1^*\}$, we have a CLT for $\hat{IC} - IC$. In fact the only leading term of $\hat{IC} - IC$ is \sqrt{h} , which only comes from the components $Y^{(m)}$ of the processes $X^{(m)}$, so the presence of $M^{(1)}$ and $M^{(2)}$ is not influential. Thus using also Theorem 3.4 in [16] and Theorem 4.2 in [7], with \xrightarrow{st} denoting stable convergence in law, we have

$$\frac{\hat{IC} - IC}{\sqrt{h\sqrt{A\hat{Var}}}} \xrightarrow{st} \mathcal{N},$$

where \mathcal{N} is a standard Gaussian r.v. and, denoted $\Delta_\ell X_\star^{(m)} := \Delta_\ell X^{(m)} I_{\{|\Delta_\ell X^{(m)}| \leq \sqrt{r_h}\}}$, $A\hat{Var} := h^{1-\frac{r+l}{2}} \sum_{i=1}^n \prod_{m=1}^2 (\Delta_i X_\star^{(m)})^2 - h^{-1} \sum_{i=1}^{n-1} \prod_{j=0}^1 \Delta_{i+j} X_\star^{(1)} \prod_{j=0}^1 \Delta_{i+j} X_\star^{(2)}$

$$\xrightarrow{P} \int_0^T (1 + \rho_t^2) (\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt. \quad \square$$

4 Conclusion

The Truncation method is much used in the literature to estimate either IV or IC , given discrete observations of a SM with jumps. This paper contributes in determining new properties of the method. Namely, the exact convergence speed is computed in the bivariate case under the special, but representative, case where the small jumps of the two components of the considered SM are stable-like and their dependence structure is parametric. It turns out that both the jumps activity indices and the dependence degree impact the speed, when the jumps have infinite variation. The intuition is as follows. TRV efficiently estimates IV only in the presence of FV jumps, however the efficiency is lost if the jumps are more highly active, as they constitute noise when estimating IV . In a bivariate framework not only the frenetic behavior of the small jumps makes TRC slower, but also their dependence increases their synergy in making noise and thus increases the asymptotic estimation bias.

The speed found here coincides with the one found in [14] in the univariate case, and the speed in the bivariate worst case scenario is computed. The conclusion is that, with iV jumps, TRC is efficient if the jumps indices are moderately high, and in this case, if the jumps are not symmetric, the Jacod&Todorov ([11]) estimator is less efficient (even if it is rate efficient, the factor in front of the rate is not the lowest possible). However in the other cases TRC is asymptotically less reliable and the Jacod&Todorov estimator is preferable as it is rate efficient.

The result of this paper is relevant in risk management and contagion modeling, where we need to estimate IC and the occurring of common jumps. In fact the indication of which estimator is more efficient in each context is important, since a less efficient estimator has an asymptotically higher bias, so is potentially less reliable in finite samples.

5 Appendix

5.1 Proof of Theorem 3.1

We aim to know the asymptotic behavior of $\sum_{i=1}^n \xi_i$. Even if the rvs ξ_i are iid, it is not trivial to establish a CLT. However, to reach our objective, for many choices of the parameters $\gamma, \alpha_1, \alpha_2$ and u we can avoid to verify Lindeberg-type conditions. Let us start considering

$$\tilde{\xi}_i := \frac{\xi_i - E[\xi_1]}{\sqrt{n\text{Var}(\xi_1)}} :$$

we know that $\sum_{i=1}^n \tilde{\xi}_i$ is always a tight sequence, because $\sqrt{n\text{Var}(\xi_1)}$ is the L^2 norm of the centered $\sum_{i=1}^n (\xi_i - E[\xi_1])$. In the next theorem 5.3 the tightness of $\sum_{i=1}^n \tilde{\xi}_i$ is stated, with explicit indication of only the leading terms of $nE[\xi_1]$ and $\sqrt{n\text{Var}(\xi_1)}$. Computation of these last two quantities requires the computation of objects of type $\int_{0 \leq x, y \leq \varepsilon} x^k y^m \mathbf{v}_\gamma(dx, dy)$, which is done based on Remark 4.1 and Lemma 4.2.

Now, when $\frac{\sqrt{n\text{Var}(\xi_1)}}{nE[\xi_1]} \rightarrow 0$ then the tightness of $\sum_{i=1}^n \tilde{\xi}_i$ implies that $\frac{\sum_{i=1}^n \xi_i}{nE[\xi_1]} \xrightarrow{P} 1$, that is $\sum_{i=1}^n \xi_i \approx nE[\xi_1]$. On the other hand, if $\frac{\sqrt{n\text{Var}(\xi_1)}}{nE[\xi_1]} \rightarrow +\infty$, the tightness of $\sum_{i=1}^n \tilde{\xi}_i$ only allows us to say that $\forall \eta > 0 \exists K_\eta$ such that with probability larger than $1 - \eta$, for all sufficiently large n we have $|\sum_{i=1}^n \xi_i| \leq K_\eta \sqrt{n\text{Var}(\xi_1)}$, but $\sum_{i=1}^n \xi_i$ could tend to 0 faster than $\sqrt{n\text{Var}(\xi_1)}$. Proposition 5.4 tells us that $\frac{\sqrt{n\text{Var}(\xi_1)}}{nE[\xi_1]} \rightarrow 0$ in all the cases of our interest (we concentrate on values of u close to 1/2) but one. However the CLT expressed by Theorem 5.5 below gives us the exact asymptotic behavior of $\sum_{i=1}^n \xi_i$ even in the last case. The statement of Theorem 3.1 then follows. \square

Remark 5.1. Note that when $\varepsilon < 1$ and $k, m \geq 1$ the integral $\int_{0 \leq x, y \leq \varepsilon} x^k y^m \mathbf{v}_\perp(dx, dy)$ is zero, because the independent components of L have no common jumps. It follows that under assumption A5, for both $k \geq 1$ and $m \geq 1$, we have

$$\int_{0 \leq x, y \leq \varepsilon} x^k y^m \mathbf{v}_\gamma(dx, dy) = (1 - \gamma) \int_{0 \leq x, y \leq \varepsilon} x^k y^m dC_{\parallel}(U_1(x), U_2(y)).$$

From the definition of Lebesgue integral and simple computations the following holds true.

Lemma 5.2. i) Given the expression of C_{\parallel} and (2.1), for $0 < \alpha_1 \leq \alpha_2, 0 < c_1 \leq c_2$, if $\varepsilon < e^{-\frac{1}{\alpha_1}}$ then for any Borel functions g s.t. $g\left(\left(\frac{\alpha_1 u}{c_1}\right)^{-\frac{1}{\alpha_1}}, \left(\frac{\alpha_2 u}{c_2}\right)^{-\frac{1}{\alpha_2}}\right)$ is Lebesgue-integrable we have

$$\int_{0 \leq x_1, x_2 \leq \varepsilon} g(x_1, x_2) \mathbf{v}_{\parallel}(dx_1, dx_2) = \int_{\frac{c_2 \varepsilon^{-\alpha_2}}{\alpha_2}}^{+\infty} g\left(\left(\frac{\alpha_1 u}{c_1}\right)^{-\frac{1}{\alpha_1}}, \left(\frac{\alpha_2 u}{c_2}\right)^{-\frac{1}{\alpha_2}}\right) du$$

ii) for $m, k \geq 1$ note that $\frac{k}{\alpha_1} + \frac{m}{\alpha_2} - 1 > 0$, and in particular we have

$$\int_{0 \leq x_1, x_2 \leq \varepsilon} x_1^k x_2^m \mathbf{v}_{\parallel}(dx_1, dx_2) = C(k, m) \varepsilon^{m+k\frac{\alpha_2}{\alpha_1} - \alpha_2};$$

iii) for $k \geq 2$ and $m = 1, 2$ we have:

$$\int_{0 < x_m \leq \varepsilon} x_m^k \mathbf{v}_{\perp}(dx_1, dx_2) = \int_{0 < x_m \leq \varepsilon} x_m^k \mathbf{v}^{(m)}(dx_m) = C_m(k) \varepsilon^{k - \alpha_m};$$

for $k, m = 1, 2, \ell \geq 2$

$$\int_{0 \leq x_1, x_2 \leq \varepsilon} x_1^k \mathbf{v}_{\parallel}(dx_1, dx_2) = C(k, 0) \varepsilon^{\frac{\alpha_2}{\alpha_1} k - \alpha_2}; \quad \int_{0 \leq x_1, x_2 \leq \varepsilon} x_2^\ell \mathbf{v}_{\parallel}(dx_1, dx_2) = C(0, \ell) \varepsilon^{\ell - \alpha_2};$$

iv) for $m = 1, 2$

$$A_m^\varepsilon := \int_{\varepsilon \leq x_m \leq 1} x_m \mathbf{v}^{(m)}(dx_m) = c_{A_m} \left[(1 - \varepsilon^{1 - \alpha_m}) I_{\alpha_m \neq 1} + I_{\alpha_m = 1} \ln \frac{1}{\varepsilon} \right]. \quad \square$$

Note that for $\varepsilon < 1$, $c_{A_m}(1 - \varepsilon^{1-\alpha_m}) > 0$ for any $\alpha_m \in]0, 2[$ and that $C(0, m) = \frac{c_2}{m-\alpha_2} = C_2(m)$. The reason why $\int_{0 \leq x_1, x_2 \leq \varepsilon} \mathcal{X}_1^k \mathbf{v}_{\parallel}(dx_1, dx_2)$ depends also on α_2 is that the jump sizes of the parallel component of M are connected by $x_2 = f(x_1)$. If $\alpha_1 \leq \alpha_2$ and $0 < c_1 \leq c_2$ then for sufficiently small ε we have $U_1(\varepsilon) \leq U_2(\varepsilon)$, thus $\varepsilon \geq U_1^{-1}(U_2(\varepsilon)) = f^{-1}(\varepsilon)$. It follows that by binding both $x_1 \leq \varepsilon$ and $x_2 = f(x_1) \leq \varepsilon$ we impose that $x_1 \leq f^{-1}(\varepsilon) \wedge \varepsilon = f^{-1}(\varepsilon)$, which is a bound depending on α_2 .

Theorem 5.3. *Assume A2-A5, $0 < \alpha_1 \leq \alpha_2 < 2, \alpha_2 \geq 1, 0 < c_1 \leq c_2$. Take $\varepsilon = h^u$, any $u \in]0, \frac{1}{2}[$, and define*

$$x_* := \frac{1 + 2u - \sqrt{-4(2\alpha_2 - 1)u^2 + 4u + 1}}{2u} \in (\alpha_2 u, \alpha_2).$$

Then as $\varepsilon \rightarrow 0$ the following quotients are tight:

i) if $\gamma \in (0, 1)$:

$$\frac{\sum_i \xi_i - T(1 - \gamma)C(1, 1)\varepsilon^{1 + \frac{\alpha_2}{\alpha_1} - \alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - Thc_{A_1} c_{A_2} F_0(\varepsilon)}{\sqrt{T}\varepsilon^{1 - \alpha_2/2} \sqrt{h\varepsilon^{2 - \alpha_1} \gamma C_1(2)C(0, 2)I_{\{\alpha_1 \leq x_*\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}} (1 - \gamma)C(2, 2)I_{\{\alpha_1 \geq x_*\}}}} \quad (5.1)$$

ii) If $\gamma = 1$:

$$\frac{\sum_i \xi_i - Thc_{A_1} c_{A_2} F_1(\varepsilon)}{\sqrt{T}\sqrt{h\varepsilon^{2 - \alpha_1/2 - \alpha_2/2}} \sqrt{C_1(2)C_2(2)}}, \quad (5.2)$$

iii) If $\gamma = 0$: with $G := C(2, 2) - 2c_{A_1}C(1, 2) + c_{A_1}^2 C(0, 2)$ we have

$$\frac{\sum_i \xi_i - TC(1, 1)\varepsilon^{1 + \frac{\alpha_2}{\alpha_1} - \alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - Thc_{A_1} c_{A_2} F_0(\varepsilon)}{\sqrt{T}\varepsilon^{1 - \alpha_2/2} \sqrt{h^2 c_{A_1}^2 C(0, 2)I_{\{\alpha_1 < \alpha_2 u\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}} [C(2, 2)I_{\{\alpha_1 > \alpha_2 u\}} + GI_{\{\alpha_1 = \alpha_2 u\}}]}}. \quad (5.3)$$

Remarks on the Theorem statement.

- The term $-4(2\alpha_2 - 1)u^2 + 4u + 1$ within x_* turns out to be strictly positive for all $u \in (0, \frac{1}{2})$, $\alpha_2 < 2$. Also, for any α_1, α_2 as in the assumptions recall that $1 + \frac{\alpha_2}{\alpha_1} - \alpha_2 > 0$.
- The numerator in each quotient is always the difference of $\sum_i \xi_i$ with the leading terms of its (tending to zero) mean. There are parameters choices such that $E[\sum_i \xi_i]$ (or $\sqrt{n\text{Var}(\xi_1)}$) has two asymptotically equivalent leading terms.
- As for the denominator in i), the case $\alpha_1 = \alpha_2$ falls within the region $\alpha_1 \geq x_*$.

Proof of Theorem 5.3 Define

$$X_m^\varepsilon := \int_0^h \int_{|x| \leq \varepsilon} x \tilde{\mu}^{(m)}(dx, dt)$$

and recall A_m^ε in iv) of Lemma 5.2: each ξ_i , $i = 1..n$, has the same law as $(X_1^\varepsilon - hA_1^\varepsilon)(X_2^\varepsilon - hA_2^\varepsilon)$. For simplicity we write A_m in place of A_m^ε . We are going to compute $E[\sum_{i=1}^n \xi_i]$ and $\text{Var}[\sum_{i=1}^n \xi_i]$, we thus need to compute the moments $E[(X_1^\varepsilon)^k (X_2^\varepsilon)^m]$, with $k = 2, 1, 0, m = 2, 1, 0$. The bivariate process $X^\varepsilon = (X_1^\varepsilon, X_2^\varepsilon)$ is Lévy with Lévy measure $\mathbf{v}_\varepsilon(dx_1, dx_2) = I_{\{0 \leq x_1, x_2 \leq \varepsilon\}} \mathbf{v}_\gamma(dx_1, dx_2)$, and note that, for small ε , $0 \leq x_1, x_2 \leq \varepsilon \Rightarrow x_1^2 + x_2^2 \leq 1$, so we reach the desired moments by differentiating the characteristic function $\varphi(u_1, u_2) = E[e^{iu_1 X_1^\varepsilon + iu_2 X_2^\varepsilon}] = \exp\{h \int (e^{iu_1 x_1 + iu_2 x_2} - 1 - iu_1 x_1 - iu_2 x_2) \mathbf{v}_\varepsilon(dx_1, dx_2)\}$, then evaluating it at $(0, 0)$, recalling the expression of \mathbf{v}_γ and using Lemma 5.2. In particular we have:

$$E[X_1^\varepsilon] = E[X_2^\varepsilon] = 0$$

$$E\left[\left(X_1^\varepsilon\right)^2\right] = h \int_{\mathbb{R}^2} x_1^2 v_\varepsilon(dx_1, dx_2) = \gamma C_1(2) h \varepsilon^{2-\alpha_1} + (1-\gamma) C(2,0) h \varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2}.$$

Note that if $\gamma \in (0, 1)$ then as $\varepsilon \rightarrow 0$ we have $E\left[\left(X_1^\varepsilon\right)^2\right] \approx h \varepsilon^{2-\alpha_1} \mathcal{A}$, where $\mathcal{A} = \gamma C_1(2) I_{\{\alpha_1 \leq \alpha_2\}} + (1-\gamma) C(2,0) I_{\{\alpha_1 = \alpha_2\}}$. In fact, with $\phi := \frac{\alpha_2}{\alpha_1} \in [1, +\infty)$, the quotient $\varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2} / \varepsilon^{2-\alpha_1} = \varepsilon^{2\phi-\alpha_1\phi-2+\alpha_1} = \varepsilon^{(2-\alpha_1)(\phi-1)}$ has an exponent which is non-negative for all $\alpha_1, \alpha_2 \in (0, 2)$, and zero for $\alpha_1 = \alpha_2$.

$$\begin{aligned} E\left[\left(X_2^\varepsilon\right)^2\right] &= h \int_{\mathbb{R}^2} x_2^2 v_\varepsilon(dx_1, dx_2) = h C(0,2) \varepsilon^{2-\alpha_2} \\ E\left[X_1^\varepsilon X_2^\varepsilon\right] &= h \int_{\mathbb{R}^2} x_1 x_2 v_\varepsilon(dx_1, dx_2) = h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} C(1,1) (1-\gamma) \\ E\left[\left(X_1^\varepsilon\right)^2 X_2^\varepsilon\right] &= h \int_{\mathbb{R}^2} x_1^2 x_2 v_\varepsilon(dx_1, dx_2) = h \varepsilon^{1+2\frac{\alpha_2}{\alpha_1}-\alpha_2} C(2,1) (1-\gamma) \\ E\left[X_1^\varepsilon \left(X_2^\varepsilon\right)^2\right] &= h \int_{\mathbb{R}^2} x_1 x_2^2 v_\varepsilon(dx_1, dx_2) = h \varepsilon^{2+\frac{\alpha_2}{\alpha_1}-\alpha_2} C(1,2) (1-\gamma) \\ E\left[\left(X_1^\varepsilon\right)^2 \left(X_2^\varepsilon\right)^2\right] &= 2E^2\left[X_1^\varepsilon X_2^\varepsilon\right] + h \int_{\mathbb{R}^2} x_1^2 x_2^2 v_\varepsilon(dx_1, dx_2) + h^2 \int_{\mathbb{R}^2} x_1^2 v_\varepsilon(dx_1, dx_2) \\ &\quad \cdot \int_{\mathbb{R}^2} x_2^2 v_\varepsilon(dx_1, dx_2) \sim (1-\gamma) h \varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-\alpha_2} C(2,2) + h C(0,2) \varepsilon^{2-\alpha_2} E\left[\left(X_1^\varepsilon\right)^2\right]. \end{aligned}$$

Let us first concentrate on $E[\sum_i \xi_i]$. From the above we reach that

$$\begin{aligned} E[\xi_i] &= E[X_1^\varepsilon X_2^\varepsilon] + h^2 A_1 A_2 = (1-\gamma) C(1,1) h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} + c_{A_1} c_{A_2} h^2 \left[\ln^2 \frac{1}{\varepsilon} I_{\alpha_1 = \alpha_2 = 1} + \right. \\ &\quad \left. \ln \frac{1}{\varepsilon} (1 - \varepsilon^{1-\alpha_2}) I_{\alpha_1 = 1 < \alpha_2} + (1 - \varepsilon^{1-\alpha_1}) \log \frac{1}{\varepsilon} I_{\{\alpha_1 < \alpha_2 = 1\}} + \prod_{q=1}^2 (1 - \varepsilon^{1-\alpha_q}) I_{\alpha_q \neq 1} \right]. \quad (5.4) \end{aligned}$$

Note that since $\varepsilon = h^u$, as $h \rightarrow 0$ we have $E[\xi_i] \rightarrow 0$.

i) and iii). If $\gamma \in [0, 1)$, then we have the following leading terms in the expression of $E[\xi_i]$, when $h \rightarrow 0$: when both $\alpha_m = 1$, for sufficiently small h we have $h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} \gg h^2 \ln^2 \frac{1}{\varepsilon}$ so the leading term is $h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$, coming from $E[X_1^\varepsilon X_2^\varepsilon]$; when $\alpha_1 = 1 < \alpha_2$, then the leading term is still $h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$; when $\alpha_1 < \alpha_2 = 1$, the leading term is $h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$ when $u = \alpha_2 u < \alpha_1$, while is $h^2 (1 - \varepsilon^{1-\alpha_1}) \log \frac{1}{\varepsilon} \sim h^2 \log \frac{1}{\varepsilon}$, coming from $h^2 A_1 A_2$, otherwise. When both $\alpha_m \neq 1$ then under our framework we necessarily have $\alpha_2 > 1$; note that $\alpha_2 u < 1$; if $\alpha_1 > 1$ then the leading term turns out to be $h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$; If $\alpha_1 < 1$: $h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$ is the only leading term only if $\alpha_2 u < \alpha_1$; when $\alpha_2 u = \alpha_1$ (and still $\alpha_2 > 1$) then $h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} \sim h^2 (1 - \varepsilon^{1-\alpha_1}) (1 - \varepsilon^{1-\alpha_2}) \sim -h^2 \varepsilon^{1-\alpha_2}$; when $\alpha_2 u > \alpha_1$ then the leading term is $-h^2 \varepsilon^{1-\alpha_2}$. However $\{\alpha_1 = \alpha_2 = 1\} \cup \{\alpha_1 = 1 < \alpha_2\} \cup \{u < \alpha_1 < \alpha_2 = 1\} \cup \{\alpha_1 \neq 1, \alpha_2 > 1, \alpha_2 u < \alpha_1\} = \{\alpha_1 > \alpha_2 u\}$ and here is where the only leading term is $E\left[X_1^\varepsilon X_2^\varepsilon\right]$; then $\{\alpha_1 = u, \alpha_2 = 1\} \cup \{\alpha_1 = \alpha_2 u < 1 < \alpha_2\} = \{\alpha_1 = \alpha_2 u\}$ and here: if $\alpha_2 > 1$ then $E\left[X_1^\varepsilon X_2^\varepsilon\right]$ and $h^2 A_1 A_2$ have the same speed $h \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$; while if $\alpha_2 = 1$ then only $h^2 A_1 A_2 \sim h^2 \log \frac{1}{\varepsilon}$ is leading; finally $\{\alpha_1 < \alpha_2 u < 1 = \alpha_2\} \cup \{\alpha_1 < \alpha_2 u < 1 < \alpha_2\} = \{\alpha_1 < \alpha_2 u\}$ and here the only leading term is $h^2 A_1 A_2$. Thus

$$E\left[\sum_i \xi_i\right] \sim T(1-\gamma) C(1,1) \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - T h c_{A_1} c_{A_2} F_0(\varepsilon). \quad (5.5)$$

ii) If $\gamma = 1$, then $nE[\xi_1] = nh^2A_1A_2$, and again the leading term is different for different choices of α_1, α_2 . We have

$$nE[\xi_1] \sim Thc_{A_1c_{A_2}}F_1(\varepsilon). \quad (5.6)$$

As for $Var(\xi_i)$: in the general case $\gamma \in [0, 1]$, writing X_m for X_m^ε , $Var(\xi_i)$ is given by

$$\begin{aligned} E[X_1^2X_2^2] - 2hA_2E[X_1^2X_2] - 2hA_1E[X_1X_2^2] + h^2A_2^2E[X_1^2] + h^2A_1^2E[X_2^2] + 2h^2A_1A_2E[X_1X_2] \\ - E^2[X_1X_2] = h^2 \int_{0 \leq x_1, x_2 \leq \varepsilon} x_1^2 d\nu_\gamma \int_{0 \leq x_1, x_2 \leq \varepsilon} x_2^2 d\nu_\gamma + E^2[X_1X_2] + h \int_{0 \leq x_1, x_2 \leq \varepsilon} x_1^2 x_2^2 d\nu_\gamma + \\ - 2hA_2E[X_1^2X_2] - 2hA_1E[X_1X_2^2] + h^2A_2^2E[X_1^2] + h^2A_1^2E[X_2^2] + 2h^2A_1A_2E[X_1X_2] \\ := \sum_{\ell=1}^8 V_\ell, \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} V_1 := h^2 \int_{0 \leq x_1, x_2 \leq \varepsilon} x_1^2 d\nu_\gamma \int_{0 \leq x_1, x_2 \leq \varepsilon} x_2^2 d\nu_\gamma; \quad V_2 := E^2[X_1X_2]; \quad V_3 := h \int_{0 \leq x_1, x_2 \leq \varepsilon} x_1^2 x_2^2 d\nu_\gamma; \\ V_4 := -2hA_2E[X_1^2X_2]; \quad V_5 := -2hA_1E[X_1X_2^2]; \quad V_6 := h^2A_2^2E[X_1^2]; \\ V_7 := h^2A_1^2E[X_2^2]; \quad V_8 := 2h^2A_1A_2E[X_1X_2]. \end{aligned}$$

As $\varepsilon \rightarrow 0$ all these terms tend to zero: we now establish the leading ones and we only keep them.

i) If $\gamma \in (0, 1)$, we have the following properties: $V_1 \approx h^2\varepsilon^{4-\alpha_1-\alpha_2} \mathcal{A}C(0, 2) \gg$

$$V_6 \approx h^3c_{A_2}^2 [(1 - \varepsilon^{1-\alpha_2})^2 I_{\alpha_2 \neq 1} + \ln^2 \frac{1}{\varepsilon} I_{\alpha_2=1}] \mathcal{A} \varepsilon^{2-\alpha_1}; \quad V_2 = (1 - \gamma)^2 C^2(1, 1) h^2.$$

$$\cdot \varepsilon^{2(\frac{\alpha_2}{\alpha_1} + 1 - \alpha_2)} \text{ and } V_4 = -2(1 - \gamma)h^2c_{A_2} [(1 - \varepsilon^{1-\alpha_2}) I_{\alpha_2 \neq 1} + \ln \frac{1}{\varepsilon} I_{\alpha_2=1}] C(2, 1) \varepsilon^{1 + \frac{2\alpha_2}{\alpha_1} - \alpha_2}$$

are negligible wrt $V_3 = (1 - \gamma)C(2, 2)h\varepsilon^{2 + 2\frac{\alpha_2}{\alpha_1} - \alpha_2}$; recalling that we chose $\alpha_1 \leq \alpha_2$ and we only are interested in the case where at least $\alpha_2 \geq 1$, we have that

$$\begin{aligned} V_8 = 2(1 - \gamma)C(1, 1)c_{A_1c_{A_2}}h^3\varepsilon^{1 + \frac{\alpha_2}{\alpha_1} - \alpha_2} \left[(1 - \varepsilon^{1-\alpha_1})(1 - \varepsilon^{1-\alpha_2}) I_{\alpha_1, \alpha_2 \neq 1} + \right. \\ \left. (1 - \varepsilon^{1-\alpha_2}) \ln \frac{1}{\varepsilon} I_{\alpha_1=1 < \alpha_2} + (1 - \varepsilon^{1-\alpha_1}) \ln \frac{1}{\varepsilon} I_{\alpha_1 < 1 = \alpha_2} + \ln^2 \frac{1}{\varepsilon} I_{\alpha_1 = \alpha_2 = 1} \right] \\ \ll V_5 = -2h^2c_{A_1} [(1 - \varepsilon^{1-\alpha_1}) I_{\alpha_1 \neq 1} + \ln \frac{1}{\varepsilon} I_{\alpha_1=1}] (1 - \gamma)C(1, 2) \varepsilon^{2 + \frac{\alpha_2}{\alpha_1} - \alpha_2}; \\ V_7 = h^3c_{A_1}^2 [(1 - \varepsilon^{1-\alpha_1})^2 I_{\alpha_1 \neq 1} + \ln^2 \frac{1}{\varepsilon} I_{\alpha_1=1}] C(0, 2) \varepsilon^{2-\alpha_2} \ll V_1, \end{aligned}$$

so we are left with $Var(\xi_j) \approx V_1 + V_3 + V_5$. Now, as $h \rightarrow 0$, we have that V_1/V_5 tends to 0 if $\alpha_2 = \alpha_1 = 1$; to K if $\alpha_2 = \alpha_1 > 1$; to ∞ if either $\alpha_1 < 1 \leq \alpha_2$ or $\alpha_1 = 1 < \alpha_2$ or $1 < \alpha_1 < \alpha_2$, and

$$\frac{V_3}{V_5} \rightarrow \begin{cases} 0 & \text{if } \alpha_1 < \alpha_2 u \\ K & \text{if } \alpha_1 = \alpha_2 u \\ \infty & \text{if } \alpha_1 > \alpha_2 u \end{cases} \quad \frac{V_1}{V_3} \rightarrow \begin{cases} 0 & \text{if } \alpha_1 \in (x_*, 2) \\ K & \text{if } \alpha_1 = x_* \\ \infty & \text{if } \alpha_1 \in (0, x_*). \end{cases}$$

By considering the different regions $\alpha_1 < \alpha_2 u$; $\alpha_1 = \alpha_2 u$; $\alpha_1 \in (\alpha_2 u, x_*)$; $\alpha_1 = x_*$; $\alpha_1 \in (x_*, 2)$; we find that V_5 is never the leading term in $V_1 + V_3 + V_5$, V_1 is the only leading term for $\alpha_1 \in (0, x_*)$; $V_1 \sim V_3$ are leading for $\alpha_1 = x_*$; and V_3 is the only leading term for $\alpha_1 \in (x_*, 2)$. However if $\alpha_1 \leq x_*$ then necessarily $\alpha_1 < \alpha_2$ so \mathcal{A} becomes $\gamma C_1(2)$ and

$$Var(\xi_i) \approx h^2\gamma C_1(2)C(0, 2)\varepsilon^{4-\alpha_1-\alpha_2} I_{\{\alpha_1 \leq x_*\}} + h(1 - \gamma)C(2, 2)\varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 \geq x_*\}}$$

$$= h\varepsilon^{2-\alpha_2} [h\varepsilon^{2-\alpha_1} \gamma C_1(2)C(0,2)I_{\{\alpha_1 \leq x_*\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}} (1-\gamma)C(2,2)I_{\{\alpha_1 \geq x_*\}}]$$

so, recalling (5.5), (5.1) follows.

ii) If $\gamma = 1$, then it turns out that $\text{Var}(\xi_1) = V_1 + V_6 + V_7 \approx V_1 = E[X_1^2 X_2^2] =$

$$= h^2 \int_{0 < x_1 \leq \varepsilon} x_1^2 v_\perp(dx_1) \int_{0 < x_1 \leq \varepsilon} x_1^2 v_\perp(dx_1) = h^2 \varepsilon^{4-\alpha_1-\alpha_2} C_1(2)C_2(2), \quad (5.8)$$

and thus, recalling (5.6), (5.2) is verified.

iii) If $\gamma = 0$ then $V_1 = h^2 \varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-2\alpha_2} C(2,0)C(0,2)$ and $V_6 = h^3 c_{A_2}^2 [(1-\varepsilon^{1-\alpha_2})^2 I_{\alpha_2 \neq 1} + \ln^2 \frac{1}{\varepsilon} I_{\alpha_2=1}] C(2,0) \varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2}$, for all $\alpha_1 \leq \alpha_2$, while the other terms V_ℓ remain the same as in i) but with $\gamma = 0$. It turns out that still $V_6 \ll V_1$, while $V_1 \ll V_3$, thus $\text{Var}(\xi_1) \approx V_3 + V_5 + V_7$, and more precisely

$$\text{Var}(\xi_1) \sim \begin{cases} V_3 & \text{if } \alpha_1 > \alpha_2 u \\ V_3 \sim V_5 \sim V_7 & \text{if } \alpha_1 = \alpha_2 u \\ V_7 & \text{if } \alpha_1 < \alpha_2 u, \end{cases}$$

and, recalling (5.5), (5.3) follows. \square

Proposition 5.4. Assume $0 < \alpha_1 \leq \alpha_2 < 2$, $\alpha_2 \geq 1$, $0 < c_1 \leq c_2$, $u \in (0, \frac{1}{2})$. As $h \rightarrow 0$ we have $\frac{\sqrt{n\text{Var}(\xi_1)}}{nE[\xi_1]} \rightarrow 0$ in the following cases:

i) for $\gamma \in [0, 1)$: for any choices of α_1, α_2 and u as in the assumptions;

ii) for $\gamma = 1$: on $\{\alpha_1 < 1, \alpha_2 \geq 1\} \cup \{\alpha_1 = 1 < \alpha_2\}$ iff $u \in (\frac{1}{2+\alpha_2-\alpha_1}, \frac{1}{2})$; on $\{1 < \alpha_1 \leq \alpha_2\}$ iff $u \in (\frac{1}{\alpha_1+\alpha_2}, \frac{1}{2})$.

We have $\frac{\sqrt{n\text{Var}(\xi_1)}}{nE[\xi_1]} \rightarrow +\infty$ in the following case:

iii) for $\gamma = 1$: on $\{\alpha_1 = \alpha_2 = 1\}$, any $u \in (0, \frac{1}{2})$.

Proof. i) Case $\gamma \in (0, 1)$. We compute $\frac{\sqrt{n\text{Var}(\xi_1)}}{nE[\xi_1]}$ by using the information (leading terms of $nE[\xi_1]$ and of $\sqrt{n\text{Var}(\xi_1)}$) summarized in (5.1) in the four different cases 1) $\alpha_1 \in (0, \alpha_2 u]$, $\alpha_2 > 1$; 2) $\alpha_1 \in (0, \alpha_2 u]$, $\alpha_2 = 1$; 3) $\alpha_1 \in (\alpha_2 u, x_*]$; 4) $\alpha_1 \in (x_*, \alpha_2]$. In the cases 1), 2), 3) we have $\alpha_1 \leq x_* < \alpha_2$, thus $\alpha_1 \neq \alpha_2$, and we reach that a sufficient condition for $\frac{\sqrt{n\text{Var}(\xi_1)}}{nE[\xi_1]} \rightarrow 0$ is $u \in (\frac{1}{2+\alpha_2-\alpha_1}, \frac{1}{2})$. However $x_* < 2 + \alpha_2 - 1/u$, thus if $\alpha_1 \leq x_*$ then $\alpha_1 < 2 + \alpha_2 - 1/u$, which is equivalent to $u > \frac{1}{2+\alpha_2-\alpha_1}$. On the other hand, in the case 4) we reach $\frac{\sqrt{n\text{Var}(\xi_1)}}{nE[\xi_1]} \rightarrow 0$ for any $u \in (0, 1/2)$.

Case $\gamma = 0$. We now look at (5.3). Here we separately study the regions $\{\alpha_1 > \alpha_2 u\}$; $\{\alpha_1 = \alpha_2 u\}$; $\{\alpha_1 < \alpha_2 u, \alpha_2 > 1\}$; $\{\alpha_1 < \alpha_2 u, \alpha_2 = 1\}$ and conclude.

ii) and iii). For $\gamma = 1$ we look at (5.2) and we separately study the regions $\{\alpha_1 < 1 < \alpha_2\}$; $\{\alpha_1 < 1 = \alpha_2\}$; $\{\alpha_1 = 1 < \alpha_2\}$; $\{\alpha_1 = \alpha_2 = 1\}$; $\{1 < \alpha_1 \leq \alpha_2\}$; and reach the results. \square

Theorem 5.5. When $\gamma = 1 = \alpha_1 = \alpha_2$: $\forall u \in (0, \frac{1}{2})$, with \xrightarrow{d} denoting convergence in distribution, we have

$$\frac{\sum_{i=1}^n \xi_i - nE[\xi_1]}{\sqrt{n\text{Var}(\xi_1)}} \xrightarrow{d} \mathcal{N}.$$

Remark. A CLT for $\sum_{i=1}^n \xi_i$ also holds in the case of completely dependent small jumps, i.e. $\gamma = 0$ (see [7], Thm 4.4).

Proof of Theorem 5.5. Under $\gamma = 1 = \alpha_1 = \alpha_2$, $M^{(1)}$ and $M^{(2)}$ are independent, and $n\text{Var}(\xi_1) \sim h\varepsilon^2$. By the Lindeberg-Feller Theorem, it is sufficient to show that for all $\delta > 0$ we have $nE[\xi_1^2 I_{\{|\xi_1| > \delta\}}] \rightarrow 0$. We begin evaluating $P\{|\xi_1| > \delta\}$: by using that we have

$\frac{nE[\xi_1]}{\sqrt{n\text{Var}(\xi_1)}} \rightarrow 0$, $hA_1 = hA_2$ and $X_1^\varepsilon = X_1$ has the same law as $X_2^\varepsilon = X_2$, for sufficiently small h we obtain

$$\begin{aligned} P\{|\tilde{\xi}_1| > \delta\} &\leq P\left\{|\xi_1| > \frac{\delta}{2}\sqrt{n\text{Var}(\xi_1)}\right\} = P\left\{|M_h^{(1)}| |M_h^{(2)}| > \frac{\delta}{2}\sqrt{n\text{Var}(\xi_1)}\right\} \leq \\ &P\left\{|X_1||X_2| + hA_2|X_1| + hA_1|X_2| + h^2A_1A_2 > \frac{\delta}{2}\sqrt{n\text{Var}(\xi_1)}\right\} \leq P\left\{|X_1||X_2| > \right. \\ &\left. \frac{\delta}{8}\sqrt{n\text{Var}(\xi_1)}\right\} + 2P\left\{hA_2|X_1| > \frac{\delta}{8}\sqrt{n\text{Var}(\xi_1)}\right\} + P\left\{h^2A_1A_2 > \frac{\delta}{8}\sqrt{n\text{Var}(\xi_1)}\right\}. \end{aligned} \quad (5.9)$$

Now, for sufficiently small h the last term is 0, because $\frac{h^2A_1A_2}{\sqrt{n\text{Var}(\xi_1)}} = \frac{h^2 \log^2 \frac{1}{\varepsilon}}{\sqrt{h\varepsilon}} = h^{\frac{3}{2}-u}$. $\log^2 \frac{1}{\varepsilon} \rightarrow 0$. We now evaluate the other 2 probabilities in (5.9) to establish their magnitude orders: since X_1X_2 is centered, by the Čebyšev inequality, using that $\alpha_1 = 1$, we have

$$\begin{aligned} P\left\{|X_1X_2| > \frac{\delta}{8}\sqrt{n\text{Var}(\xi_1)}\right\} &\leq \frac{\text{Var}[|X_1X_2|]}{Kh\varepsilon^2} = \frac{E^2X_1^2}{Kh\varepsilon^2} \sim \frac{(h\varepsilon^{2-\alpha_1})^2}{Kh\varepsilon^2} = Kh; \\ P\left\{hA_2|X_1| > \frac{\delta}{8}\sqrt{n\text{Var}(\xi_1)}\right\} &= P\left\{|X_1| > K\frac{\delta}{8}h^{u-\frac{1}{2}}\right\} \leq Kh^{2-u}\log^2 \frac{1}{\varepsilon}. \end{aligned}$$

Noting that $\frac{h^{2-u}\log^2 \frac{1}{\varepsilon}}{h} \rightarrow 0$, it follows that $P\{|\tilde{\xi}_1| > \delta\} \leq Kh$. Now, for any conjugate exponents p, q ,

$$nE[\tilde{\xi}_1^{2p} I_{\{|\tilde{\xi}_1| > \delta\}}] \leq nE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]P^{\frac{1}{q}}\{|\tilde{\xi}_1| > \delta\} \leq KnE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]h^{\frac{1}{q}}.$$

We now evaluate $E[\tilde{\xi}_1^{2p}] =$

$$E\left[\left(\frac{\xi_1}{\sqrt{n\text{Var}(\xi_1)}} - \frac{E[\xi_1]}{\sqrt{n\text{Var}(\xi_1)}}\right)^{2p}\right] \leq KE\left[\left(\frac{\xi_1}{\sqrt{n\text{Var}(\xi_1)}}\right)^{2p}\right] + K\left(\frac{E[\xi_1]}{\sqrt{n\text{Var}(\xi_1)}}\right)^{2p}.$$

Reaching $E[\xi_1]$ from the expression of $nE[\xi_1]$ above (5.6) and using the expression for $\sqrt{n\text{Var}(\xi_1)}$, the last term equals

$$K\left(h^{\frac{3}{2}-u}\log^2 \frac{1}{\varepsilon}\right)^{2p}$$

On the other hand

$$E\left[\frac{\xi_1^{2p}}{(n\text{Var}(\xi_1))^p}\right] \leq K\left(\frac{E[(X_1X_2)^{2p}]}{(n\text{Var}(\xi_1))^p} + 2\frac{E[(hA_2)^{2p}X_1^{2p}]}{(n\text{Var}(\xi_1))^p} + \frac{E[(h^2A_1A_2)^{2p}]}{(n\text{Var}(\xi_1))^p}\right);$$

the last term contributes with $\left(h^{3/2-u}\log^2 \frac{1}{\varepsilon}\right)^{2p}$; the second term, by the Burkholder-Davis-Gundy inequality and recalling that $\alpha_1 = 1$, is dominated by

$$K\left(\frac{h^2 \log^2 \frac{1}{\varepsilon} \int_{t_{i-1}}^{t_i} \int_0^\varepsilon x^2 \nu^{(1)}(dx)}{h\varepsilon^2}\right)^p = \left(h^{2-u}\log^2 \frac{1}{\varepsilon}\right)^p;$$

and the first term is

$$\frac{E[X_1^{2p}X_2^{2p}]}{(h\varepsilon^2)^p} = \frac{E^2[X_1^{2p}]}{(h\varepsilon^2)^p} \leq K\frac{(h\varepsilon)^{2p}}{(h\varepsilon^2)^p} = Kh^p.$$

Thus

$$E[\tilde{\xi}_1^{2p}] \leq K\left(\left(h^{3/2-u}\log^2 \frac{1}{\varepsilon}\right)^{2p} + \left(h^{2-u}\log^2 \frac{1}{\varepsilon}\right)^p + h^p\right) \sim h^p. \quad \square$$

It follows that we have $nE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]h^{\frac{1}{q}} \leq Knhh^{\frac{1}{q}} \rightarrow 0$.

5.2 Proof of Theorem 3.2

We remark that under A1 we have (point iii) within the proof of Theorem 1 in [13]) that a.s.

$$\sup_{1 \leq j \leq n} \frac{|\Delta_j D^{(m)}|}{\sqrt{2h \log \frac{1}{h}}} \leq K_m(\omega) < \infty, \quad m = 1, 2, \quad (5.10)$$

where $K_m := \sup_{s \in [0, T]} |a|_s + \sup_{s \in [0, T]} |\sigma|_s + 1$ are finite random variables.

By using a localization procedure similar to the one in [9] (sec. 3.6.3) we can assume wlg that the coefficients $a^{(m)}, \sigma^{(m)}, \rho$ in (1.1) are bounded. In particular, we can take K_m to be constants.

In the following denote, for $m = 1, 2$,

$$N_t^{(m)} = \sum_{s \leq t} I_{\{|\Delta X_s^{(m)}| > 1\}}, \quad \tilde{N}_t^{(m)} = \sum_{s \leq t} I_{\{|\Delta X_s^{(m)}| > \sqrt{r_h}\}}, \quad \tilde{V}_t^{(m)} = \sum_{s \leq t} I_{\{|\Delta M_s^{(m)}| > \sqrt{r_h}\}}, \quad \theta_m = hr_h^{-\frac{\alpha_m}{2}}$$

For U a rv, we denote $\|U\|_\ell = E^{\frac{1}{\ell}}[|U|^\ell]$.

To prove Theorem 3.2 we write

$$\hat{IC} - IC = \sum_{k=1}^4 I_k, \quad (5.11)$$

where

$$\begin{aligned} I_1 &= \left[\sum_{i=1}^n \Delta_i Y^{(1)} \Delta_i Y^{(2)} I_{\{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}\}} I_{\{|\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}} - IC \right], \\ I_2 &= \sum_{i=1}^n \Delta_i Y^{(1)} \Delta_i Y^{(2)} \left(I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} - I_{\{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}\}} I_{\{|\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}} \right), \\ I_3 &= \sum_{i=1}^n (\Delta_i Y^{(1)} \Delta_i M^{(2)} + \Delta_i Y^{(2)} \Delta_i M^{(1)}) I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}, \\ I_4 &= \sum_{i=1}^n \Delta_i M^{(1)} \Delta_i M^{(2)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}. \end{aligned}$$

We know that $I_1/\sqrt{h} \xrightarrow{st} U$, with U mixed Gaussian rv ([16]). We are going to show that:

$$I_2 \sim \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i \tilde{V}^{(2)}| \geq 1\}} \sim \theta_2 = hr_h^{-\frac{\alpha_2}{2}}, \quad I_3 \ll \sqrt{h}$$

and I_4 is the sum of $\sum_{i=1}^n \xi_i$ with some other terms which however are negligible wrt one of the terms \sqrt{h} , θ_2 or $\sum_{i=1}^n \xi_i$. That will prove (3.5). Term I_2 turns out to be the most complicated, and in order to deal with it we need the following Lemmas 5.7, 5.8, 5.9, and 5.10. Remark 5.6 is used to prove such Lemmas.

It then turns out that none of the terms appearing in (3.5) is always negligible, while depending on the combination of the parameters $\gamma, \alpha_1, \alpha_2$ the leading term is different, and we show (3.7, 3.8, 3.9).

Remark 5.6. 1. (Lemma 2 in [1]: note that the expansion (24) and the estimate (50), on which the proof is based, hold for any stable process and any stability index in $(0, 2)$, thanks to (2.4.6), (2.4.8) in the cited book of Zolotarev and to the expansion of $p^0(1, x)$ at page 89 in [18]) If \tilde{L} is a symmetric stable process with $\tilde{N}_t = \sum_{s \leq t} \Delta \tilde{L}_{\{|\Delta \tilde{L}_s| > \varepsilon\}}$ and Lévy density $F(dx) = \frac{c}{|x|^{1+\alpha}} dx$, if $\tilde{\theta} = h\varepsilon^{-\alpha}$, then:

$$P\left\{ \left| \Delta_i \tilde{L} - \sum_{s \in [t_{i-1}, t_i]} \Delta \tilde{L}_s I_{\{|\Delta \tilde{L}_s| > \varepsilon\}} \right| > \varepsilon \right\} + P\{|\Delta_i \tilde{L}| > \varepsilon, \Delta_i \tilde{N} = 0\} + P\{|\Delta_i \tilde{L}| \leq \varepsilon, \Delta_i \tilde{N} = 1\} \leq K\tilde{\theta}^{\frac{4}{3}}$$

2. ([6], ch.3, Prop. 3.7) For any Lévy process V with Lévy measure ν , then $\sum_{s \leq t} I_{\{|\Delta V_s| > \varepsilon\}}$ is a Poisson process with parameter $t\nu\{|x| > \varepsilon\} = tU(\varepsilon)$, where $U(x)$ gives the tail of the jumps sizes measure; it follows that if $\nu(dx) = a|x|^{-1-\alpha} \cdot I_{x < 0} + bx^{-1-\alpha} I_{x > 0}$ with $a, b \geq 0$ and $(a, b) \neq (0, 0)$, then with $p \in (0, 1)$:
- $$P\{\sum_{s \in [t_{i-1}, t_i]} I_{\{|\Delta V_s| > \varepsilon\}} = 1\} \sim \tilde{\theta},$$

$$P\{\sum_{s \in [t_{i-1}, t_i]} I_{\{|\Delta V_s| > \varepsilon\}} \geq 2\} \sim \tilde{\theta}^2, P\{\sum_{s \in [t_{i-1}, t_i]} I_{\{|\Delta V_s| \in (\varepsilon(1-p), \varepsilon]\}} = 1\} \sim \tilde{\theta}((1-p)^{-\alpha} - 1). \square$$

Let us recall that each $M^{(m)}$ is given by the small jumps of a *one-sided* stable process $L^{(m)}$.

Lemma 5.7. *Let L be a one-sided α -stable process with characteristic triplet $(z, 0, c \cdot I_{\{x > 0\}} x^{-1-\alpha} dx)$, let $H_1 := (L_t - zt)_t$, take $\varepsilon = \varepsilon(h)$ s.t. $h/\varepsilon(h) \rightarrow 0$, any constant $p \in (0, 1)$ s.t. $p > |z|h/\varepsilon$ and any $q \in (0, 1-p)$. For $m = 1, 2, i = 1..n$ we have the following.*

1. $P\{\Delta_i N^{(m)} \neq 0, (\Delta_i M^{(m)})^2 > r_h\} \leq K \frac{h^2}{r_h}, P(|\Delta_i M^{(m)}| > K\sqrt{r_h}) \leq K\theta_m.$
2. $P\{|\Delta_i L| > \varepsilon, \sum_{s \in [t_{i-1}, t_i]} I_{\{|\Delta L_s| > \varepsilon\}} = 0\} \leq K\tilde{\theta}^{4/3} + K\tilde{\theta}(q^{-\alpha} - 1).$
3. $P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(m)} = 0\} \leq K\theta_m^{4/3} + K\theta_m(q^{-\alpha_m} - 1).$
4. $P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s \in [t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \varepsilon\}} = 1\} \leq K[\tilde{\theta}^{4/3} + \tilde{\theta}(1 - (1+2p)^{-\alpha})]$
 $P\{|\Delta_i L| \leq \varepsilon, \sum_{s \in [t_{i-1}, t_i]} I_{\{|\Delta L_s| > \varepsilon\}} = 1\} \leq K[\tilde{\theta}^{4/3} + \tilde{\theta}(1 - (1+2p)^{-\alpha})].$
5. With $\varepsilon = \sqrt{r_h}$ we have $P\{|\Delta_i M^{(m)}| \leq \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \geq 1\} \leq K\theta_m^{4/3} + K\theta_m(1 - (1+2p)^{-\alpha_m}).$

Proof. Point 1. By the independence of $N^{(m)}, M^{(m)}$ and using the Markov inequality for $P\{(\Delta_i M^{(m)})^2 > r_h\}$, we reach $P\{\Delta_i N^{(m)} \neq 0, (\Delta_i M^{(m)})^2 > r_h\} \leq Kh \frac{\int_0^1 x^2 \nu^{(m)}(dx)}{r_h} = K \frac{h^2}{r_h}$. The second inequality is a trivial consequence of Lemma 6 in [1], as $M^{(m)}$ is a semimartingale following the same model as $X^{(m)}$ in (1.1) with $a \equiv \sigma \equiv J^{(m)} \equiv 0$.

Point 2: the idea here is to look at H_1 as half of a symmetric stable process. More precisely, take an independent and identically distributed copy H_2 of H_1 , then $\tilde{L} = H_1 - H_2$ is a symmetric α -stable process. Let us fix any p as in the assumptions and call \tilde{L}' and H'_ℓ the processes \tilde{L}, H_ℓ deprived of their jumps bigger than ε , $\ell = 1, 2$, e.g. $H'_{\ell t} = H_{\ell t} - \sum_{s \leq t} \Delta H_{\ell s} I_{\{|\Delta H_{\ell s}| > \varepsilon\}}$. Note that if $|\Delta_i L| > \varepsilon$ then $|\Delta_i H_1| = |\Delta_i L - zh| > |\Delta_i L| - |z|h > \varepsilon - |z|h > \varepsilon(1-p)$, and also that the jumps of L and H_1 are the same and are positive, thus

$$P\left\{|\Delta_i L| > \varepsilon, \sum_{s \in [t_{i-1}, t_i]} I_{\{|\Delta L_s| > \varepsilon\}} = 0\right\} \leq P\left\{|\Delta_i H_1| > \varepsilon(1-p), \sum_{s \in [t_{i-1}, t_i]} I_{\{\Delta H_{1s} > \varepsilon\}} = 0\right\} =$$

$$P\left\{|\Delta_i H'_1| > \varepsilon(1-p), \sum_{s \in [t_{i-1}, t_i]} I_{\{\Delta H_{1s} > \varepsilon\}} = 0\right\} \leq P\{|\Delta_i H'_1| > \varepsilon(1-p)\} = \quad (5.12)$$

$$P\{|\Delta_i H'_1| > \varepsilon(1-p), |\Delta_i H'_2| \leq \varepsilon(1-p-q)\} + P\{|\Delta_i H'_1| > \varepsilon(1-p), |\Delta_i H'_2| > \varepsilon(1-p-q)\}$$

Now on the first set of the last display we have $|\Delta_i \tilde{L}'| = |\Delta_i H'_1 - \Delta_i H'_2| > |\Delta_i H'_1| - |\Delta_i H'_2| > \varepsilon(1-p) - \varepsilon(1-p-q) = \varepsilon q$, while the probability of the second set, by the independence of H'_1 and H'_2 , is $P\{|\Delta_i H'_1| > \varepsilon(1-p)\}P\{|\Delta_i H'_2| > \varepsilon(1-p-q)\}$ which is dominated by $K\tilde{\theta}^2$ by (66) in [1], applied with $a \equiv \sigma \equiv J \equiv 0, M_t = \int_0^t \int_0^\varepsilon x \tilde{\mu}(dx) - t \int_\varepsilon^1 x \nu(dx)$, ν the Lévy measure of L and $\tilde{\mu}$ the compensated jump measure of L . It follows that (5.12) is dominated by

$$P\{|\Delta_i \tilde{L}'| > \varepsilon q\} + K\tilde{\theta}^2 : \quad (5.13)$$

note that $P\{|\Delta_i \tilde{L}'| > \varepsilon q\} = P\{|\Delta_i \tilde{L}'| > \varepsilon q, \Delta_i \tilde{N} = 0\} + P\{|\Delta_i \tilde{L}'| > \varepsilon q, \Delta_i \tilde{N} \geq 1\}$: by the independence of $\Delta_i \tilde{L}'$ on $\Delta_i \tilde{N}$ and using (66) in [1] and Remark 5.6, point 2, we have

$$P\{|\Delta_i \tilde{L}'| > \varepsilon q, \Delta_i \tilde{N} \geq 1\} = P\{|\Delta_i \tilde{L}'| > \varepsilon q\} P\{\Delta_i \tilde{N} \geq 1\} \leq K \tilde{\theta}^2.$$

Therefore (5.13) is dominated by $P\{|\Delta_i \tilde{L}'| > \varepsilon q, \Delta_i \tilde{N} = 0\} + K \tilde{\theta}^2$; since $q < 1$, the last quantity is dominated by

$$P\{|\Delta_i \tilde{L}'| > \varepsilon q, \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta \tilde{L}_s| > \varepsilon q\}} = 0\} + P\left\{\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta \tilde{L}_s| \in (\varepsilon q, \varepsilon]\}} \geq 1\right\} + K \tilde{\theta}^2: \quad (5.14)$$

using Remark 5.6, point 1 with εq in place of ε for the first term, Remark 5.6 point 2 for the second term and the fact that $\tilde{\theta}^2 \ll \tilde{\theta}^{4/3}$, we reach our thesis.

Point 3 is a consequence of point 2. Let us denote $L^{(m)}$ with L . We have

$$\begin{aligned} & P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(m)} = 0\} = P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(m)} = 0, \\ & \sum_{s \in]t_{i-1}, t_i]} I_{\{\Delta L_s > 1\}} = 0\} + P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(m)} = 0, \sum_{s \in]t_{i-1}, t_i]} I_{\{\Delta L_s > 1\}} \geq 1\} \\ & \leq P\{|\Delta_i H_1'| > \sqrt{r_h}(1-p), \sum_{s \in]t_{i-1}, t_i]} I_{\{\Delta H_{1s} > \sqrt{r_h}\}} = 0\} + P\left\{\sum_{s \in]t_{i-1}, t_i]} I_{\{\Delta L_s > 1\}} \geq 1\right\} \end{aligned}$$

the first term is bounded by the one in (5.12) with $\varepsilon = \sqrt{r_h}$, while the second one involves the Poisson process counting the jumps of L bigger than 1 within $]t_{i-1}, t_i]$, which has parameter $hU(1)$, thus the thesis follows.

Point 4. With the same notations as at point 2, we have

$$\begin{aligned} & P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \varepsilon\}} = 1\} = P\left\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \varepsilon\}} \right. \\ & \left. = 1, |\Delta_i H_2| > \varepsilon p\right\} + P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \varepsilon\}} = 1, |\Delta_i H_2| \leq \varepsilon p\}: \quad (5.15) \end{aligned}$$

the first term of the right hand side (rhs) is dominated by $P\{\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \varepsilon\}} = 1, |\Delta_i H_2| > \varepsilon p\} = P\{\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \varepsilon\}} = 1\} P\{|\Delta_i H_2| > \varepsilon p\} \leq K \tilde{\theta}^2$, having used the independence and Remark 5.6 point 2 and (66) in [1]. As for the second term, on $\{|\Delta_i H_1| \leq \varepsilon(1+p), |\Delta_i H_2| \leq \varepsilon p\}$ we have $|\Delta_i \tilde{L}| = |\Delta_i H_1 - \Delta_i H_2| \leq |\Delta_i H_1| + |\Delta_i H_2| \leq \varepsilon(1+p) + \varepsilon p = \varepsilon(1+2p)$. Moreover, by their independence, the two H_ℓ have no common jumps, so a jump of H_2 cannot neutralize any jumps of H_1 , thus $\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \varepsilon\}} = 1 \Rightarrow \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta \tilde{L}_s| > \varepsilon\}} \geq 1$. Since $P\{\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta \tilde{L}_s| > \varepsilon\}} \geq 2\} \leq K \tilde{\theta}^2$, it follows that (5.15) is dominated by

$$K \tilde{\theta}^2 + P\{|\Delta_i \tilde{L}| \leq \varepsilon(1+2p), \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta \tilde{L}_s| > \varepsilon\}} = 1\} \leq K \tilde{\theta}^2 +$$

$$P\{|\Delta_i \tilde{L}| \leq \varepsilon(1+2p), \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta \tilde{L}_s| > \varepsilon(1+2p)\}} = 1\} + P\left\{\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta \tilde{L}_s| \in (\varepsilon, \varepsilon(1+2p))\}} = 1\right\}$$

and the first thesis follows by Remark 5.6 point 1, with $\varepsilon(1+2p)$ in place of ε , point 2 and the fact that $\tilde{\theta}^2 \ll \tilde{\theta}^{4/3}$. The second inequality at point 4 follows from the previous one. In fact if $|\Delta_i L| \leq \varepsilon$ then $|\Delta_i H_1| = |\Delta_i L - z h| \leq |\Delta_i L| + |z| h < \varepsilon(1+p)$, further L and H_1 do exactly the same jumps, thus

$$P\{|\Delta_i L| \leq \varepsilon, \sum_{s \in]t_{i-1}, t_i]} I_{\{L_s > \varepsilon\}} = 1\} \leq P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s \in]t_{i-1}, t_i]} I_{\{H_{1s} > \varepsilon\}} = 1\}.$$

Point 5 follows from point 4. Let us again denote $L^{(m)}$ with L . We have

$$\begin{aligned} & P\{|\Delta_i M^{(m)}| \leq \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \geq 1\} = \quad (5.16) \\ & P\{|\Delta_i M^{(m)}| \leq \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \geq 1, \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta L_s| > 1\}} = 0\} + \\ & P\{|\Delta_i M^{(m)}| \leq \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \geq 1, \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta L_s| > 1\}} \geq 1\} : \end{aligned}$$

the second term of the rhs is bounded by Kh , as at Point 3. On the set at the first term the jumps of M coincide with the jumps of L , and the very M coincides with H_1 . Thus the last display is dominated by

$$\begin{aligned} & P\{|\Delta_i H_1| \leq \sqrt{r_h}(1+p), \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \sqrt{r_h}\}} \geq 1, \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta L_s| > 1\}} = 0\} + Kh \leq \\ & P\{|\Delta_i H_1| \leq \sqrt{r_h}(1+p), \sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \sqrt{r_h}\}} = 1\} + P\{\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta H_{1s}| > \sqrt{r_h}\}} \geq 2\} + Kh, \end{aligned}$$

and the thesis follows by Lemma 5.7, point 4, Remark 5.6, point 2 and $Kh \ll \theta_m$. \square

Lemma 5.8. *Let, for $i=1..n$, $A_i \subset \Omega$ be independent on $W^{(1)}$ and $W^{(2)}$ and s.t. $\forall i, P(A_i) \leq \theta_m$. If each $\sigma^{(j)}$ satisfies (3.3), then*

$$i) \quad \frac{1}{\theta_m} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{A_i} \approx \frac{1}{\theta_m} \sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{A_i}.$$

$$ii) \text{ Any } P(A_i) \text{ is, we have} \quad E[|\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{A_i}|] \leq KP(A_i).$$

Proof. i) Denote $\sigma_i := \sigma_{t_i}$. We have $\sigma_s = \sigma_{i-1} + (\sigma_s - \sigma_{i-1})$, thus

$$\begin{aligned} & \frac{1}{\theta_m} \sum_{i=1}^n \left[\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} - \sigma_{i-1}^{(1)} \Delta_i W^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} \right] I_{A_i} = \quad (5.17) \\ & \frac{1}{\theta_m} \sum_{i=1}^n \left[\sigma_{i-1}^{(1)} \Delta_i W^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)} + \int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} + \right. \\ & \quad \left. \int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)} \right] I_{A_i}. \end{aligned}$$

Firstly note that

$$E \left[\int_{t_{i-1}}^{t_i} (\sigma_s^{(m)} - \sigma_{i-1}^{(m)})^2 ds | A_i \right] = \frac{E \left[\int_{t_{i-1}}^{t_i} (\sigma_s^{(m)} - \sigma_{i-1}^{(m)})^2 ds I_{A_i} \right]}{P(A_i)} \leq \frac{E \left[\int_{t_{i-1}}^{t_i} (\sigma_s^{(m)} - \sigma_{i-1}^{(m)})^2 ds \right]}{P(A_i)}$$

is bounded by $Kh^2/P(A_i)$. It follows that $E \left[|\sigma_{i-1}^{(1)} \Delta_i W^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)}| \right] =$

$$\begin{aligned} & E \left[|\sigma_{i-1}^{(1)} \Delta_i W^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)}| | A_i \right] P(A_i) \leq \\ & \sqrt{E \left[|\sigma_{i-1}^{(1)} \Delta_i W^{(1)}|^2 | A_i \right]} \sqrt{E \left[\left(\int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)} \right)^2 | A_i \right]} P(A_i); \quad (5.18) \end{aligned}$$

since $W^{(m)}$ is independent on A_i , its law under P and under $P(\cdot | A_i)$ is the same, thus it keeps its martingale property also under $P(\cdot | A_i)$. However, any bounded càdlàg integrand η is,

the stochastic integral $\eta \cdot W^{(m)}$ is a martingale under $P(\cdot | A_i)$, thus $E \left[\left| \int_{t_{i-1}}^{t_i} \eta_s dW_s^{(m)} \right|^2 \middle| A_i \right] = E \left[\int_{t_{i-1}}^{t_i} \eta_s^2 ds \middle| A_i \right]$, and (5.18) coincides with

$$\sqrt{E \left[(\sigma_{i-1}^{(1)})^2 h \middle| A_i \right]} \sqrt{E \left[\int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)})^2 ds \middle| A_i \right]} P(A_i) \leq K \sqrt{h} \sqrt{\frac{h^2}{P(A_i)}} P(A_i),$$

which equals $Kh\sqrt{hP(A_i)}$. Therefore the norm $\|\cdot\|_1$ of the first term in the rhs of (5.17) is bounded by

$$\frac{1}{\theta_m} \sum_{i=1}^n E \left[\left| \sigma_{i-1}^{(1)} \Delta_i W^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)} \right| \middle| A_i \right] \leq K \frac{nh\sqrt{hP(A_i)}}{\theta_m} \leq K \sqrt{\frac{h}{\theta_m}} \rightarrow 0.$$

We reach the same result also for the second term in the rhs of (5.17). Finally

$$\begin{aligned} & \frac{1}{\theta_m} \sum_{i=1}^n E \left[\left| \int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)} \right| \middle| A_i \right] \leq \\ & \frac{1}{\theta_m} \sum_{i=1}^n \sqrt{E \left[\left(\int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)} \right)^2 \middle| A_i \right]} \sqrt{E \left[\int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)})^2 ds \middle| A_i \right]} P(A_i) \leq \\ & K \frac{1}{\theta_m} \sum_{i=1}^n \frac{h^2}{P(A_i)} P(A_i) \leq K \varepsilon^{\alpha_m} \rightarrow 0. \end{aligned}$$

ii) Similarly, $E \left[\left| \sum_{i=1}^n \sigma_{i-1}^{(1)} \Delta_i W^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} \right| \middle| A_i \right] \leq \sum_{i=1}^n E \left[\left| \sigma_{i-1}^{(1)} \Delta_i W^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} \right| \middle| A_i \right] P(A_i) \leq K \sum_{i=1}^n h P(A_i) \leq K P(A_i)$. \square

Lemma 5.9. With \xrightarrow{ucp} denoting convergence in probability uniformly on $[0, T]$ and $IC_t := \int_0^t \rho_s \sigma_s^{(1)} \sigma_s^{(2)} ds$, we have

$$\frac{1}{\theta_m} \sum_{i=1}^{\lfloor t/h \rfloor} \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \geq 1\}} \xrightarrow{ucp} \frac{c_m}{\alpha_m} IC_t.$$

Proof. By the independence of each $W^{(j)}$ on $\tilde{V}^{(m)}$, using Lemma 5.8 and Remark 5.6 point 2, we have that the left hand side (lhs) of the above display has the same asymptotic behavior (in the \approx sense) as

$$\frac{1}{\theta_m} \sum_{i=1}^{\lfloor t/h \rfloor} \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \geq 1\}} := \sum_{i=1}^{\lfloor t/h \rfloor} \eta_i.$$

However we have

$$\sum_{i=1}^{\lfloor t/h \rfloor} E_{i-1}[\eta_i] = \frac{1}{\theta_m} \sum_{i=1}^{\lfloor t/h \rfloor} \sigma_{t_{i-1}}^{(1)} \sigma_{t_{i-1}}^{(2)} E_{i-1}[\Delta_i W^{(1)} \Delta_i W^{(2)}] P_{i-1}\{\Delta_i \tilde{V}^{(m)} \geq 1\} :$$

$E_{i-1}[\Delta_i W^{(1)} \Delta_i W^{(2)}] = E_{i-1}[\int_{t_{i-1}}^{t_i} \rho_s ds]$, and $P_{i-1}\{\Delta_i \tilde{V}^{(m)} \geq 1\} = 1 - e^{-\lambda_m h}$ with $\lambda_m = c_m \frac{r_h}{\alpha_m} \frac{-\alpha_m}{2}$, and $|1 - e^{-\lambda_m h} - \lambda_m h| \leq K \theta_m^2$, thus the last display has the same limit in probability as

$$\frac{1}{\theta_m} \sum_{i=1}^{\lfloor t/h \rfloor} \sigma_{t_{i-1}}^{(1)} \sigma_{t_{i-1}}^{(2)} E_{i-1} \left[\int_{t_{i-1}}^{t_i} \rho_s ds \right] \lambda_m h = \sum_{i=1}^{\lfloor t/h \rfloor} \sigma_{t_{i-1}}^{(1)} \sigma_{t_{i-1}}^{(2)} E_{i-1} \left[\int_{t_{i-1}}^{t_i} \rho_s ds \right] \frac{c_m}{a_m}. \quad (5.19)$$

Further,

$$E \left[\sum_{i=1}^{\lfloor t/h \rfloor} |\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)}| \cdot \left| E_{i-1} \left[\int_{t_{i-1}}^{t_i} \rho_s ds \right] - \rho_{t_{i-1}} h \right| \right]$$

$$\leq KE \left[\sum_{i=1}^{\lfloor t/h \rfloor} |\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)}| E_{i-1} \left[\int_{t_{i-1}}^{t_i} |\rho_s - \rho_{t_{i-1}}| ds \right] \right] \leq Knh^2 \rightarrow 0,$$

and this implies that (5.19) has the same limit in probability as

$$\sum_{i=1}^{\lfloor t/h \rfloor} \sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} \rho_{t_{i-1}} h \xrightarrow{P} \frac{c_m}{\alpha_m} IC.$$

However by separating $\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} \rho_{i-1} = (\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} \rho_{i-1})^+ - (\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} \rho_{i-1})^-$ and applying the reasoning indicated in [9], just before (3.5), we reach that such a convergence is also ucp. Further

$$\sum_{i=1}^{\lfloor t/h \rfloor} E_{i-1}[\eta_i^2] = \frac{1}{\theta_m^2} \sum_{i=1}^{\lfloor t/h \rfloor} (\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)})^2 E_{i-1}[(\Delta_i W^{(1)})^2 (\Delta_i W^{(2)})^2] P_{i-1} \{ \Delta_i \tilde{V}^{(m)} \geq 1 \} :$$

by using the expression of $W^{(2)}$ given after (1.1) we find that $E_{i-1}[(\Delta_i W^{(1)})^2 (\Delta_i W^{(2)})^2] \leq Kh^2$, thus

$$\sum_{i=1}^{\lfloor t/h \rfloor} E_{i-1}[\eta_i^2] \leq K \frac{1}{\theta_m^2} h \theta_m \sum_{i=1}^{\lfloor t/h \rfloor} (\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)})^2 h \sim \frac{h}{\theta_m} \int_0^t (\sigma_s^{(1)} \sigma_s^{(2)})^2 ds \sim \varepsilon^{\alpha_m} \rightarrow 0.$$

Thus, by Lemma 4.2 in [9], the thesis follows. \square

Lemma 5.10. *We have*

$$\frac{1}{\theta_1} \sum_{i=1}^{\lfloor t/h \rfloor} \sigma_{i-1}^{(1)} \Delta_i W^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} I_{\{\Delta_i \tilde{V}^{(1)} \geq 1, \Delta_i \tilde{V}^{(2)} \geq 1\}} \xrightarrow{ucp} (1-\gamma) \frac{c_1}{\alpha_1} IC_t.$$

Proof. Let us start by proving that

$$P\{\Delta_i \tilde{V}^{(1)} \geq 1, \Delta_i \tilde{V}^{(2)} \geq 1\} \approx (1-\gamma) \theta_1 \frac{c_1}{\alpha_1}. \quad (5.20)$$

In fact, with $\varepsilon = \sqrt{r_h}$, such a probability equals $P\left\{ \mu(\lfloor t_{i-1}, t_i \rfloor \times (\varepsilon, 1] \times (\varepsilon, 1]) \geq 1 \right\} = 1 - e^{-\lambda h} \approx \lambda h$, where $\lambda = \nu_\gamma(\lfloor t_{i-1}, t_i \rfloor \times (\varepsilon, 1] \times (\varepsilon, 1])$. In view of (2.2) and the shape of f , which is increasing, concave and with $f(1) > 1$, due to our choice of the parameters, we have $\lambda = (1-\gamma) \int_{(\varepsilon, 1] \times (\varepsilon, 1]} \mathbf{1}_{\nu_\parallel}(dx_1, dx_2) : \nu_\parallel$ only weights the points (x_1, x_2) with $x_2 = f(x_1)$, and $x_1 \wedge f(x_1) > \varepsilon$ means that $x_1 > f^{-1}(\varepsilon) \vee \varepsilon = \varepsilon$, while $x_1 \vee f(x_1) \leq 1$ means that $x_1 \leq f^{-1}(1) \wedge 1 = f^{-1}(1)$, thus $\lambda = (1-\gamma) \nu_\parallel((\varepsilon, f^{-1}(1)]) = (1-\gamma) \left[c_1 \frac{\varepsilon^{-\alpha_1}}{\alpha_1} - \frac{c_2}{\alpha_2} \right]$, having used that $f = U_2^{-1} \circ U_1$. However if $\gamma \neq 1$ then the leading term of $P\{\Delta_i \tilde{V}^{(1)} \geq 1, \Delta_i \tilde{V}^{(2)} \geq 1\}$ is $(1-\gamma) \theta_1 \frac{c_1}{\alpha_1}$. If $\gamma = 1$ then $\lambda = 0$, and (5.20) is verified.

Let us now define

$$\sum_{i=1}^{\lfloor t/h \rfloor} \frac{1}{\theta_1} \sigma_{i-1}^{(1)} \Delta_i W^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} I_{\{\Delta_i \tilde{V}^{(1)} \geq 1, \Delta_i \tilde{V}^{(2)} \geq 1\}} = \sum_{i=1}^{\lfloor t/h \rfloor} \chi_i :$$

by the independence of each $W^{(m)}$ on each $\tilde{V}^{(\ell)}$, we have

$$\sum_{i=1}^{\lfloor t/h \rfloor} E_{i-1}[\chi_i] \approx \sum_{i=1}^{\lfloor t/h \rfloor} \sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} E_{i-1} \left[\int_{t_{i-1}}^{t_i} \rho_s ds \right] (1-\gamma) \frac{c_1}{\alpha_1} \xrightarrow{P} (1-\gamma) \frac{c_1}{\alpha_1} IC_t \cdot I_{\{\gamma \in [0, 1)\}};$$

as in the previous Lemma, we reach that such a convergence is also ucp. Further, $\sum_{i=1}^{\lfloor t/h \rfloor} E_{i-1}[\chi_i^2] \leq K \frac{h\theta_1}{\theta_2^2} \leq K\varepsilon^{\alpha_1} \rightarrow 0$, so, by Lemma 4.2 in [9], the thesis follows. \square

Continuation of the proof of Theorem 3.2. From now on take a $p \in (0, 1)$, h sufficiently small and s.t. $p > \sqrt{h \ln \frac{1}{h}} / \sqrt{r_h}$, $q \in (0, 1 - p)$. Recall (5.11), and let us start showing that $I_2 \approx \theta_2$.

Term I_2 . Write $I_2 := I_{2,1} - \mathcal{J}$, where

$$I_{2,1} = \sum_{i=1}^n \Delta_i Y^{(1)} \Delta_i Y^{(2)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\} \cap \{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}^c},$$

$$\mathcal{J} = \sum_{i=1}^n \Delta_i Y^{(1)} \Delta_i Y^{(2)} \left(I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}^c \cap \{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}} \right).$$

We first show that $I_{2,1} \ll \theta_2$. In fact, for each i , on the set highlighted by the indicator we have $|\Delta_i Y^{(m)}| > 2\sqrt{r_h}$ for at least one $m \in \{1, 2\}$, and, using (5.10), we have $|\Delta_i J^{(m)}| + K\sqrt{h \ln \frac{1}{h}} \geq |\Delta_i D^{(m)} + \Delta_i J^{(m)}| = |\Delta_i Y^{(m)}| > 2\sqrt{r_h}$, which implies that $|\Delta_i J^{(m)}| \geq 2\sqrt{r_h}(1 - p)$, thus $|\Delta_i J^{(m)}| \neq 0$. However $|\Delta_i X^{(m)}| \leq \sqrt{r_h}$, and so $|\Delta_i J^{(m)} + \Delta_i M^{(m)}| - |\Delta_i D^{(m)}| < |\Delta_i X^{(m)}| \leq \sqrt{r_h}$ implies on one hand that $|\Delta_i J^{(m)} + \Delta_i M^{(m)}| < \sqrt{r_h}(1 + p)$, and on the other hand that, since we consider a sufficiently small h , that $1 - |\Delta_i M^{(m)}| < |\Delta_i J^{(m)}| - |\Delta_i M^{(m)}| < |\Delta_i J^{(m)} + \Delta_i M^{(m)}| < \sqrt{r_h}(1 + p)$, and thus $|\Delta_i M^{(m)}| > 1 - \sqrt{r_h}(1 + p) > \sqrt{r_h}$. It follows that $\forall i = 1..n$ there is an index m_i s.t. $\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}^c \cap \{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}^c \subset \{\Delta_i N^{(m_i)} \neq 0, \Delta_i M^{(m_i)} > \sqrt{r_h}\}$, thus, using Lemma 5.7 point 1, $P\left\{\frac{I_{2,1}}{\theta_2} \neq 0\right\} \leq \sum_{i=1}^n P\{\Delta_i N^{(m_i)} \neq 0, \Delta_i M^{(m_i)} > \sqrt{r_h}\} \leq K \frac{h}{r_h} \rightarrow 0$, which implies that $\frac{I_{2,1}}{\theta_2} \xrightarrow{P} 0$.

As for term \mathcal{J} , on $\{|\Delta_i Y^{(m)}| \leq 2\sqrt{r_h}\}$ we have $|\Delta_i J^{(m)}| - |\Delta_i D^{(m)}| < |\Delta_i Y^{(m)}| \leq 2\sqrt{r_h}$ and thus $|\Delta_i J^{(m)}| < 2\sqrt{r_h}(1 + p) < 1$, which implies that $\Delta_i J^{(m)} = 0$, i.e. $\Delta_i Y^{(m)} = \Delta_i D^{(m)}$. Thus, calling

$$\mathcal{B}_i = \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}^c \cap \{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\},$$

we have $\mathcal{J} = \sum_{i=1}^n \Delta_i Y^{(1)} \Delta_i Y^{(2)} I_{\mathcal{B}_i} = \sum_{i=1}^n \Delta_i D^{(1)} \Delta_i D^{(2)} I_{\mathcal{B}_i} := \sum_{k=2}^4 I_{2,k}$, where

$$I_{2,2} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} a_s^{(1)} ds \int_{t_{i-1}}^{t_i} a_s^{(2)} ds I_{\mathcal{B}_i}, \quad I_{2,4} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\mathcal{B}_i},$$

$$I_{2,3} = \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} a_s^{(2)} ds \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} + \int_{t_{i-1}}^{t_i} a_s^{(1)} ds \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \right) I_{\mathcal{B}_i}.$$

We show that $I_{2,4}$ is the leading term and it asymptotically behaves as θ_2 . As for $I_{2,2}$, by the boundedness of each $a^{(m)}$ we have $E\left[\frac{I_{2,2}}{\theta_2}\right] \leq K \frac{h}{\theta_2} \rightarrow 0$.

As for $I_{2,3}$, note that on $\{|\Delta_i X^{(m)}| > \sqrt{r_h}, |\Delta_i J^{(m)}| = 0\}$ we have $|\Delta_i M^{(m)}| + K\sqrt{h \ln \frac{1}{h}} > |\Delta_i M^{(m)}| + |\Delta_i D^{(m)}| \geq |\Delta_i D^{(m)} + \Delta_i M^{(m)}| = |\Delta_i X^{(m)}| > \sqrt{r_h}$ thus $|\Delta_i M^{(m)}| > \sqrt{r_h} - K\sqrt{h \ln \frac{1}{h}} > \sqrt{r_h}(1 - p)$. Using also Lemma 5.7 point 1 and noting that $\theta_1 \leq \theta_2$, it follows that

$$\frac{E[|I_{2,3}|]}{\theta_2} \leq \frac{K}{\theta_2} \sum_{i=1}^n h \sqrt{h \ln \frac{1}{h}} \left(P\{|\Delta_i M^{(1)}| > K\sqrt{r_h}\} + P\{|\Delta_i M^{(2)}| > K\sqrt{r_h}\} \right) \leq K \sqrt{h \ln \frac{1}{h}},$$

which tends to 0.

We now show that $I_{2,4} \approx \theta_2$. We proceed in two steps: 1) we show that

$$\frac{I_{2,4}}{\theta_2} \approx \frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \left(I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}\}} + I_{\{|\Delta_i X^{(2)}| > \sqrt{r_h}\}} \right) - B_n, \quad (5.21)$$

where $B_n \approx \theta_1 / \theta_2 (1 - \gamma) \frac{c_1}{\alpha_1} IC$; 2) we deal with each $\frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)}$.

$\int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(m)}| > \sqrt{r_h}\}}$ and we conclude. In order to deal with 1) we need two sub-steps: 1a) we show that

$$\frac{I_{2,4}}{\theta_2} \approx \frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}^c}. \quad (5.22)$$

Then, since $I_{(A \cap B)^c} = I_{A^c} + I_{B^c} - I_{A^c \cap B^c}$, each term of the sum at the numerator in (5.22) coincides with

$$\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \left[I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}\}} + I_{\{|\Delta_i X^{(2)}| > \sqrt{r_h}\}} - I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}, |\Delta_i X^{(2)}| > \sqrt{r_h}\}} \right] \quad (5.23)$$

so, with

$$B_n := \frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}, |\Delta_i X^{(2)}| > \sqrt{r_h}\}}, \quad (5.24)$$

(5.21) holds true.

2a) we then need to show $B_n \approx \frac{\theta_1}{\theta_2} (1 - \gamma) \frac{c_1}{\alpha_1} IC$. This part is cumbersome: we first show that

$$B_n \approx \frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i \tilde{V}^{(1)}| \geq 1, |\Delta_i \tilde{V}^{(2)}| \geq 1\}}, \quad (5.25)$$

then we apply Lemmas 5.8 and 5.10 to conclude. Let us proceed.

1a) Let us show (5.22). As argued just after the expression of $I_{2,1}$, $\{|\Delta_i Y^{(m)}| > 2\sqrt{r_h}\} \subset \{\Delta_i N^{(m)} \neq 0\}$, thus

$$\begin{aligned} & \frac{1}{\theta_2} \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \right| I_{\{|\Delta_i X^{(m)}| \leq \sqrt{r_h}, m=1,2\}^c \cap \{|\Delta_i Y^{(m)}| \leq 2\sqrt{r_h}, m=1,2\}^c} \leq \\ & \frac{1}{\theta_2} \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \right| \left(I_{\{\Delta_i N^{(1)} \neq 0\}} + I_{\{\Delta_i N^{(2)} \neq 0\}} \right) : \quad (5.26) \\ & \frac{\sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \right| I_{\{\Delta_i N^{(m)} \neq 0\}}}{\theta_2} \leq \frac{K}{\theta_2} \sum_{i=1}^n h \ln \frac{1}{h} I_{\{\Delta_i N^{(m)} \neq 0\}} \end{aligned}$$

has expectation bounded by $K \frac{h \ln \frac{1}{h}}{\theta_2} = \varepsilon^{\alpha_2} \ln \frac{1}{h} \rightarrow 0$, thus (5.22) follows.

1b) To prove (5.25) we show that

$$\frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \left(I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}, |\Delta_i X^{(2)}| > \sqrt{r_h}\}} - I_{\{|\Delta_i \tilde{V}^{(1)}| \geq 1, |\Delta_i \tilde{V}^{(2)}| \geq 1\}} \right) \quad (5.27)$$

tends to 0. In fact

$$\begin{aligned} & I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}, |\Delta_i X^{(2)}| > \sqrt{r_h}\}} - I_{\{|\Delta_i \tilde{V}^{(1)}| \geq 1, |\Delta_i \tilde{V}^{(2)}| \geq 1\}} = \quad (5.28) \\ & = I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}, |\Delta_i X^{(2)}| > \sqrt{r_h}, \Delta_i \tilde{V}^{(1)} = 0\} \cup \{|\Delta_i X^{(1)}| > \sqrt{r_h}, |\Delta_i X^{(2)}| > \sqrt{r_h}, \Delta_i \tilde{V}^{(2)} = 0\}} \\ & \quad - I_{\{|\Delta_i \tilde{V}^{(1)}| \geq 1, |\Delta_i \tilde{V}^{(2)}| \geq 1, |\Delta_i X^{(1)}| \leq \sqrt{r_h}\} \cup \{|\Delta_i \tilde{V}^{(1)}| \geq 1, |\Delta_i \tilde{V}^{(2)}| \geq 1, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \leq \end{aligned}$$

$$I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}, \Delta_i \tilde{V}^{(1)} = 0\} \cup \{|\Delta_i X^{(2)}| > \sqrt{r_h}, \Delta_i \tilde{V}^{(2)} = 0\}} + I_{\{\Delta_i \tilde{V}^{(1)} \geq 1, |\Delta_i X^{(1)}| \leq \sqrt{r_h}\} \cup \{\Delta_i \tilde{V}^{(2)} \geq 1, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}$$

On $\{|\Delta_i X^{(\ell)}| > \sqrt{r_h}, \Delta_i \tilde{V}^{(\ell)} = 0\}$ either $\Delta_i J^{(\ell)} \neq 0$ or $\Delta_i J^{(\ell)} = 0$. In this last case, as above (5.21), $|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p)$, thus

$$I_{\{|\Delta_i X^{(\ell)}| > \sqrt{r_h}, \Delta_i \tilde{V}^{(\ell)} = 0\}} = I_{\mathcal{D}_1^\ell} - I_{\mathcal{D}_2^\ell} + I_{\mathcal{D}_3^\ell} \leq \sum_{i=1}^3 I_{\mathcal{D}_i^\ell},$$

where $\mathcal{D}_1^\ell := \{|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(\ell)} = 0, \Delta_i J^{(\ell)} = 0\}$, $\mathcal{D}_2^\ell := \{|\Delta_i X^{(\ell)}| \leq \sqrt{r_h}, \Delta_i \tilde{V}^{(\ell)} = 0, \Delta_i J^{(\ell)} = 0, |\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p)\}$, $\mathcal{D}_3^\ell := \{|\Delta_i X^{(\ell)}| > \sqrt{r_h}, \Delta_i \tilde{V}^{(\ell)} = 0, \Delta_i J^{(\ell)} \neq 0\}$. Further also on $\{\Delta_i \tilde{V}^{(\ell)} \geq 1, |\Delta_i X^{(\ell)}| \leq \sqrt{r_h}\}$, either $\Delta_i J^{(\ell)} \neq 0$ or $\Delta_i J^{(\ell)} = 0$, and in this last case we have $|\Delta_i M^{(\ell)}| = |\Delta_i X^{(\ell)} - \Delta_i D^{(\ell)}| \leq |\Delta_i X^{(\ell)}| + |\Delta_i D^{(\ell)}| \leq \sqrt{r_h}(1 + \sqrt{h \ln \frac{1}{h}} / \sqrt{r_h}) < \sqrt{r_h}(1 + h^\eta)$, with $0 < \eta < 1/2 - u$. Thus

$$I_{\{\Delta_i \tilde{V}^{(1)} \geq 1, |\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} = I_{\mathcal{D}_4^\ell} - I_{\mathcal{D}_5^\ell} + I_{\mathcal{D}_6^\ell} \leq \sum_{i=4}^6 I_{\mathcal{D}_i^\ell},$$

where $\mathcal{D}_4^\ell := \{|\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+h^\eta), \Delta_i \tilde{V}^{(\ell)} \geq 1, \Delta_i J^{(\ell)} = 0\}$, $\mathcal{D}_5^\ell := \{|\Delta_i X^{(\ell)}| > \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+h^\eta), \Delta_i \tilde{V}^{(\ell)} \geq 1, \Delta_i J^{(\ell)} = 0\}$, $\mathcal{D}_6^\ell := \{|\Delta_i X^{(\ell)}| \leq \sqrt{r_h}, \Delta_i \tilde{V}^{(\ell)} \geq 1, \Delta_i J^{(\ell)} \neq 0\}$. We show that

$$\frac{1}{\theta_2} \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \right| I_{\mathcal{D}_k^\ell} \xrightarrow{P} 0 \quad (5.29)$$

for all $k = 1..6$, and $\ell = 1, 2$.

Firstly, as for (5.26), the terms with \mathcal{D}_3^ℓ and \mathcal{D}_6^ℓ are negligible. As for the terms with \mathcal{D}_5^ℓ , since $\{|\Delta_i X^{(\ell)}| > \sqrt{r_h}\}$ and $\Delta_i J^{(\ell)} = 0$, we have $|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p)$, which leads to $\sqrt{r_h}(1-p) < |\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+h^\eta)$. However, by Lemma 6 in [1], we have, with $\phi := \eta \wedge \alpha_{mu} \wedge (1 - \alpha_{mu} - 2\eta) > 0$,

$$P\{(1-p)h^u < |\Delta_i M^{(m)}| \leq h^u(1+h^\eta)\} \leq Kh^{1-\alpha_{mu}u+\phi}.$$

Applying the Hölder inequality with conjugate exponents $s_1, s_2 > 1$ we reach that

$$\begin{aligned} & \frac{1}{\theta_2} \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \right| I_{\mathcal{D}_5^\ell} \leq \\ & K \sum_{i=1}^n h \frac{P^{s_2} \{ \sqrt{r_h}(1-p) < |\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+h^\eta) \}}{\theta_2} \leq Kh^{(1-\alpha_{2u}+\phi)\frac{1}{s_2} - (1-\alpha_{2u})}, \end{aligned}$$

which, for s_2 properly chosen close to 1, tends to 0, since $1 - \alpha_{2u} + \phi > 1 - \alpha_{2u}$.

As for the terms with \mathcal{D}_2^ℓ , we have that on $\{|\Delta_i X^{(\ell)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p)\}$ either $|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1+h^\eta)$, which leads to $\Delta_i J^{(\ell)} \neq 0$ and thus to a negligible term, or $\sqrt{r_h}(1-p) < |\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+h^\eta)$, which also leads, by the same reasoning as just above, to a negligible term.

Finally, using again the negligibility of $\frac{\sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \right| I_{\{\Delta_i J^{(\ell)} \neq 0\}}}{\theta_2}$, regarding the terms with \mathcal{D}_1^ℓ and \mathcal{D}_4^ℓ we have

$$\begin{aligned} & \sum_{i=1}^n \frac{\prod_{q=1}^2 \int_{t_{i-1}}^{t_i} \sigma_s^{(q)} dW_s^{(q)} I_{\{\Delta_i J^{(\ell)} = 0\}}}{\theta_2} I_{\{|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(\ell)} = 0\} \cup \{|\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+h^\eta), \Delta_i \tilde{V}^{(\ell)} \geq 1\}} \\ & \approx \sum_{i=1}^n \frac{\prod_{q=1}^2 \int_{t_{i-1}}^{t_i} \sigma_s^{(q)} dW_s^{(q)}}{\theta_2} I_{\{|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(\ell)} = 0\} \cup \{\Delta_i \tilde{V}^{(\ell)} \geq 1, |\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+h^\eta)\}} \end{aligned}$$

$$\approx \frac{1}{\theta_2} \sum_{i=1}^n \sigma_{i-1}^{(1)} \Delta_i W^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} I_{\{|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(\ell)} = 0\} \cup \{|\Delta_i \tilde{V}^{(\ell)}| \geq 1, |\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+h^\eta)\}},$$

and writing $E[\sum_{i=1}^n |\eta_i|] = E[\sum_{i=1}^n E_{i-1} |\eta_i|]$, using the independence, the Hölder inequality for $E[|\sigma_{i-1}^{(1)} \Delta_i W^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)}|]$, Lemma 5.7 points 3 and 5, and recalling that $\theta_1 \leq \theta_2$, then $\forall p \in (0, 1)$, $p > h^\eta > \sqrt{h \ln \frac{1}{h}} / \sqrt{r_h}$, $\forall q \in (0, 1-p)$ the expectation of the sum of the absolute values of the terms in the previous display is dominated by

$$K \frac{(\theta_2^{\frac{4}{3}} + \theta_2(q^{-\alpha_2} - 1) + \theta_2(1 - (1+2p)^{-\alpha_2}))}{\theta_2} \rightarrow K(q^{-\alpha_2} - (1+2p)^{-\alpha_2}),$$

as $h \rightarrow 0$. However if we take $p \rightarrow 0$ and $q \rightarrow 1$ we obtain that the last limit is zero, thus (5.29) is verified, and also (5.25) is shown.

Now, by Lemma 5.8 and Lemma 5.10, the lhs of (5.25) has the same limit as

$$\frac{1}{\theta_2} \sum_{i=1}^n \sigma_{i-1}^{(1)} \Delta_i W^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} I_{\{|\Delta_i \tilde{V}^{(1)}| \geq 1, |\Delta_i \tilde{V}^{(2)}| \geq 1\}} \approx \frac{\theta_1}{\theta_2} (1-\gamma) \frac{c_1}{\alpha_1} IC,$$

thus step 1) is concluded.

2) Now, by reasoning exactly as for (5.25) and then using Lemma 5.9 we have

$$\begin{aligned} & \frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(m)}| > \sqrt{r_h}\}} \sim \\ & \frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \geq 1\}} \approx \frac{\theta_m}{\theta_2} \frac{c_m}{\alpha_m} IC. \end{aligned}$$

It follows, by (5.21), that

$$\frac{I_{2,4}}{\theta_2} \approx IC \left[\frac{\theta_1}{\theta_2} \frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2} - \frac{\theta_1}{\theta_2} (1-\gamma) \frac{c_1}{\alpha_1} \right]:$$

if $\alpha_1 < \alpha_2$ we deduce that $I_2 \sim I_{2,4} \sim \theta_2$, while if $\alpha_1 = \alpha_2 = \alpha$, then $I_{2,4}/\theta_2 \approx c_1/\alpha + c_2/\alpha - (1-\gamma)c_1/\alpha = (c_2 + \gamma c_1)/\alpha$, which is always non zero because $\gamma \geq 0$ and $c_m > 0$, so again $I_2 \sim I_{2,4} \sim \theta_2$.

Term I_3 . We now show that I_3 in (5.11) is negligible wrt \sqrt{h} . Here we adjust to the bivariate case the proof given in [5] for the univariate case. I_3/\sqrt{h} is the sum of two terms of type $\frac{1}{\sqrt{h}} \sum_{i=1}^n \Delta_i Y^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}$ with $(m, \ell) \in \{(1, 2), (2, 1)\}$, that we can treat at the same time. The last expression equals

$$\sum_{i=1}^n \frac{\Delta_i D^{(m)} \Delta_i M^{(\ell)}}{\sqrt{h}} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} + \sum_{i=1}^n \frac{\Delta_i J^{(m)} \Delta_i M^{(\ell)}}{\sqrt{h}} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}. \quad (5.30)$$

As for the second term, as already commented just after the definition of $I_{2,1}$, on $\{|\Delta_i X^{(m)}| \leq \sqrt{r_h}, \Delta_i J^{(m)} \neq 0\}$ we have $\{|\Delta_i M^{(m)}| > \sqrt{r_h}\}$, thus, by Lemma 5.7 point 1,

$$P \left\{ \frac{1}{\sqrt{h}} \sum_{i=1}^n \Delta_i J^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \neq 0 \right\} \leq \sum_{i=1}^n \{ \Delta_i J^{(m)} \neq 0, |\Delta_i M^{(m)}| > \sqrt{r_h} \},$$

which tends to 0, thus the second term of (5.30) tends to 0 in probability.

As for the first term, on $\{|\Delta_i X^{(\ell)}| \leq \sqrt{r_h}\}$ we have $|\Delta_i X^{(\ell)}| > |\Delta_i Z^{(\ell)}| - |\Delta_i D^{(\ell)}|$ then $|\Delta_i Z^{(\ell)}| < |\Delta_i X^{(\ell)}| + |\Delta_i D^{(\ell)}| \leq \sqrt{r_h} + \sqrt{h \ln \frac{1}{h}} \leq 2\sqrt{r_h}$, thus the first term is dominated by

$$\frac{1}{\sqrt{h}} \sum_{i=1}^n \Delta_i D^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i Z^{(\ell)}| \leq 2\sqrt{r_h}\}}:$$

since, similarly as above, the terms where $\Delta_i J^{(\ell)} \neq 0$ are negligible, and $\{|\Delta_i Z^{(\ell)}| \leq 2\sqrt{r_h}, \Delta_i J^{(\ell)} = 0\} = \{|\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}, \Delta_i J^{(\ell)} = 0\}$, we are left with $\frac{1}{\sqrt{h}} \sum_{i=1}^n \Delta_i D^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} \cdot I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}, \Delta_i J^{(\ell)} = 0\}}$. However again the last sum with $\{\Delta_i J^{(\ell)} \neq 0\}$ in place of $\{\Delta_i J^{(\ell)} = 0\}$ is negligible, because on $\{|\Delta_i X^{(\ell)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}, \Delta_i J^{(\ell)} \neq 0\}$ we still have $|\Delta_i M^{(\ell)}| > \sqrt{r_h}$. So we remain with

$$\frac{1}{\sqrt{h}} \sum_{i=1}^n \Delta_i D^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}\}}. \quad (5.31)$$

Now, by Lemma 3.1 in [5] we know that on $|\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}$ we have $\Delta_i M^{(\ell)} = \Delta_i M^{(\ell)h} - h \int_{2v_h}^1 x v^{(\ell)}(dx)$, where $\Delta_i M^{(\ell)h} = \int_{t_{i-1}}^{t_i} \int_{0 < x \leq 2v_h} x \tilde{\mu}^\ell(dx, ds)$, and v_h is a given sequence satisfying $0 < v_h \leq r_h^{1/4}$. As a consequence, exactly as in (43) of [5], the component

$$\frac{1}{\sqrt{h}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} a_s^{(m)} ds \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}\}}$$

of (5.31) tends to zero in probability. Now we show the negligibility of

$$\frac{1}{\sqrt{h}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \Delta_i M^{(\ell)h} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}\}} :$$

in fact, by the independence of $W^{(m)}$ on $\tilde{\mu}^{(\ell)}$ also $[\int_0^{\cdot} \sigma_s^{(m)} dW_s^{(m)}, M^{(\ell)h}] \equiv 0$, and the squared norm $\|\cdot\|_2^2$ of the last display is dominated by

$$\begin{aligned} \frac{1}{h} E \left[\left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \Delta_i M^{(\ell)h} \right)^2 \right] &= \frac{1}{h} \sum_{i=1}^n E \left[\left(\int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \right)^2 \left(\Delta_i M^{(\ell)h} \right)^2 \right] \\ &\leq \frac{K}{h} n \cdot h \ln \left(\frac{1}{h} \right) \cdot h \int_0^{r_h^{1/4}} x^2 v^{(\ell)}(dx) \leq K r_h^{\frac{2-\alpha_\ell}{4}} \log \frac{1}{h} \rightarrow 0. \end{aligned}$$

Finally we show the negligibility also of

$$\frac{1}{\sqrt{h}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} h \int_{2v_h}^1 x v^{(\ell)}(dx) I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}\}} :$$

in fact recall that $\int_{2v_h}^1 x v^{(\ell)}(dx) = c_{A_\ell} \left[(1 - (2v_h)^{1-\alpha_\ell}) I_{\alpha_\ell \neq 1} + \ln \frac{1}{2v_h} I_{\alpha_\ell = 1} \right]$ is positive for all the values of $\alpha_\ell \in (0, 2)$, so the norm $\|\cdot\|_1$ of the last display is dominated by

$$\sqrt{h} c_{A_\ell} \left[(1 - (2v_h)^{1-\alpha_\ell}) I_{\alpha_\ell \neq 1} + \ln \frac{1}{2v_h} I_{\alpha_\ell = 1} \right] E \left[\left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \right| \right] \quad (5.32)$$

and noting that if $i \neq j$ then $E \left[\int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \int_{t_{j-1}}^{t_j} \sigma_s^{(m)} dW_s^{(m)} \right] = E \left[\int \sigma_s^{(m)} I_{s \in [t_{i-1}, t_i]} \sigma_s^{(m)} \cdot I_{s \in [t_{j-1}, t_j]} ds \right] = 0$, and that

$$E \left[\left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \right| \right] \leq \left\| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \right\|_2 = \sqrt{E \left[\sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \right)^2 \right]} = O(1),$$

it follows that (5.32) is dominated by $K \sqrt{h} \left[1 - (2v_h)^{1-\alpha_\ell} I_{\alpha_\ell \neq 1} + \ln \frac{1}{2v_h} I_{\alpha_\ell = 1} \right] \rightarrow 0$.

Term I_4 . We now deal with I_4 of (5.11). We have

$$I_4 = \sum_{i=1}^n \Delta_i M^{(1)} \Delta_i M^{(2)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}$$

$$\begin{aligned}
&= \sum_{i=1}^n \Delta_i M^{(1)} \Delta_i M^{(2)} \left[I_{\{\Delta_i \tilde{N}^{(1)}=0, \Delta_i \tilde{N}^{(2)}=0\}} + I_{\{\Delta_i \tilde{N}^{(1)}=0, \Delta_i \tilde{N}^{(2)}=0\}^c} \right] I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \\
&= \sum_{i=1}^n \Delta_i M^{(1)} \Delta_i M^{(2)} \left[I_{\{\Delta_i \tilde{N}^{(1)}=0, \Delta_i \tilde{N}^{(2)}=0\}} - I_{\{\Delta_i \tilde{N}^{(1)}=0, \Delta_i \tilde{N}^{(2)}=0\} \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}^c} \right. \\
&\quad \left. + I_{\{\Delta_i \tilde{N}^{(1)}=0, \Delta_i \tilde{N}^{(2)}=0\}^c \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \right].
\end{aligned}$$

However, where both $\Delta_i \tilde{N}^{(1)} = 0, \Delta_i \tilde{N}^{(2)} = 0$, we have $\Delta_i M^{(1)} \Delta_i M^{(2)} = \xi_i$, thus $I_4 = \sum_{k=1}^4 I_{4,k}$, where $I_{4,1} = \sum_{i=1}^n \xi_i$, $I_{4,2} = -\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde{N}^{(1)}=0, \Delta_i \tilde{N}^{(2)}=0\}^c}$,

$$I_{4,3} = -\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde{N}^{(1)}=0, \Delta_i \tilde{N}^{(2)}=0\} \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}^c},$$

$$I_{4,4} = \sum_{i=1}^n \Delta_i M^{(1)} \Delta_i M^{(2)} I_{\{\Delta_i \tilde{N}^{(1)}=0, \Delta_i \tilde{N}^{(2)}=0\}^c \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}.$$

We are going to show that the terms $I_{4,2}, I_{4,4}$ are negligible wrt θ_2 , while $I_{4,3}$ is negligible either wrt θ_2 or wrt $\sum_{i=1}^n \xi_i$, depending on the parameters values. As for $I_{4,2}$, using again that $I_{A \cup B} = I_A + I_B - I_{A \cap B}$, it is sufficient to show that both $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde{N}^{(\ell)} \geq 1\}} \ll \theta_2$, for $\ell = 1, 2$ and $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1\}} \ll \theta_2$. Using the independence of ξ_i on $\Delta_i \tilde{N}^{(\ell)}$, we reach that

$$E_{i-1}[\xi_i I_{\{\Delta_i \tilde{N}^{(\ell)} \geq 1\}}] = KE[\xi_i] \theta_\ell, \quad E_{i-1}[\xi_i^2 I_{\{\Delta_i \tilde{N}^{(\ell)} \geq 1\}}] \leq KE[\xi_i^2] \theta_\ell.$$

Thus, if, for any ℓ , we call

$$\sum_{i=1}^n \frac{\xi_i I_{\{\Delta_i \tilde{N}^{(\ell)} \geq 1\}}}{\theta_2} := \sum_{i=1}^n \chi_i,$$

we have that $\forall t \geq 0, \sum_{i \leq t} E_{i-1}[\chi_i] \leq K \sum_{i \leq t} E[\xi_i] \leq Kn E[\xi_1]$, which, by looking at Theorem 5.3, tends to zero in all the cases $\gamma \in [0, 1]$. Further, $\sum_{i \leq t} E_{i-1}[\chi_i]$ is positive for all t , and increasing in t , thus the convergence is also ucp. Moreover $\forall t \geq 0, \sum_{i \leq t} E_{i-1}[\chi_i^2] \leq n E[\xi_1^2] / \theta_2 \leq n \text{Var}(\xi_1) / (K \theta_2)$, with $K \in (0, 1)$, having used that, since ξ_1 is not constant, then $E^2[\xi_i] < E[\xi_i^2]$. Using now for $n \text{Var}(\xi_1)$ the expressions at the denominators of (5.1), (5.2), (5.3) it is verified that under our assumptions $n \text{Var}(\xi_1) / \theta_2 \rightarrow 0$ in all the cases $\gamma \in [0, 1]$. We remark that for the case $\gamma \in (0, 1)$ and $\alpha_1 \geq x_*$ condition $u > 1 / [2(1 + \alpha_2 / \alpha_1)]$ is needed, however it is implied by our assumption (3.4). It follows that $\sum_{i=1}^n \chi_i \xrightarrow{ucp} 0$, that is $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde{N}^{(\ell)} \geq 1\}} \ll \theta_2$.

If we now call $P\{\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1\} := \theta_{1,2} \leq \theta_2$, and

$$\sum_{i=1}^n \chi_i := \sum_{i=1}^n \frac{\xi_i I_{\{\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1\}}}{\theta_2},$$

we have $\sum_{i \leq t} E_{i-1}[\chi_i] = \left[\frac{t}{h} \right] E[\xi_1] \frac{\theta_{1,2}}{\theta_2} \leq \left[\frac{t}{h} \right] E[\xi_1] \xrightarrow{ucp} 0$, and $\sum_{i \leq t} E_{i-1}[\chi_i^2] \leq Kn \cdot \text{Var}(\xi_1) / \theta_2 \rightarrow 0$, so again $\sum_{i=1}^n \chi_i \xrightarrow{ucp} 0$ and $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1\}} \ll \theta_2$.

We now show that $I_{4,3}$ is negligible either wrt θ_2 or wrt $\sum_{i=1}^n \xi_i$. Each term of the sum is counted only if both $\Delta_i \tilde{N}^{(j)} = 0, j = 1, 2$ but $|\Delta_i X^{(\ell)}| > \sqrt{r_h}$ for at least one index ℓ . Note that if $\Delta_i \tilde{N}^{(\ell)} = 0$ then $\Delta_i J^{(\ell)} = 0$ and $\Delta_i \tilde{V}^{(\ell)} = 0$. However, as commented for $I_{2,3}$, we have $\{|\Delta_i X^{(\ell)}| > \sqrt{r_h}, \Delta_i J^{(\ell)} = 0\} \subset \{|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p)\}$, and $P\{\Delta_i \tilde{V}^{(\ell)} = 0, |\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p)\} \leq P\{\Delta_i \tilde{V}^{(\ell)} = 0, |\Delta_i M^{(\ell)}| > \sqrt{r_h}\} + P\{|\Delta_i M^{(\ell)}| \in (\sqrt{r_h}(1-p), \sqrt{r_h})\} \leq P\{\Delta_i \tilde{V}^{(\ell)} = 0, |\Delta_i M^{(\ell)}| > \sqrt{r_h}\} + P\{|\Delta_i M^{(\ell)}| \in (\sqrt{r_h}(1-p), \sqrt{r_h}(1+h^\eta))\} \leq K \theta_2^{4/3} + K \theta_2 h^\phi \approx K \theta_2 h^\phi$,

with $\eta > 0$, and having used the result in Lemma 5.7, point 3 with 1 in place of $1 - p$, which holds for any $q < 1$, and Lemma 6 in [1]. It follows that

$$E[|I_{4,3}|] \leq K \sum_{i=1}^n \|\xi_i\|_2 \sqrt{\theta_2 h^\phi} \leq K \sqrt{n \text{Var}(\xi_i)} \sqrt{n} \theta_2^{\frac{1}{2}} h^{\frac{\phi}{2}} = K \sqrt{n \text{Var}(\xi_i)} \varepsilon^{-\frac{\alpha_2}{2}} h^{\frac{\phi}{2}} := a_n :$$

looking at (5.1), (5.2), (5.3), depending on the different choices of γ , α_1 , α_2 we have the following: for $\gamma \in (0, 1)$ and $\alpha_1 \leq x_*$, then $a_n \ll \theta_2$ iff $u > 1/(4 - \alpha_1)$, however this last condition is implied by (3.4); for $\gamma \in (0, 1)$ and $\alpha_1 > x_*$ then, using also Proposition 5.4, $a_n \ll nE[\xi_1] \approx \sum_{i=1}^n \xi_i$; if $\gamma = 1$ then $a_n \ll \theta_2$ iff $u > 1/(4 - \alpha_1)$; if $\gamma = 0$ and either $\alpha_1 < \alpha_2 u$ or $(\alpha_1 = \alpha_2 u, \alpha_2 = 1)$ then $a_n \ll \theta_2$; if $\gamma = 0$ and either $(\alpha_1 = \alpha_2 u, \alpha_2 > 1)$ or $\alpha_1 > \alpha_2 u$ then $a_n \ll nE[\xi_1] \approx \sum_{i=1}^n \xi_i$.

Finally we show that $I_{4,4}$ is negligible wrt to θ_2 : we check this when the summands satisfy the three cases $(\Delta_i \tilde{N}^{(2)} = 0, \Delta_i \tilde{N}^{(1)} \geq 1)$; $(\Delta_i \tilde{N}^{(2)} \geq 1, \Delta_i \tilde{N}^{(1)} = 0)$; $(\Delta_i \tilde{N}^{(2)} \geq 1, \Delta_i \tilde{N}^{(1)} \geq 1)$, which are dealt with similarly. For the indices i such that $\Delta_i \tilde{N}^{(2)} = 0$ and $\Delta_i \tilde{N}^{(1)} \geq 1$, then the terms with $\Delta_i J^{(1)} \neq 0$, as previously, do not contribute to $I_{4,4}/\theta_2$, since $|\Delta_i X^{(1)}| \leq \sqrt{r_h}$, and thus $|\Delta_i M^{(1)}| > \sqrt{r_h}(1 - p)$. We then remain with the terms where $\Delta_i J^{(1)} = 0$ and, since $|\Delta_i X^{(1)}| \leq \sqrt{r_h}$, we have $|\Delta_i M^{(1)}| \leq \sqrt{r_h}(1 + p)$. On the other hand on $\{\Delta_i \tilde{N}^{(2)} = 0\}$ we have $\Delta_i J^{(2)} = 0$, and thus also $|\Delta_i M^{(2)}| \leq \sqrt{r_h}(1 + p)$. It follows that, as for (5.31), $\Delta_i M^{(\ell)} = \Delta_i M^{(\ell)h} - h \int_{2v_h}^1 x v^{(\ell)}(dx)$. Now

$$\begin{aligned} \frac{1}{\theta_2} E \left[\left| \sum_{i=1}^n \Delta_i M^{(1)h} \Delta_i M^{(2)h} I_{\{\Delta_i \tilde{N}^{(1)} \geq 1\}} \right| \right] &\leq \frac{1}{\theta_2} \left\| \sum_{i=1}^n \Delta_i M^{(1)h} \Delta_i M^{(2)h} I_{\{\Delta_i \tilde{N}^{(1)} \geq 1\}} \right\|_2 = \\ \frac{1}{\theta_2} \sqrt{\sum_{i=1}^n E[(\Delta_i M^{(1)h} \Delta_i M^{(2)h})^2] P\{\Delta_i \tilde{N}^{(1)} \geq 1\}} &\leq \frac{1}{\theta_2} \sqrt{nh^2 r_h^{1 - \frac{\alpha_1}{4} - \frac{\alpha_2}{4}} \theta_1} \leq \sqrt{r_h^{1 - \frac{\alpha_1}{4} + \frac{3}{4} \alpha_2}} \end{aligned}$$

which tends to 0, having used: the independence among the increments and the independence of the $\Delta_i M^{(\ell)h}$ with $\tilde{N}^{(1)}$, and the Hölder and the Burkholder-Davis-Gundy inequalities to reach that $E[(\Delta_i M^{(1)h})^2 (\Delta_i M^{(2)h})^2] \leq \int_{i-1}^i \int_{x \leq v_h} x_1^2 v^{(1)}(dx_1) \cdot$

$\int_{i-1}^i \int_{x \leq v_h} x_2^2 v^{(2)}(dx_2) = h^2 r_h^{1 - \frac{\alpha_1}{4} - \frac{\alpha_2}{4}}$. Further

$$\begin{aligned} \frac{E \left[\left| \sum_{i=1}^n h^2 \int_{2v_h}^1 x v^{(2)}(dx) \int_{2v_h}^1 x v^{(2)}(dx) I_{\{\Delta_i \tilde{N}^{(1)} \geq 1\}} \right| \right]}{\theta_2} &\leq \\ Kh \prod_{\ell=1,2} \left[1 - (2v_h)^{1 - \alpha_\ell} |I_{\alpha_\ell \neq 1} + I_{\alpha_\ell = 1} \ln \frac{1}{2v_h}| \right] & \end{aligned}$$

which in the worst case of $\alpha_1, \alpha_2 > 1$ is dominated by $Kh v_h^{1 - \alpha_1} v_h^{1 - \alpha_2} \leq h^{1 + \frac{\alpha_1}{4}(1 - \alpha_1)} \cdot h^{\frac{\alpha_2}{4}(1 - \alpha_2)} \rightarrow 0$. It follows that $\frac{1}{\theta_2} E \left[\left| \sum_{i=1}^n \Delta_i M^{(1)} \Delta_i M^{(2)} I_{\{\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} = 0, |\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} \right| \right] \rightarrow 0$, and thus

$$E \left[\frac{1}{\theta_2} \left| \sum_{i=1}^n \Delta_i M^{(1)} \Delta_i M^{(2)} I_{\{\Delta_i \tilde{N}^{(2)} = 0, \Delta_i \tilde{N}^{(1)} \geq 1\} \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \right| \right] \rightarrow 0.$$

For the indices i such that $\Delta_i \tilde{N}^{(2)} \geq 1$ and $\Delta_i \tilde{N}^{(1)} = 0$, we reason similarly as above and obtain that

$$E \left[\frac{1}{\theta_2} \sum_{i=1}^n |\Delta_i M^{(1)} \Delta_i M^{(2)}| I_{\{\Delta_i \tilde{N}^{(2)} \geq 1, \Delta_i \tilde{N}^{(1)} = 0\} \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \right] \rightarrow 0.$$

For the indices i such that $\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1$, again the terms with one $\Delta_i J^{(\ell)} \neq 0$, are negligible and we remain with the terms where both $\Delta_i J^{(\ell)} = 0$, thus we reach that both $|\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1 + p)$ and, as above,

$$E \left[\frac{1}{\theta_2} \sum_{i=1}^n |\Delta_i M^{(1)} \Delta_i M^{(2)}| I_{\{\Delta_i \tilde{N}^{(2)} \geq 1, \Delta_i \tilde{N}^{(1)} \geq 1\} \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \right] \rightarrow 0,$$

and the proof of the negligibility of $I_{4,4}$ wrt θ_2 is completed.

We thus obtained that $\hat{I}C - IC \sim \sqrt{h} + \sum_{i=1}^n \xi_i + \theta_2$. Now we are going to make this more explicit. In (3.2) we compared \sqrt{h} with $\sum_{i=1}^n \xi_i$. As for θ_2 versus \sqrt{h} we have that:

$$\theta_2 \ll \sqrt{h} \text{ if } \alpha_2 < \frac{1}{2u}; \theta_2 \sim \sqrt{h} \text{ if } \alpha_2 = \frac{1}{2u}; \theta_2 \gg \sqrt{h} \text{ if } \alpha_2 > \frac{1}{2u}.$$

Comparing now θ_2 with $\sum_{i=1}^n \xi_i$, we reach that

$$\begin{aligned} &\text{when } \gamma = 1, \theta_2 \gg \sum_{i=1}^n \xi_i, \text{ for } \alpha_2 = \alpha_1 = 1: \text{ if } u > \frac{1}{4}; \text{ for } (\alpha_1, \alpha_2) \neq (1, 1): \forall u \in (0, \frac{1}{2}) \\ &\text{when } \gamma \in [0, 1), \theta_2 \gg \sum_{i=1}^n \xi_i, \text{ for } \alpha_1 \leq \alpha_2 u: \text{ any } u \in (0, \frac{1}{2}); \\ &\quad \text{for } \alpha_2 > \alpha_1 > \alpha_2 u: \text{ iff } u > \frac{1}{1 + \frac{\alpha_2}{\alpha_1}} \\ &\text{when } \gamma \in [0, 1), \theta_2 \ll \sum_{i=1}^n \xi_i, \text{ for } \alpha_1 = \alpha_2: \text{ any } u \in (0, \frac{1}{2}). \end{aligned}$$

It follows that

$$\begin{aligned} \hat{I}C - IC &\sim I_{\alpha_2 \geq \frac{1}{2u}} \left(\theta_2 \left[I_{\{\gamma=1\}} + I_{\{\gamma \in [0,1), \alpha_1 \leq \alpha_2 u\}} + I_{\{\gamma \in [0,1), \alpha_2 > \alpha_1 > \alpha_2 u, u \geq \frac{\alpha_1}{\alpha_1 + \alpha_2}\}} \right] \right. \\ &\quad \left. + \sum_{i=1}^n \xi_i \left[I_{\{\gamma \in [0,1), \alpha_2 > \alpha_1 > \alpha_2 u, u < \frac{\alpha_1}{\alpha_1 + \alpha_2}\}} + I_{\{\gamma \in [0,1), \alpha_2 = \alpha_1 > \alpha_2 u\}} \right] \right) \\ &\quad + I_{\alpha_2 < \frac{1}{2u}} \left(\sqrt{h} \left[I_{\{\gamma=1, \alpha_1 < \alpha_1^{**}\}} + I_{\{\gamma \in [0,1), \alpha_1 \leq \alpha_1^*\}} \right] + \sum_{i=1}^n \xi_i \left[I_{\{\gamma=1, \alpha_1 \geq \alpha_1^{**}\}} + I_{\{\gamma \in [0,1), \alpha_1 > \alpha_1^*\}} \right] \right). \end{aligned}$$

However: note that $u < \frac{\alpha_1}{\alpha_1 + \alpha_2}$ implies $\alpha_1 > \alpha_2 u$; if $\alpha_1 = \alpha_2$ then on one hand $\alpha_1 > \alpha_2 u$, since $\alpha_2 > \alpha_2 u$, on the other hand $u < \frac{\alpha_1}{\alpha_1 + \alpha_2}$; $\alpha_2 < 1/(2u) \Rightarrow \alpha_1 < \alpha_1^{**}$; $u \geq \frac{\alpha_1}{\alpha_1 + \alpha_2}$ implies $\alpha_2 > \alpha_1$. Thus the above display simplifies and (3.7, 3.8, 3.9) follow. \square

5.3 Proofs of the Remarks to the main results

Statement (3.2). Defined

$$\alpha_1^* := \frac{\alpha_2 u}{\alpha_2 u - u + 1/2} \in (2u, 1), \quad \alpha_1^{**} := \frac{1 + 2u(2 - \alpha_2)}{2u} > \frac{1}{2u} > 1,$$

we have that:

$$\begin{cases} \text{if } \gamma \in [0, 1): & \sum_i \xi_i \ll \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^*; \\ \text{if } \gamma = 1: & \sum_i \xi_i \ll \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^{**}. \end{cases}$$

Proof. We heavily use Proposition 5.4. In the case $\gamma \in [0, 1)$ we have $\sum_{i=1}^n \xi_i \sim nE[\xi_1]$. Using (5.5) we have that on $\{\alpha_1 \leq \alpha_2 u, \alpha_2 = 1\} \cup \{\alpha_1 \leq \alpha_2 u, \alpha_2 > 1\}$ both $\alpha_1 < \alpha_1^*$ and $nE[\xi_1]/\sqrt{h} \rightarrow 0$. On $\{\alpha_1 > \alpha_2 u\}$ then $nE[\xi_1]/\sqrt{h} \rightarrow 0$ iff $\alpha_1 < \alpha_1^*$.

In the case $\gamma = 1$ then on $\{\alpha_1 < 1 \leq \alpha_2\} \cup \{\alpha_1 = 1 < \alpha_2\}$ we have $\alpha_1 < \alpha_1^{**}$. If $u > 1/(2 + \alpha_2 - \alpha_1)$ then $\sum_{i=1}^n \xi_i \sim nE[\xi_1]$ and $nE[\xi_1]/\sqrt{h} \rightarrow 0$; if $u \leq 1/(2 + \alpha_2 - \alpha_1)$ then $\sum_{i=1}^n \xi_i/\sqrt{h} \sim \sqrt{n\text{Var}(\xi_1)}/\sqrt{h} \rightarrow 0$. On $\{1 < \alpha_1 \leq \alpha_2\}$: if $u > \frac{1}{\alpha_1 + \alpha_2}$ then $\sum_{i=1}^n \xi_i \sim nE[\xi_1] \ll \sqrt{h}$ iff $\alpha_1 < \alpha_1^{**}$. On the other hand $u \leq \frac{1}{\alpha_1 + \alpha_2}$ is equivalent to $\alpha_1 \leq 1/u - \alpha_2$, which is less than α_1^{**} , if $u > 1/4$, and if $u \leq \frac{1}{\alpha_1 + \alpha_2}$ then $\sum_{i=1}^n \xi_i \sim \sqrt{n\text{Var}(\xi_1)} \ll \sqrt{h}$. Finally, when $\alpha_1 = \alpha_2 = 1$ then $\sum_{i=1}^n \xi_i/\sqrt{h} \sim \sqrt{n\text{Var}(\xi_1)}/\sqrt{h} = \sqrt{h\varepsilon^2}/\sqrt{h} \rightarrow 0$, and $\alpha_1 = 1 < \alpha_1^{**}$. \square

Remark x) to Theorem 3.2. For fixed h , the convergence speed is a function $s(\gamma, \alpha_1, \alpha_2, u)$ of our parameters. Such a function is smooth most of the times, however it has some singularities (as is evident in Figure 1).

In fact when $\alpha_2 \geq \alpha_1(1/u - 1)$ and $\gamma \in [0, 1)$: if $\alpha_1 \neq \alpha_2$ but the two indices are close and above $1/(2u)$, then $s = h\varepsilon^{-\alpha_2} = h^{1-\alpha_2u}$ while at $\alpha_1 = \alpha_2$ the function s jumps at $\varepsilon^{2-\alpha_2} = h^{2u-\alpha_2u}$. The jump would disappear if it was $u = 1/2$.

On the contrary, we have smoothness at $\alpha_1 = \alpha_1^*$ if $\alpha_2 < 1/(2u)$: in fact if α_1 is much less than α_2 (case $\alpha_1 \leq \alpha_1^* < 1 \leq \alpha_2 < \frac{1}{2u}$) then $s = h^{1/2}$; for α_1 at α_1^* we have $s = \sqrt{h} = \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$, and with $\alpha_1 \in (\alpha_1^*, \alpha_2]$ still is $s = \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$. When $\gamma = 1$ we have smoothness at $\alpha_2 = 1/(2u)$: in fact when $\alpha_2 = 1/(2u)$ we have $\sqrt{h} = h\varepsilon^{-\alpha_2}$. \square

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