Nonstationary Cross-Validation
(*preliminary first draft*)

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Abstract

Cross-validation is the most common data-driven procedure for choosing the smoothing parameter in nonparametric regression. For the case of iid or strong mixing data, it is known that the bandwidth chosen by cross-validation is optimal with respect to the mean integrated squared error. However, the properties of cross-validated bandwidths in the context of nonstationary regressions have not been established yet. This is the subject of the current paper. Assuming β-recurrent (stationary or nonstationary) Markov chains, we show that the cross-validated bandwidth is optimal with respect to the average squared error. The accuracy of the method is analyzed via a Monte Carlo study. Its practical usefulness (in a highly-persistent, possibly nonstationary environment) is illustrated by virtue of an application to nonlinear predictive systems for market returns.

Keywords: Bandwidth Selection, Cross-Validation, Nonstationary Autoregression, Predictive Regression, Recurrence.

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1 Introduction

The vast literature on unit roots and cointegration has largely focused on linear models. While it is well-known that the limiting behavior of partial sums, and affine functionals of them, can be approximated by Gaussian processes, much less is known about the asymptotic behavior of functional estimators of nonstationary time series. Nonparametric regression with nonstationary discrete-time processes has, in fact, been receiving attention only in recent years.

Several financial time-series central to asset pricing, like the short-term rate, the dividend-to-price ratio or, more generally, any financial ratio whose denominator depends on the market's price level, are highly persistent and often depend on their past in a non-linear fashion. The joint presence of *possibly* nonstationary behaviour and nonlinearities of unknown form provides econometric content to nonparametric regressions with *possibly* nonstationary time series. The word "possibly" is an important qualifier. While strict nonstationarity is, in many cases, economically unpalatable, allowing for stationary as well as for nonstationary behaviour is a satisfactory (theoretical and empirical) way to accommodate a broad array of levels, especially including very high levels, of persistence in one unified framework.

Coherently with this observation, the literature on nonparametric autoregression has focused on β-recurrent (stationary for $\beta = 1$ and nonstationary for $\beta < 1$) Markov chains. In this framework, the number of regenerations of a recurrent chain has heavily been used to derive the limiting behavior of the number of visits around a given point and related estimators (see, e.g., Karlsen and Tjøstheim (2001), Moloche (2001), Gao et al. (2011)). Schienle (2010) considers the case of many regressors. Guerre (2004) derives convergence rates for a somewhat more general class of chains.

This succesful literature has established consistency and asymptotic mixed normality for local constant and local linear kernel estimators of nonstationary autoregressions and nonstationary cointegration. However, it has been largely silent about practical guidance on bandwidth selection.

The two most common approaches for bandwidth selection in functional problems are (often ad-hoc) "plug-in" methods and cross-validation. In the β-recurrent case, the bandwidth conditions for consistency and asymptotic mixed normality depend on the generally unknown parameter β. In light of the slow (i.e., logarithmic) convergence rate of

existing estimates of β (Karlsen and Tjøstheim (2001)), plug-in methods appear even more ad-hoc than in the more classical stationary environment. On the other hand, bandwidths chosen by cross-validation have been used broadly in empirical work. However, their properties have not been established yet in a (possibly) nonstationary context. This is the subject of the current paper.

For the case of identically distributed and independent observations, it has been shown that "leave-one-out" cross-validated bandwidths are optimal with respect to the Mean Integrated Squared Error (MISE), see, e.g., Härdle and Marron (1985) and Härdle (1986). More explicitly, the ratio of the cross-validated bandwidth and the infeasible bandwidth that minimizes the estimator's MISE converges in probability to one. Optimality with respect to MISE has also been established in the case of strong mixing observations (Härdle and Vieu (1992), Kim and Cox (1996), and Xia and Li (2002)).

This paper shows that, for β-recurrent Markov chains, the bandwidth chosen via "leave-one-out" cross-validation is asymptotically optimal with respect to the Average Squared Error (ASE). In other words, the ratio of the cross-validated bandwidth and the infeasible bandwidth that minimizes the ASE converges in probability to one. Differently from stationary environments, the asymptotic equivalence between the *random* cross-validated bandwidth sequence and a *deterministic* sequence minimizing the estimator's MISE does not hold. This is easily explained. In the β-recurrent case the effective sample size is random, the asymptotic variance of conditional mean estimates is also random, so is the MISE. Of course, should $\beta = 1$, the ASE would converge uniformly to a deterministic limit, the MISE, and our findings would again deliver equivalence between an adaptive random sequence and a deterministic sequence, optimal with respect to the MISE, as a subcase of a more general result.

After studying the case of bandwidth selection both for the conditional mean function and for the conditional variance function, we apply the methods to a mean-variance problem relying on (possibly) nonlinear predictability. The high persistence of successful predictive variables, like the dividend-to-price ratio, justifies our emphasis on bandwidth selection procedures capable to handle such a persistence.

The remainder of the paper is organized as follows. Section 2 defines the framework. Section 3 reports our main result establishing optimality of cross-validation with respect to the ASE. Here, we focus on the conditional mean function. Section 4 turns to the conditional variance function. Section 5 reports the findings of a Monte Carlo study in

which we analyze the variance and bias of nonparametric kernel estimators based on cross-validated bandwidths. In Section 6 we discuss an empirical illustration in which cross-validated bandwidths are used in predictive systems for market returns and a nonlinear (mean-variance) portfolio allocation problem. All proofs are gathered in the appendix.

2 The Model

Intuitively, one can estimate conditional moments, evaluated at a given point, only if this specific point is visited infinitely often as the sample size grows. For this reason, it is natural to focus attention on irreducible recurrent chains, i.e., chains satisfying the property that, at any point in time, the neighborhood of each point has a strictly positive probability of being visited and, eventually, will be visited an infinite number of times over time.

For positive recurrent (ergodic) chains, the expected time between two consecutive visits is finite. Hence, the time spent in the neighborhood of a point grows linearly with the sample size, n say. For null recurrent (nonstationary) chains, the expected time between two consecutive visits is infinite. Therefore, the time spent in the neighborhood of a point grows at a rate, generally unknown, which is slower than n.

Since, up to some mild regularity conditions, positive recurrent chains are strongly mixing, consistency and asymptotic normality follow by, e.g., the work of Robinson (1983). In this case, bandwidth selection may be implemented by virtue of cross-validation, whose optimality properties have been thoroughly established (Härdle and Vieu (1992) and Kim and Cox (1996), for kernel regressions, and Xia and Li (2002), for local linear estimators).

Nonparametric regression with null recurrent chains, however, poses substantial theoretical challenges since the amount of time spent in the neighborhood of a point is not only unknown but also random. In an influential contribution, Karlsen and Tjøstheim (2001) derive consistency and mixed asymptotic normality for conditional moment estimators in the case of recurrent Markov chains.

To clarify ideas, let $\mu(X_{t-1}) = \mathrm{E}\left(X_t|X_{t-1}\right)$ and $\sigma^2(X_{t-1}) = \mathrm{var}\left(X_t|X_{t-1}\right)$, so that X_t can be written as

$$X_t = \mu(X_{t-1}) + \sigma(X_{t-1})u_t. \tag{1}$$

Eq. (1) allows for nonlinearities of unknown form in both the conditional mean and the conditional variance. In the presence of iid shocks, which we assume below, the resulting

time series can also be interpreted as a discretized diffusion process.

Consider the Nadaraya-Watson kernel estimator

$$\widehat{\mu}_{h_n}(x) = \frac{\frac{1}{h_n}\sum_{i=2}^{n} X_i K\left(\frac{X_{i-1}-x}{h_n}\right)}{\frac{1}{h_n}\sum_{i=2}^{n} K\left(\frac{X_{i-1}-x}{h_n}\right)}. \tag{2}$$

where K is a kernel function and h_n a bandwidth parameter. The limiting properties of $\widehat{\mu}_{n,h}(x)$ have been established by Karlsen and Tjøstheim (2001, Theorem 5.4) under Assumption 1 below, which largely corresponds to their Assumption B_0-B_4.

Assumption 1.

1. $\{X_t,\, t \geq 0\}$ is a β-recurrent, ϕ-irreducible Markov chain on a general state space $(\mathbf{E}, \mathcal{E})$ with transition probability P. Let $\beta \in (0, 1]$.

2. The invariant measure π_s has a locally twice continuously differentiable density p_s, which is strictly positive and bounded on every small set.

3. The kernel function K is a bounded density with compact support satisfying $\int uK(u)du = 0$ and $K_2 = \int K^2(u)du < \infty$. The set $\mathcal{N}_x = \{y : K_{h=1}(y - x) \neq 0\}$ is a small set.

4. We have $\lim_{h\downarrow 0} \overline{\lim}_{y\to x} P(y, A_h) = 0$ for all sets $A_h \in \mathcal{E}$ so that $A_h \downarrow \emptyset$ when $h \downarrow 0$.

5. The functions $\mu(x)$ and $\sigma^2(x)$ are locally twice continuously differentiable.

Let Assumption 1 hold. For $x \in \mathcal{C}$, where $\mathcal{C}$ is a compact set, and if $h_n n^{\beta-\varepsilon} \to \infty$ for some $\varepsilon > 0$, then

$$\left(h_n \sum_{i=2}^{n} K\left(\frac{X_{i-1}-x}{h_n}\right)\right)^{1/2} (\widehat{\mu}_{h_n}(x) - \mu(x) - b_{h_n}(x)) \xrightarrow{d} N\left(0, \sigma^2(x)\int K(u)^2 \mathrm{d}u\right), \tag{3}$$

where $b_{h_n}(x)$ denotes a bias term that converges to zero under the additional condition $h_n n^{\beta/5+\varepsilon} \to 0$ (Theorem 5.4 in Karlsen and Tjøstheim (2001)).

The bandwidth rate conditions (and, implicitly, the estimator's convergence rate) depend on the generally unknown degree of recurrence β. Although β can be estimated,

existing estimators only converge at a logarithmic rate and, thus, may not be overly useful in practice (Remark 3.7 in Karlsen and Tjøstheim (2001)).

In essence, one remains with the problem of adaptively choosing the bandwidth. Because β is unknown, in general, and cannot be estimated reliably, "plug-in" methods cannot be implemented effectively. Cross-validation appears as a viable alternative, one which is broadly employed in empirical work. In what follows, we set the stage for studying the properties of cross-validated bandwidths in a (possibly) nonstationary environment, i.e., for $\beta \leq 1$.

Asymptotic mixed normality for β-recurrent chains is shown via split chains, i.e., by splitting the chain into identically and independently distributed components (Nummelin (1984), Chapter 4). We will use split chains in what follows. Assume Assumption 1 holds. Write the denominator in Eq. (2) as

$$\frac{1}{h_n}\sum_{i=2}^{n} K\left(\frac{X_{i-1}-x}{h_n}\right) = U_{0,x,h} + \sum_{k=1}^{T_n} U_{k,x,h} + U_{n,x,h}.$$

We abbreviate by writing $K_{x,h}(y) := K((y-x)/h)/h$ and $U_{k,x,h} := U_k(K_{x,h})$, where

$$U_k(f) := \begin{cases} \sum_{j=1}^{\tau_0} f(X_{j-1}), & k = 0 \\ \sum_{j=\tau_{k-1}+1}^{\tau_k} f(X_{j-1}), & 1 \leq k < n \\ \sum_{j=\tau_{T_n}+1}^{n} f(X_{j-1}), & k = n \end{cases} \tag{4}$$

Let $\mathcal{F} := \{f = hK_{x,h} : x \in \mathcal{C}, 0 < h \leq 1\}$ with envelope F, i.e. $f \leq F$ for all $f \in \mathcal{F}$.

For any h and x, the random variables $U_{k,x,h}$, $k \geq 1$, are independent and identically distributed. The quantity T_n denotes the number of complete regenerations from time 0 to time n and τ_k, $k = 0, \ldots, n$, are the regeneration times. T_n is a random quantity playing the same role as the sample size in stationary problems. Note that $U_{k,x,h}$ and the "effective" sample size T_n are, in general, not independent.

For a compact set $\mathcal{C}$,[1] now define

$$T_n(\mathcal{C}) = \sum_{i=1}^{n} 1\{X_i \in \mathcal{C}\}, \tag{5}$$

[1] In the β-recurrent case, under rather general regularity conditions, compact sets are small sets (see, e.g., Feigin and Tweedie (1985)). A set A is small, if there exists a positive measure λ, positive constant b and an integer $m \geq 1$, such that $P^m \geq 1\{A\}b \oplus \lambda$, where P is the measure governing the chain. For a detailed description of small sets and related properties, see Nummelin (1984) or Section 3 in Karlsen and Tjøstheim (2001)

which represents the random number of visits to $\mathcal{C}$. For some constants $0 < \underline{c}, \overline{c} < \infty$, $\frac{1}{5} < \overline{\eta} < \underline{\eta} < 1$, and $1 - 2\underline{\eta} + \overline{\eta} > 0$, let

$$H_n := \left\{ h \in \mathbb{R} : \ \underline{c}\, T_n(\mathcal{C})^{-\underline{\eta}} \leq h \leq \overline{c}\, T_n(\mathcal{C})^{-\overline{\eta}} \right\} \tag{6}$$

be a set of bandwidths.

The set H_n is the feasible set over which the cross-validated bandwidth will be chosen. This set is random. This is in contrast with the independent, or the strong mixing, case in which it is a function of n. In both of these cases, the time between two consecutive visits to any compact set is finite, and so the number of visits to $\mathcal{C}$ grows at the same rate as the sample size n. In the general β-recurrent case, the "effective" sample size is given by the number of regenerations T_n, however T_n is not observable, while $T_n(\mathcal{C})$ is. Since, from Remark 3.5 in Karlsen and Tjøstheim (2001), T_n and $T_n(\mathcal{C})$ are of the same almost-sure order (defined in Lemma 3.4 of Karlsen and Tjøstheim (2001)), we define the feasible bandwidth set in terms of $T_n(\mathcal{C})$. This definition ensures that any sequence of bandwidths in H_n satisfies the rate restrictions (reported above and below Eq. (3)) sufficient for consistency, asymptotic normality, and zero asymptotic bias of nonparametric conditional mean (and variance) estimators. We note that the conditions governing H_n largely correspond to those of Härdle (1986) which are, however, stated in term of n and are for stationary processes.

Define, now, the "leave-one-out" estimator

$$\widehat{\mu}_{j,h}(X_{j-1}) = \frac{\sum_{i=2, i\neq j}^{n} X_i K_h\left(X_{i-1} - X_{j-1}\right)}{\sum_{i=2, i\neq j}^{n} K_h\left(X_{i-1} - X_{j-1}\right)} \qquad \text{for } j = 2, \ldots, n. \tag{7}$$

The cross-validation (CV) criterion is

$$\mathrm{CV}_n(h) = \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(X_j - \widehat{\mu}_{j,h}(X_{j-1})\right)^2 1\left\{X_{j-1} \in \mathcal{C}\right\}. \tag{8}$$

Hence, the cross-validated bandwidth is

$$\widehat{h}_n = \arg\min_{h \in H_n} CV_n(h), \tag{9}$$

with H_n defined in Eq. (6).

We emphasize that the CV-criterion is computed by averaging nonparametric residuals over a compact set. This is also true in the independent case (Härdle and Marron (1985) or Härdle (1986)). The reason for constraining the CV-criterion over a compact set will become apparent in the next section.

3 Optimal Bandwidth for Conditional Mean Estimates

To state the properties of the CV-bandwidth $\widehat{h}_n$ in Eq. (9) we first introduce some definitions and assumptions.

By a slight abuse of notation, we use the symbol P for probabilities calculated on the probability space of the Markov chain as well as those calculated on the extended probability space of the corresponding split chain (see section 4.4 of Nummelin (1984) for details).

Let E denote expectations with respect to π_s the invariant measure of the Markov chain $\{X_k : k \geq 0\}$, i.e., $\pi_s g = EU_k(g)$ for $g \in L^1(\pi_s)$. For two functions l, u, let $[l, u]$ be the set of all functions f such that $l \leq f \leq u$. $[l, u]$ is called an ε-bracket in $L^p(\pi_s)$, $p \geq 1$, if $l, u \in L^p(\pi_s)$ and $\pi_s|u - l|^p \leq \varepsilon^p$. The bracketing number $N_{[]}(\varepsilon, \mathcal{F}, L^p(\pi_s))$ is the smallest number of ε-brackets in $L^p(\pi_s)$ covering $\mathcal{F}$.

Assumption 2.

1. There are constants c_1 and c_2 so that $N_{[]}(\varepsilon, \mathcal{F}, L^1(\pi_s)) \leq c_1 \varepsilon^{-c_2} \quad \forall \varepsilon \in (0, 1]$.
2. The innovations u_t in Eq. (1) are iid with $\mathrm{E}\,(u_t^{2\kappa}) < \infty$, $\kappa > \frac{\beta - \epsilon}{(\beta - \varepsilon)\overline{\eta} - 4\varepsilon}$, $\overline{\eta}$ defined in Eq. (16), and $\varepsilon > 0$ arbitrarily small.
3. $\sup_{f \in \mathcal{F}} E(U_k(f)^2) < \infty$.

Assumption 2.1 on the bracketing number is standard and satisfied, for example, by the Euclidean classes discussed in Nolan and Pollard (1987). To see this, consider the stationary case, $\beta = 1$. The assumption is implied by a standard bracketing condition on the space of bounded kernel functions. Let $N_{[]}(\varepsilon, \mathcal{F}, L^1(P)$ be the smallest number of brackets $[l, u] = \{f : l \leq f \leq u\}$ with $E|u(X_k) - l(X_k)| < \varepsilon$ that cover the space of bounded kernel functions. Then, by Lemma 2.2 in Nolan and Pollard (1987), $N_{[]}(\varepsilon, \mathcal{F}, L^1(\pi_s)) \leq c_1 \varepsilon^{-c_2}$ for some constants $c_1, c_2 > 0$. Therefore, Assumption 2.1 holds.

Following Härdle and Marron (1985, p. 1466), we define optimality of a bandwidth sequence relative to a given criterion $d(\hat{\mu}_h, \mu)$ that measures the distance between the estimator $\hat{\mu}_h$ and μ. Specifically, we say that $\hat{h}_n$ is optimal with respect to $d(\hat{\mu}_h, \mu)$ if

$$\left| \frac{d\left(\widehat{\mu}_{\widehat{h}_n}, \mu\right)}{\inf_{h \in H_n} d\left(\widehat{\mu}_h, \mu\right)} \right| = o_p(1).$$

Intuitively, $\widehat{h}_n$ is optimal for $d\left(\widehat{\mu}_h, \mu\right)$ if it is equivalent to the argmin of the latter as the sample size gets large.

The criterion we consider is the Average Squared Error[2] ("ASE"):

$$d_{A,h}\left(\widehat{\mu}, \mu\right) = \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\widehat{\mu}_h(X_{j-1}) - \mu(X_{j-1})\right)^2 1\left\{X_{j-1} \in \mathcal{C}\right\}. \tag{10}$$

This criterion is natural in that it amounts to a least-squares metric, measuring squared differences between the nonparametric estimates and the unknown true function. Its argmin is

$$\widetilde{h}_n = \arg\min_{h \in H_n} d_{A,h}\left(\widehat{\mu}, \mu\right).$$

Remark 1: The cross-validated criterion in Eq. (8) and the ASE in Eq. (10) are defined over a compact set $\mathcal{C}$. It is easy to see that this is not specific to the nonstationary case but is also important in the stationary case. In the stationary case, the cross-validated bandwidth is shown to be asymptotically equivalent to the *random* sequence minimizing the ASE (defined there as $\frac{T_n(\mathcal{C})}{n} d_{A,h}\left(\widehat{\mu}, \mu\right)$) and the *deterministic* sequence minimizing the mean-integrated squared error or MISE:

$$\begin{aligned} \text{MISE}\left(h\right) &\overset{asy}{\sim} \text{E}\left(\int \left(\left(\widehat{\mu}_h(x) - \mu(x)\right)^2\right) f(x) w(x) \mathrm{d}x\right) \\ &= \frac{1}{nh} \int K^2(u) \mathrm{d}u \int \sigma^2(x) \frac{1}{f(x)} w(x) \mathrm{d}x \\ &\quad + \int \left[\int K(u) \left(\mu(x - hu) - \mu(x)\right) f(x - hu) \mathrm{d}u\right]^2 \frac{1}{f(x)} w(x) \mathrm{d}x. \end{aligned} \tag{11}$$

Under stationarity, if $w(x) = 1$ and the support is the entire real line, $\int_{\mathbb{R}} \sigma^2(x) \frac{1}{f(x)} \mathrm{d}x$ may however not be finite. Hence, the MISE criterion would not be well-posed.

Remark 2: In nonstationary problems, the effective sample size T_n is a random variable. Hence, there is a difference between our assumed bandwidth set H_n, which is random, and the set assumed in stationary problems which is, instead, deterministic.

Remark 3: Due to the random variance of the kernel estimator (see, e.g., Eq. (3)), the MISE would not have a deterministic form in nonstationary environments. Therefore, in general, one would not be able to show that the cross-validated bandwidth is, as in

[2]For the sake of simplicity, all of the proofs are caried out for $w(X_{i-1}) = 1\left\{X_{j-1} \in \mathcal{C}\right\}$. However, the same result would follow for any weighting scheme trimming away observations outside of a small set $\mathcal{C}$.

the stationary case, asymptotically equivalent to a deterministic sequence minimizing the estimator's MISE.

Before presenting our main result in Theorem 1 below, we introduce a series of four Lemmas.

Lemma 1: *Define*

$$\widehat{p}_h(x) = \frac{1}{T_n h} \sum_{j=1}^{n} K\left(\frac{X_{j-1} - x}{h}\right). \tag{12}$$

Under Assumptions 1.1–1.4 and 2:

$$\sup_{x \in \mathcal{C},\, h \in H_n} |\widehat{p}_h(x) - p_s(x)| = o_{a.s.}(1),$$

where $p_s(x)$ *is defined in Assumption 1.2.*

Lemma 1 establishes uniform convergence of $\widehat{p}_h(x)$ to its deterministic counterpart, i.e., the density associated to the invariant measure π_s. This allows us to handle the contribution of the denominator in Eq. (2) and work with $\widehat{\mu}_h \frac{\widehat{p}_h}{p_s}$ rather than with $\widehat{\mu}_h$.

Lemma 2: *Let*

$$b_{\mathcal{C}}(x,h) = \frac{1}{p_s(x)} \left(\int K(u) \left(\mu(x - hu) - \mu(x)\right) p_s(x - hu) \mathrm{d}u \right).$$

Define

$$\Omega_{n,h} = \left\{ \varpi : d_{A,h}(\widehat{\mu}, \mu) \geq \alpha C \left(\frac{1}{h n^{\beta+\varepsilon}} + \inf_{x \in \mathcal{C}} b_{\mathcal{C}}^2(x,h) \right) \right\},$$

for some $0 < \alpha \leq 1$ *and some constant* C*. Denote by* $\Omega_{n,h}^c$ *the complement of* $\Omega_{n,h}$*. Let Assumptions 1 and 2 hold. Uniformly in* $h \in H_n$*, we have*

$$\lim_{n \to \infty} \Pr\left\{\Omega_{n,h}^c\right\} = 0.$$

Lemma 2 establishes a lower bound in probability for the ASE. This bound will prove useful in handling the denominator in the ratio of the main quantity in Lemma 3. Define, now,

$$\overline{d}_{A,h}\left(\widehat{\mu}, \mu\right) = \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\widehat{\mu}_{j,h}(X_{j-1}) - \mu(X_{j-1})\right)^2 1\left\{X_{j-1} \in \mathcal{C}\right\}, \tag{13}$$

with $\widehat{\mu}_{j,h}(X_{j-1})$ as in Eq. (7).

Lemma 3: *Let Assumptions 1 and 2 hold. Then,*

$$\sup_{h\in H_n}\left|\frac{d_{A,h}\left(\widehat{\mu},\mu\right)-\overline{d}_{A,h}\left(\widehat{\mu},\mu\right)}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|=o_p(1),$$

with $d_{A,h}\left(\widehat{\mu},\mu\right)$ and $\overline{d}_{A,h}\left(\widehat{\mu},\mu\right)$ defined as in Eq. (10) and Eq. (13), respectively.

Lemma 3 simply states that the average mean squared error criterion computed using $\widehat{\mu}_h(X_{j-1})$ and the "leave-one-out" estimator $\widehat{\mu}_{j,h}(X_{j-1})$ are asymptotically equivalent (uniformly in h).

Lemma 4: *Let Assumptions 1 and 2 hold. Define*

$$\text{Cross}(h)=\overline{d}_{A,h}\left(\widehat{\mu},\mu\right)-CV(h)-\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(X_j-\mu(X_{j-1})\right)^2 1\left\{X_{j-1}\in\mathcal{C}\right\}, \tag{14}$$

with $CV(h)$ as in Eq. (8). Then,

$$\sup_{h\in H_n}\left|\frac{\text{Cross}(h)}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|=o_p(1).$$

Note that $\text{Cross}(h)=\frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\sigma(X_{j-1})u_j\left(\widehat{\mu}_{h,j}(X_{j-1})-\mu(X_{j-1})\right)$. Lemma 4 establishes that this term converges to zero faster than the average squared error criterion.

Lemma 1 through 4 lead to the final result yielding asymptotic equivalence between the cross-validated bandwidth, $\widehat{h}_n$, and the optimal ASE bandwidth, $\widetilde{h}_n$.

Theorem 1: *Let Assumptions 1 and 2 hold. Then, the cross-validated bandwidth $\widehat{h}_n$ in Eq. (9) is asymptotically optimal with respect to $d_{A,h}\left(\widehat{\mu},\mu\right)$, i.e.,*

$$\left|\frac{d_{A,\widehat{h}_n}\left(\widehat{\mu},\mu\right)}{\inf_{h\in H_n}d_{A,h}\left(\widehat{\mu},\mu\right)}-1\right|=o_p(1).$$

Corollary 1: *Let Assumptions 1 and 2 hold. Then,*

$$\left|\frac{\widehat{h}_n}{\widetilde{h}_n}-1\right|=o_p(1),$$

where $\widetilde{h}_n=\arg\min_{h\in H_n}d_{A,h}\left(\widehat{\mu},\mu\right)$.

Remark 4: The case of nonparametric cointegration with

$$Y_t = g(X_t) + u_t$$

and u_t iid and independent of X_t can be treated as the autoregressive case discussed above. If the u_t's exhibit some dependence, or are only asymptotically uncorrelated with X_t but not strictly independent, then the statement in Theorem 1 does not directly apply. Lemma 4 would have to be modified in this case. We leave this issue for future work.

4 Optimal Bandwidth for Conditional Variance Estimates

In the previous section we established the optimality of the cross-validated bandwidth for the conditional mean estimator. In practice, one frequently also needs to optimally select the bandwidth for the conditional variance. One such example is provided in Section 6 below.

Even though the rate conditions for consistency and mixed normality of the conditional mean and variance are the same, the optimal bandwidth cannot be the same because of their different functional form. In the case of the conditional variance, the additional difficulty is that we do not know the conditional mean. Hence, it needs to be replaced by an estimated counterpart.

Define the feasible estimator

$$\widehat{\sigma}_{\xi}^{2}(x) = \frac{1}{T_n \xi} \sum_{i=2}^{n} \frac{1}{\widehat{p}_{\xi}(x)} K\left(\frac{X_{i-1} - x}{\xi}\right) \left(X_i - \widehat{\mu}_{\widehat{h}_n}(X_{i-1})\right)^2,$$

and its infeasible counterpart,

$$\widetilde{\sigma}_{\xi}^{2}(x) = \frac{1}{T_n \xi} \sum_{i=2}^{n} \frac{1}{\widehat{p}_{\xi}(x)} K\left(\frac{X_{i-1} - x}{\xi}\right) \left(X_i - \mu(X_{i-1})\right)^2,$$

where $\widehat{h}_n$ is the cross-validated bandwidth for the conditional mean and $\widehat{p}_{\xi}(x) = \frac{1}{T_n \xi} \sum_{i=2}^{n} K\left(\frac{X_{i-1}-x}{\xi}\right)$.

Similarly, we define the feasible cross-validation criterion

$$\mathrm{CV}_n(\xi) = \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\left(X_j - \widehat{\mu}_{\widehat{h}_n}(X_{j-1})\right)^2 - \widehat{\sigma}_{j,\xi}^{2}(X_{j-1}) \right) 1\left\{X_{j-1} \in \mathcal{C}\right\}$$

and its infeasible counterpart

$$\widetilde{\mathrm{CV}}_n(\xi) = \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left((X_j - \mu(X_{j-1}))^2 - \widetilde{\sigma}^2_{j,\xi}(X_{j-1}) \right) 1\{X_{j-1} \in \mathcal{C}\}, \tag{15}$$

where $\widehat{\sigma}^2_{j,\xi}(x)$ and $\widetilde{\sigma}^2_{j,\xi}(x)$ are the "leave one out" versions of $\widehat{\sigma}^2_{\xi}(x)$ and $\widetilde{\sigma}^2_{\xi}(x)$.
It is important to note that the estimation error of the conditional mean plays a twofold role. It affects the conditional variance estimator, i.e. $\widehat{\sigma}^2_{j,\xi}(X_{j-1})$ vs. $\widetilde{\sigma}^2_{j,\xi}(X_{j-1})$. It also directly affects the cross-validation criterion to be minimized through the term $\left(X_j - \widehat{\mu}_{\widehat{h}_n}(X_{j-1})\right)^2$ versus $(X_j - \mu(X_{j-1}))^2$.

Now, define the relevant cross-validated bandwidths, namely

$$\widehat{\xi}_n = \arg\min_{\xi \in \Xi_n} \mathrm{CV}_n(\xi)$$

and

$$\widetilde{\widehat{\xi}}_n = \arg\min_{\xi \in \Xi_n} \widetilde{\mathrm{CV}}_n(\xi),$$

where

$$\Xi_n \equiv \left\{ \underline{\xi}_n, \overline{\xi}_n \right\} = \left\{ \underline{c}^{\sigma} T_n(\mathcal{C})^{-\underline{\eta}}, \overline{c}^{\sigma} T_n(\mathcal{C})^{-\overline{\eta}} \right\}, \tag{16}$$

with $0 < \underline{c}^{\sigma}, \overline{c}^{\sigma} < \infty$, $\frac{1}{5} < \overline{\eta} < \underline{\eta} < 1$, and $1 - 2\underline{\eta} + \overline{\eta} > 0$. It is immediate to see that Ξ_n is of the same almost-sure order as H_n. We simply allow $\underline{c}^{\sigma}$ and $\overline{c}^{\sigma}$ to differ from $\underline{c}$ and $\overline{c}$.

Finally, we define the ASE criterion for the variance estimator, i.e.,

$$d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma\right) = \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\widetilde{\sigma}^2_{\xi}(X_{j-1}) - \sigma^2(X_{j-1}) \right)^2 1\{X_{j-1} \in \mathcal{C}\}$$

and its minimizer

$$\widetilde{\xi}_n = \arg\min_{\xi \in \Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma\right).$$

Our objective is to show that $\widehat{\xi}_n$ is asymptotically optimal with respect to $d_{A,\xi}(\widetilde{\sigma}^2, \sigma)$, thereby yielding $\left|\widehat{\xi}_n / \widetilde{\xi}_n - 1\right| = o_p(1)$. This is accomplished in two steps. First, we establish the optimality of $\widetilde{\widehat{\xi}}_n$ for $d_{A,\xi}(\widetilde{\sigma}^2, \sigma)$, i.e., we establish ASE optimality for the case in which the true conditional mean is assumed known, leading to $\left|\widetilde{\widehat{\xi}}_n / \widetilde{\xi}_n - 1\right| = o_p(1)$. Second, we show that conditional mean estimation does not affect this optimality result, i.e., $\left|\widehat{\xi}_n / \widetilde{\widehat{\xi}}_n - 1\right| = o_p(1)$.

The results in this section rely on a modified version of Assumption 2. Let $\mathcal{F}^\sigma := \{f = \xi K_{x,\xi} : x \in \mathcal{C}, 0 < \xi \leq 1\}$ with envelope F, i.e. $f \leq F$ for all $f \in \mathcal{F}^\sigma$.

Assumption 3.

1. Ξ_n is defined as in Eq. (16).
2. There are constants c_1 and c_2 so that $N_{[]}(\varepsilon, \mathcal{F}, L^1(\pi_s)) \leq c_1 \varepsilon^{-c_2} \quad \forall \varepsilon \in (0, 1]$.
3. The innovations u_t in Eq. (1) are *iid* with $\mathrm{E}\left(u_t^{4\kappa}\right) < \infty$, $\kappa > \frac{\beta-\epsilon}{(\beta-\varepsilon)\overline{\eta}-4\varepsilon}$, $\overline{\eta}$ defined in Eq. (16), and $\varepsilon > 0$ arbitrarily small.
4. $\sup_{f\in\mathcal{F}} E(U_k(f)^2) < \infty$.

Before deriving our main result in Theorem 2 below, we introduce the following four Lemmas.

Lemma 5: *Let*

$$b_{\mathcal{C}}(x,\xi) = \frac{1}{p_s(x)} \left(\int K(u) \left(\sigma^2(x - \xi u) - \sigma^2(x) \right) p_s(x - \xi u) \mathrm{d}u \right).$$

Define

$$\Omega_{n,\xi} = \left\{ \varpi : d_{A,h}(\widehat{\sigma}^2, \sigma^2) \geq \alpha C \left(\frac{1}{\xi n^{\beta+\varepsilon}} + \inf_{x\in\mathcal{C}} b_{\mathcal{C}}^2(x,\xi) \right) \right\},$$

for some $0 < \alpha \leq 1$ and some constant C. Denote by $\Omega_{n,\xi}^c$ the complement of $\Omega_{n,\xi}$. Let Assumptions 1 and 3 hold. Uniformly in $\xi \in H_n$, we have

$$\lim_{n\to\infty} \Pr\left\{\Omega_{n,\xi}^c\right\} = 0.$$

Lemma 5 parallels Lemma 2 establishing a lower bound for our definition of the ASE criterion for the variance estimator.

Lemma 6: *Let Assumptions 1 and 3 hold. Then, the cross-validated bandwidth $\widetilde{\widehat{\xi}}_n$ in Eq. (15) is asymptotically optimal with respect to $d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right)$, i.e.,*

$$\left| \frac{d_{A,\widetilde{\widehat{\xi}}_n}\left(\widetilde{\sigma}^2, \sigma^2\right)}{\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right)} - 1 \right| = o_p(1).$$

Lemma 6 establishes the asymptotic optimality of the bandwidth minimizing the infeasible CV criterion. Next, we show the same result for the feasible CV criterion. We first need to control the probability order of the conditional mean's estimation error. This is done via Lemma 7.

Lemma 7: *Let Assumptions 1 and 3 hold. Then,*

$$\sup_{x\in\mathcal{C}}\left(\left(\widehat{\mu}_{\widehat{h}_n}(x)-\mu(x)\right)^4\right)=o_p\left(d_{A,\widehat{h}_n}\left(\widehat{\mu},\mu\right)\right).$$

Now, we ought to isolate the component of $\widetilde{\mathrm{CV}}_n(\xi)-\mathrm{CV}_n(\xi)$ which does not depend on ξ and show that the remaining component is of smaller probabilty order than $\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)$. This is accomplished in Lemma 8 below. Write,

$$\begin{aligned}
&\mathrm{CV}(\xi)-\widetilde{\mathrm{CV}}(\xi)\\
=\;&\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{j,\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)^4 1_{\mathcal{C}}(X_{j-1})\\
&+\frac{4}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\left(X_j-\mu(X_{j-1})\right)\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)\right)^2 1_{\mathcal{C}}(X_{j-1})\\
&+\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})(u_j^2-1)\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)^2\right)1_{\mathcal{C}}(X_{j-1})\\
&-4\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(X_j-\mu(X_{j-1})\right)\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)^3\\
&-4\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})(u_j^2-1)\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)\right)1_{\mathcal{C}}(X_{j-1})\\
&+\widehat{\mathrm{Error}}(\xi),
\end{aligned}\tag{17}$$

where

$$
\begin{aligned}
&\widehat{\mathrm{Error}}(\xi) \\
= \;& \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\sigma}^2_{j,\xi}\left(X_{j-1}\right)-\widetilde{\sigma}^2_{j,\xi}(X_{j-1})\right)^2 1_{\mathcal{C}}(X_{j-1}) && (18)\\
&+\frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\left(\sigma^2(X_{j-1})-\widetilde{\sigma}^2_{j,\xi}(X_{j-1})\right)\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)^2\right)1_{\mathcal{C}}(X_{j-1}) && (19)\\
&+\frac{4}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\left(\sigma^2(X_{j-1})-\widetilde{\sigma}^2_{j,\xi}(X_{j-1})\right)\sigma(X_{j-1})u_j\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)\right)1_{\mathcal{C}}(X_{j-1}) && (20)\\
&-\frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\left(\sigma^2(X_{j-1})-\widetilde{\sigma}^2_{j,\xi}(X_{j-1})\right)\left(\widehat{\sigma}^2_{j,\xi}-\widetilde{\sigma}^2_{j,\xi}\right)\right)1_{\mathcal{C}}(X_{j-1}) && (21)\\
&-\frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})(u_j^2-1)\left(\widehat{\sigma}^2_{j,\xi}-\widetilde{\sigma}^2_{j,\xi}\right)\right)1_{\mathcal{C}}(X_{j-1}) && (22)\\
&+\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\left(\widehat{\sigma}^2_{j,\xi}-\widetilde{\sigma}^2_{j,\xi}\right)\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)^2\right)1_{\mathcal{C}}(X_{j-1}) && (23)\\
&+\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\left(\widehat{\sigma}^2_{j,\xi}-\widetilde{\sigma}^2_{j,\xi}\right)\sigma(X_{j-1})u_j\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)\right)1_{\mathcal{C}}(X_{j-1}) && (24)
\end{aligned}
$$

Lemma 8: *Let Assumptions 1 and 3 hold. Then,*

$$\sup_{\xi\in\Xi_n}\left|\frac{\widehat{\mathrm{Error}}(\xi)}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}\right|=o_p(1).$$

Lemma 8 shows that the estimation error component of the cross-validation criterion is of sufficiently small probability order. We can finally turn to the optimality of the bandwidth minimizing the *feasible* criterion.

Theorem 2: *Let Assumptions 1 and 3 hold. Then, the cross-validated bandwidth $\widehat{\xi}_n$ is asymptotically optimal with respect to $d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)$, i.e.,*

$$\left|\frac{d_{A,\widehat{\xi}_n}\left(\widetilde{\sigma}^2,\sigma^2\right)}{\inf_{\xi\in\Xi_n}d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}-1\right|=o_p(1).$$

Corollary 2: *Let Assumptions 1 and 3 hold. Then,*

$$\left|\frac{\widehat{\xi}_n}{\widetilde{\xi}_n}-1\right|=o_p(1).$$

5 Simulations

In this section, we report the results of a simulation experiment that illustrates the finite sample performance of our proposed bandwidth selection procedure. We generate data from four different models: an autoregressive process

$$X_t = \mu(X_{t-1}) + u_t,$$

with a linear, $\mu(x) = \rho x$, or a nonlinear, $\mu(x) = \rho x^2/10$, mean function, and a nonlinear predictive regression,

$$\begin{aligned} Y_t &= \mu(X_t) + u_t \\ X_t &= \rho X_{t-1} + \varepsilon_t \end{aligned}$$

with a linear, $\mu(x) = x$, or a nonlinear[3], $\mu(x) = \sum_{j=1}^{4}(-1)^{j+1}\sin(j\pi x)/j^2$, mean function. For all four models, X_t is initialized at zero ($X_0 = 0$) and (u_t, ε_t) are independent, standard normal random variables, iid across time.

We are interested in nonparametrically estimating the function $\mu(.)$. We vary the sample size, T, and the degree of persistence, ρ, in the X_t process. All results are based on $1,000$ Monte Carlo samples, the standard normal density kernel, and the set $\mathcal{C}$ set equal to the range of the data on X_t.

Since we are not aware of any other data-driven method for choosing the bandwidth in nonstationary settings, we compare our estimator to a linear least-squares estimator ("OLS"). Tables 1–4 present the bias, standard deviation ("stdev") and root mean-square error ("RMSE") of the nonparametric estimator based on cross-validated bandwidth and of the OLS estimator. The biases, standard deviations, and RMSEs are averaged values over a grid of points. Figure 1 plots the cross-validation criterion function and the selected bandwidth, each averaged over all Monte Carlo samples. Figure 2 plots the functional estimates.

In the linear specifications, both the bias and the standard deviation of the nonparametric estimates are larger than what is found for the least-squares estimates. As expected, while the standard deviation of the latter decreases with the level of persistence, the standard deviation of the former increases, due to a reduced number of visits of the process to each evaluation point. In the nonlinear specifications, the reduced biases of

[3]This is the same nonlinear design as in Hall and Horowitz (2005) and Wang and Phillips (2009).

the nonparametric estimates more than offset some increases in dispersion over the least-squares counterparts, thereby yielding root mean-square errors which are considerably smaller than in the least-squares case.

The interaction of nonlinearities and increased levels of dependence may lead to large biases and sizeable root mean-squared errors in the least-squares case. Not only do cross-validated bandwidths satisfy meaningful optimality criteria in the presence of (potential) nonstationarities, as shown by our theoretical results, they also appear to lead to an empirically meaningful trade-off between bias and variance.

6 Nonlinear Predictive Regressions

In this section, we discuss the application of our proposed bandwidth selection method to a fundamental applied problem both in the academic finance literature and the industry. We consider a prototypical portfolio allocation problem in which an investor computes the optimal allocation from predictions of stock returns by dividend-to-price ratios. We show the economic gains from using our nonparametric procedure for prediction as compared to an investor who uses linear regressions.

6.1 A Portfolio Allocation Problem

Consider an investor with quadratic utility

$$U(w) = w - \frac{\tilde{\lambda}}{2} w^2$$

over wealth w, who allocates a fraction ω_t of wealth in period t, W_t, to the market and a fraction $(1-\omega_t)$ to treasury bills. Let the investor's planning horizon be h periods, denote by $R_{t,t+h}$ and $R^f_{t,t+h}$ the h-period returns of the market and treasury bills, respectively. Then the investor's end-of-horizon wealth W_{t+h} is

$$W_{t+h} = \omega_t W_t R_{t,t+h} + (1-\omega_t) W_t R^f_{t,t+h}.$$

Suppose the excess returns $\tilde{R}_{t,t+h} := R_{t,t+h} - R^f_{t,t+h}$ evolve according to

$$\tilde{R}_{t,t+h} = \mu(X_t) + \sigma(X_t) u_{t+h} \tag{25}$$

where X_t is some predictor variable and the errors u_{t+h} satisfy $E[u_{t+h}|\mathcal{F}_t] = 0$ and $E[u_{t+h}^2|\mathcal{F}_t] = 1$. Further, suppose the investor's information set $\mathcal{F}_t$ at time t contains $(X_s, W_s, R^f_{s,s+h})$ for $s \leq t$. Then the period-t optimal allocation problem can be written as

$$\begin{aligned}\omega &= \text{argmax}_w E[U(W_{t+h}(w))|\mathcal{F}_t] \\ &= \text{argmax}_w \left\{ -\frac{\tilde{\lambda}}{2} W_t \left(\sigma^2(X_t) + \mu(X_t)^2\right) w^2 + \left(1 - \tilde{\lambda} W_t R^f_{t,t+h}\right) \mu(X_t) w \right\} \end{aligned} \tag{26}$$

As is common in the literature (e.g. Fleming et al. (2001)) we facilitate comparisons across portfolios by keeping the relative risk aversion $\gamma_t = \tilde{\lambda} W_t/(1 - \tilde{\lambda} W_t)$ constant at a value γ. Then, the optimal portfolio weights are independent of wealth. Let $\lambda := \gamma/(1+\gamma)$. Conditional on $X_t = x$ and $R^f_{t,t+h} = r$, the optimal allocation of an investor solving (26), called a "nonlinear investor", is therefore

$$\omega_{non}(x) = \frac{(1 - \lambda r)\mu(x)}{\lambda(\sigma^2(x) + \mu(x)^2)}.$$

A "linear investor" is one who assumes $\mu(x) = \alpha + \beta x$ and $\sigma(x) = \sigma$, leading to the optimal allocation

$$\omega_{lin}(x) = \frac{(1 - \lambda r)(\alpha + \beta x)}{\lambda(\sigma^2 + (\alpha + \beta x)^2)}.$$

The optimal portfolio allocation depends on the way in which the investor implements the predictive regression in (25), a problem that has received a lot of attention in literature.

6.2 Predictive Regressions

Our bandwidth selection procedure appears attractive in the context of predictive regressions such as (25) because, as we now argue, the economcis of stock return predictability suggest (possible) nonlinearities in the conditional mean and variance, extreme persistence of the predictor, and endogeneity of the predictor.

First, the dynamic dividend growth model of Campbell and Shiller (1988) justifies the standard theoretical framework for regressing stock returns on financial ratios. The log-linearization of their model leads to a linear predictive regression, but, in general, should be viewed as an approximation to a more complex nonlinear model. Little is known about the quality of such an approximation, especially for longer horizons $h > 1$.

We therefore argue that estimation procedures for the present context should not rely strongly on linearity in (25). Furthermore, the early literature generally focused on the conditional mean and set $\sigma(\cdot)$ equal to a constant. Recent implementations, however, treat the conditional variance similarly to the conditional mean and allow it to be a nonlinear function of the predictor. See the survey by Brandt (2010) for a discussion of nonlinear predictive models and their implications for portfolio allocation.

Second, auto-regressive models that are local-to-unit root have a long history in financial econometrics and are known to capture well the persistence properties of financial ratios, such as the dividend-to-price ratio (e.g. Cavanagh et al. (1995), Torous and Valkanov (2000), Valkanov (2003)). Therefore, we expect our predictor in (25) to be highly persistent.

Third, if X_t is the dividend-to-price ratio or, more generally, a financial ratio whose denominator depends on the market's price level, its variation between time $t-1$ and time t is necessarily correlated with the variation in $R_{t-1,t}$, the market return over the same period. In effect, a positive shock to prices lowers X_t while increasing $\tilde{R}_{t-1,t}$, thereby inducing a negative correlation between $\tilde{R}_{t-1,t}$ and X_t. Stambaugh (1986, 1999) provides early discussions of the importance of this correlation and its role in predictive models for stock returns.

6.3 Results

Our dataset is obtained from CRSP and is the same as that in Lewellen (2004). We compute excess returns from monthly continuously-compounded returns ($R_{t,t+1}$) on the value-weighted NYSE index net of the one-month treasury bill rate ($R^f_{t,t+1}$). The predictor X_t is the dividend-to-price ratio constructed as dividend paid during the previous year divided by the current value of the value-weighted NYSE index. The h-period excess returns $\tilde{R}_{t,t+h}$ are aggregates of 1-month excess returns: $\tilde{R}_{t,t+h} = \sum_{i=1}^{h-1} \tilde{R}_{t+i,t+i+1}$. Figure 3 displays the data.

We implement the nonparametric kernel estimators introduced above together with our proposed cross-validation criterion for choosing the bandwidth. For investment horizons h larger than one month, we modify the cross-validation criterion function so as to leave out not only one observation, but also h before and h after a particular observation. This modification controls the dependence structure of the regression residuals upon

aggregation. All calculations are based on the normal density kernel and the set $\mathcal{C}$ set equal to the range of the data on X_t. We report results for the investment horizons of one month ($h = 1$), 3 years ($h = 36$), and 5 years ($h = 60$), and risk aversion parameters $\gamma \in \{1, 5, 10\}$.

Figure 4 shows the cross-validation objective function, which is very flat near its minimum when $h = 1$, leading to large bandwidth choices, and possesses clearly separate minima for the horizons $h > 1$, leading to much smaller bandwidths. Figure 5 provides the resulting nonparametric estimates and asymptotic (pointwise) standard errors of the conditional mean and variance functions. The conditional means are increasing in the predictor and mildly nonlinear, with the nonlinearities being statistically significant at the two higher horizons. As is well-known, a high dividend-to-price ratio should predict high returns, low dividend growth, or decreasing prices. It generally predicts high returns. It also generally does so strongly over longer horizons. Leaving nonlinearities aside, our estimates exhibit overall shapes that confirm the findings in the literature. Similarly, at the short horizon we find a flat conditional variance which is consistent with classical modelling approaches in the predictability literature. However, it appears to be nonlinear, and decreasing, at longer horizons. When predicting over 3 and 5 years, a higher conditional variance seems to be associated with higher prices relative to dividends. This effect is interesting. In Figure 6, we report three scatterplots of squared de-meaned excess returns, i.e. $(\sum_{i=1}^{h-1} \tilde{R}_{t+i,t+i+1} - \widehat{\mu}_{\widehat{h}_n}(X_t))^2$, against X_t, along with least-squares estimates of the same relation. Again, the long-run conditional variance appears to decrease with increases in the dividend-to-price ratio. A least-squares specification would capture these effects while missing some nonlinearities around the mean dividend-to-price level.

We now evaluate the economic impact of taking into account the nonlinearities found in the estimates of Figure 5. We quantify the economic gains for the investor employing a nonlinear rather than a linear specification by computing an annualized fee Δ that makes the nonlinear investor indifferent between the nonlinear and the linear specification.

Specifically, we follow Fleming et al. (2001) and define Δ as the solution[4] to

$$\frac{1}{T}\sum_{t=1}^{T}\left\{R^p_{t,t+h}(\omega_{non}(X_t)) - \Delta - \frac{\lambda}{2}\left(R^p_{t,t+h}(\omega_{non}(X_t)) - \Delta\right)^2\right\}$$
$$= \frac{1}{T}\sum_{t=1}^{T}\left\{R^p_{t,t+h}(\omega_{non}(X_t)) - \frac{\lambda}{2}R^p_{t,t+h}(\omega_{non}(X_t))^2\right\} \quad (27)$$

where $R^p_{t,t+h}(\omega) := \omega R_{t,t+h} + (1-\omega)R^f_{t,t+h}$ denotes the portfolio return for weights ω. Figure 7 shows realized utilities of the nonlinear investor minus those of the linear investor plotted over time. Δ is the additional annualized return necessary for the nonlinear investor to have the same average realized utility as the linear investor. Table 5 reports the estimates of Δ. As expected the annualized fees are small for short investment horizons for which we have already seen that the conditional mean and variance are essentially linear. For the longer investment horizons, however, we found significant nonlinearities in the conditional mean and variance, and therefore the nonlinear investor requires large fees Δ to be indifferent between the nonlinear and the linear specification. The fees range from 1.5% to 6.2% annualized return on top of the portfolio return to reach indifference. These fees indicate large economic gains from nonlinear predictability.

7 Conclusions

Cross-validation is the most widely used method of bandwidth selection in nonparametric econometrics. It is employed routinely in empirical work irrespective of the level of persistence. We show that this common practice is theoretically justified. Even in the nonstationary case, which is a sub-case of our broader framework, the cross-validated bandwidth is optimal with respect to the averaged squared error criterion, i.e., the averaged squared distance between the true function and its nonparametric counterpart. Should stationarity be satisfied, the classical optimality with respect to the mean integrated squared error would be easily re-established. We provide a treatment which covers both the conditional mean and the conditional variance estimator in the context of (positive and null) recurrent Markov chain models.

[4] In general, there are two solutions to (27). We discard the solution that moves the investor's return to the decreasing side of the utility function.

Even though many economic time series may not be genuinely nonstationary, persistence is a fact of life. While extreme forms of persistence lead us to re-consider the allowed criterion for optimality, this paper shows that a meaningful notion of optimality continues to be a fundamental property of cross-validated bandwidth choices. This is, in our view, important information for nonparametric applied work in economics.

A Proofs

A.1 Proofs of Section 3

We begin with a proposition.

Proposition 1: Under Assumptions 1 and 2,

$$\sup_{x\in\mathcal{C},\, h\in H_n}\left|\frac{1}{T_n}\sum_{k=1}^{T_n}U_{k,x,h}-E\left(U_{k,x,h}\right)\right|=o_{a.s.}(1) \tag{28}$$

and

$$\sup_{x\in\mathcal{C},\, h\in H_n}\left|\frac{1}{T_n(\mathcal{C})}\sum_{k=1}^{T_n(\mathcal{C})}U_{k,x,h}-E\left(U_{k,x,h}\right)\right|=o_{a.s.}(1). \tag{29}$$

Proof. Consider some subsequence $\{a_n : n \geq 1\}$ of $\{n\}$ such that $a_n \to \infty$ as $n \to \infty$. Let

$$\tilde{H}_{a_n,n} := \left\{h \in \mathbb{R} :\ \underline{\tilde{h}}_{a_n,n} \leq h \leq \bar{\tilde{h}}_{a_n,n}\right\}$$

with $\underline{\tilde{h}}_{a_n,n} := \underline{c} a_n^{-\underline{\eta}}$ and $\bar{\tilde{h}}_{a_n,n} := \bar{c} a_n^{-\overline{\eta}}$. We first show that

$$\bar{A}_{a_n} := \sup_{x\in\mathcal{C},\, h\in\tilde{H}_{a_n,n}}\left|\frac{1}{a_n}\sum_{k=1}^{a_n}U_{k,x,h}-E\left(U_{k,x,h}\right)\right|=o_{a.s.}(1). \tag{30}$$

For a sequence of random variables $A_1, A_2, \ldots$, define the empirical average $P_n A_k := \frac{1}{a_n}\sum_{k=1}^{a_n} A_k$.

Step (1): Define the truncated version of $U_k(f)$ as $T_k(f) := U_k(f)1\{U_k(F)^2 \leq b_n\}$, where $b_n = a_n^{2(1-\overline{\eta}+\underline{\eta})}$ and $b_n \to \infty$ because $\underline{\eta} > \overline{\eta}$. Also, let R_n be the approximation error of $U_k(f)$ by $T_k(f)$:

$$R_n := \sup_{f\in\mathcal{F}_{a_n,n}}\left\{|P_n(U_k(f)-T_k(f))|+|E[U_k(f)-T_k(f)]|\right\}$$

where $\mathcal{F}_{a_n,n} := \{f = hK_{x,h} : x \in \mathcal{C}, h \in \tilde{H}_{a_n,n}\}$ defines a sequence of subsets of $\mathcal{F}$ that incorporates rate restrictions on the bandwidth h. Note that $U_k(F)^2 > b_n$ and $U_k(F) \geq 0$ imply $U_k(F) < U_k(F)^2/\sqrt{b_n}$. We have

$$\begin{aligned}
|P_n T_k(f) - P_n U_k(f)| &= P_n U_k(f) 1\{U_k(F)^2 > b_n\} \\
&\leq P_n U_k(F) 1\{U_k(F)^2 > b_n\} \\
&\leq b_n^{-1/2} P_n U_k(F)^2 1\{U_k(F)^2 > b_n\} \\
&\leq b_n^{-1/2} P_n U_k(F)^2. \qquad (31)
\end{aligned}$$

Taking expectations on both sides of Eq. (31) yields

$$|E(T_k(f)) - E(U_k(f))| \leq E|T_k(f) - U_k(f)| \leq b_n^{-1/2} E(U_k(F)^2). \qquad (32)$$

Eqs. (31) and (32) together with the fact that $U_1(f), U_2(f), \ldots$ are i.i.d. (p. 135 in Nummelin (1984)), the strong law of large numbers, and Assumption 2.3 imply

$$R_n = O_{a.s.}(b_n^{-1/2}),$$

i.e. we can replace $U_k(f)$ in Eq. (30) by the truncated variable $T_k(f)$ as long as $b_n^{-1/2}$ is a negligible higher-order term.

Step (2): We now apply a standard bracketing argument to show that the truncated process $P_n T_k(f)$ converges almost surely to its expectation uniformly over $\mathcal{F}_{a_n,n}$. To this end, notice that, for any $\xi > 0$:

$$\begin{aligned}
&P\left(\sup_{x \in \mathcal{C},\, h \in \tilde{H}_{a_n,n}} |P_n T_k(K_{x,h}) - E T_k(K_{x,h})| > \xi\right) \\
&\quad \leq P\left(\sup_{f \in \mathcal{F}_{a_n,n}} \frac{1}{\underline{\tilde{h}}_{a_n,n}} |P_n T_k(f) - E T_k(f)| > \xi\right) \\
&\quad = P\left(\sup_{f \in \mathcal{F}_{a_n,n}} P_n T_k(f) - E T_k(f) > \xi \underline{\tilde{h}}_{a_n,n}\right) \qquad (33) \\
&\qquad + P\left(\sup_{f \in \mathcal{F}_{a_n,n}} P_n T_k(f) - E T_k(f) < -\xi \underline{\tilde{h}}_{a_n,n}\right). \qquad (34)
\end{aligned}$$

Consider the first probability. By Assumption 2.1, there is a collection of $\varepsilon \underline{\tilde{h}}_{a_n,n}$-brackets $B(\varepsilon \underline{\tilde{h}}_{a_n,n}, \mathcal{F}_{a_n,n}, L^1(\pi_s))$ of cardinality $N_{[]}(\varepsilon \underline{\tilde{h}}_{a_n,n}, \mathcal{F}_{a_n,n}, L^1(\pi_s)) \leq N_{[]}(\varepsilon \underline{\tilde{h}}_{a_n,n}, \mathcal{F}, L^1(\pi_s))$ so that,

for every $f \in \mathcal{F}_{a_n,n}$, there is a bracket $[f^l, f^u]$ such that $f^l \leq f \leq f^u$ and $E[U_k(f^u) - U_k(f^l)] \leq \varepsilon \underline{\tilde{h}}_{a_n,n}$. Therefore, for any $f \in \mathcal{F}_{a_n,n}$,

$$E[T_k(f^u) - T_k(f)] \leq E\left[(U_k(f^u) - U_k(f))1\{U_k(F)^2 \leq b_n\}\right] \leq E\left[(U_k(f^u) - U_k(f))\right] \leq \varepsilon \underline{\tilde{h}}_{a_n,n}$$

or, equivalently,

$$ET_k(f) \leq ET_k(f^u) - \varepsilon \underline{\tilde{h}}_{a_n,n}. \tag{35}$$

By Assumptions 1 and 2, we can apply Lemma 4.1 in Karlsen and Tjøstheim (1998) to get $hE[U_k(K_{x,h})^2] = O(1)$ and thus, for any $f \in \mathcal{F}_{a_n,n}$, $EU_k(f)^2 = O(\bar{\tilde{h}}_{a_n,n})$. We use this fact together with Eq. (35) to bound the term in (33) as follows: letting c_j denote some constants,

$$\begin{aligned}
&P\left(\sup_{f\in\mathcal{F}_{a_n,n}} P_n\left[T_k(f) - ET_k(f)\right] > \xi \underline{\tilde{h}}_{a_n,n}\right) \\
&\quad \leq P\left(\max_{f^u:\,[f^l,f^u]\in B(\varepsilon \underline{\tilde{h}}_{a_n,n}, \mathcal{F}_{a_n,n}, L^1(\pi_s))} P_n\left[T_k(f^u) - ET_k(f^u)\right] > (\xi+\varepsilon)\underline{\tilde{h}}_{a_n,n}\right) \\
&\quad \leq N_{[]}(\varepsilon, \mathcal{F}_{a_n,n}, L^1(\pi_s)) \\
&\qquad \times \max_{f^u:\,[f^l,f^u]\in B(\varepsilon \underline{\tilde{h}}_{a_n,n}, \mathcal{F}_{a_n,n}, L^1(\pi_s))} P\left(P_n T_k(f^u) - ET_k(f^u) > (\xi+\varepsilon)\underline{\tilde{h}}_{a_n,n}\right) \\
&\quad \leq c_1 \varepsilon^{-c_2} \cdot \exp\left\{-\frac{a_n^2(\xi+\varepsilon)^2 \underline{\tilde{h}}_{a_n,n}^2}{a_n E[T_k(f^u)^2] + \frac{1}{3}\sqrt{b_n}(\xi+\varepsilon)\underline{\tilde{h}}_{a_n,n}}\right\} \\
&\quad \leq c_1 \varepsilon^{-c_2} \cdot \exp\left\{-\frac{a_n^2(\xi+\varepsilon)^2 \underline{\tilde{h}}_{a_n,n}^2}{a_n E[U_k(f^u)^2] + (\xi+\varepsilon)o(1)}\right\} \\
&\quad \leq c_4 \cdot \exp\left\{-\frac{a_n \underline{\tilde{h}}_{a_n,n}^2}{\bar{\tilde{h}}_{a_n,n}}\right\} \\
&\quad \leq c_5 \cdot \exp\left\{-a_n^{1-2\underline{\eta}+\overline{\eta}}\right\} \to 0
\end{aligned}$$

where we use $|T_k(f)| \leq \sqrt{b_n}$, the Bernstein inequality in Appendix B of Pollard (1984), and $\frac{1}{5} < \overline{\eta} < \underline{\eta} < 1$. The term in Eq. (34) is bounded in a similar fashion. Therefore, the Borel-Cantelli Lemma implies $\bar{A}_{a_n} \to 0$ a.s. as $n \to \infty$.

Step (3): By Lemma 3.2 in Karlsen and Tjøstheim (2001), we have that $T_n(\mathcal{C}) \to \infty$ a.s. as $n \to \infty$. Fix an ω such that $\bar{A}_{a_n}(\omega) \to 0$. Then $\bar{A}_{T_n(\mathcal{C})}(\omega) \to 0$ so that Eq. (29) follows because this happens for all ω outside a null set. Eq. (28) follows because $T_n(\mathcal{C})/T_n \to \pi_s 1_{\mathcal{C}}$ a.s. as $n \to \infty$. Q.E.D.

Lemma 1. *Under Assumptions 1 and 2:*

$$\sup_{x\in\mathcal{C},\, h\in H_n} |\widehat{p}_h(x) - p_s(x)| = o_{a.s.}(1).$$

Proof. The estimator $\hat{p}_h(x)$ can be decomposed as

$$\widehat{p}_h(x) = \frac{1}{T_n} U_{0,x,h} + \frac{1}{T_n}\sum_{k=1}^{T_n} U_{k,x,h} + \frac{1}{T_n} U_{n,x,h}, \tag{36}$$

Consider the middle term first. By Proposition 1,

$$\sup_{x\in\mathcal{C},\, h\in H_n} \left| \frac{1}{T_n}\sum_{k=1}^{T_n} U_{k,x,h} - E\left(U_{k,x,h}\right) \right| = o_{a.s.}(1). \tag{37}$$

Following the same steps as in the method of proof of Theorem 4.1 in Gao et al. (2011), we have

$$E\left(U_{k,x,h}\right) = \int \frac{1}{h} K\left(\frac{u-x}{h}\right) d\pi_s(u) = \int K(u) p_s(x+hu) d(u),$$

Therefore, by Assumption 1.2, $\sup_{x\in\mathcal{C}} p_s(x) < \infty$ and, thus,

$$\sup_{x\in\mathcal{C},\, h\in H_n} |E\left(U_{k,x,h}\right) - p_s(x)| = o(1). \tag{38}$$

By Eq. (37) and Eq. (38), it remains to show that the boundary terms $\sup_{x\in\mathcal{C},h\in H_n} |\frac{1}{T_n} U_{0,x,h}|$ and $\sup_{x\in\mathcal{C},h\in H_n} |\frac{1}{T_n} U_{n,x,h}|$ are negligible. To this end, notice that Proposition 1 holds with T_n replaced by T_n+1 so that $\sup_{x\in\mathcal{C},h\in H_n} |\frac{1}{T_n} U_{n,x,h}| = o_{a.s.}(1)$. The negligibility of $\sup_{x\in\mathcal{C},h\in H_n} |\frac{1}{T_n} U_{0,x,h}|$ follows similarly as in the proof of Theorem 5.1 in Karlsen and Tjøstheim (2001, p. 405-406). Q.E.D.

Let

$$\overline{\mathcal{F}} := \{f : \mathbb{R}^2 \to \mathbb{R} :\ f(x_1, x_2) = x_2 h K_{x,h}(x_1),\ x \in \mathcal{C},\ 0 < h \leq 1\}.$$

This set has envelope $\overline{F}(x_1, x_2) := |x_2| F(x_1)$, i.e. $|f| \leq \overline{F}$ for all $f \in \overline{\mathcal{F}}$, where $F(x) := c1_D(x)$ and the constant c and the set $D \in \mathcal{E}$ are chosen such that F is the envelope of $\mathcal{F}$. For any $f \in \overline{\mathcal{F}}$, define

$$\overline{U}_k(f) := \begin{cases} \sum_{j=1}^{\tau_0} f(X_{j-1}, X_j), & k = 0 \\ \sum_{j=\tau_{k-1}+1}^{\tau_k} f(X_{j-1}, X_j), & 1 \leq k < n \\ \sum_{j=\tau_{T_n}+1}^{n} f(X_{j-1}, X_j), & k = n \end{cases}$$

and

$$\overline{U}_{k,x,h} := \begin{cases} \frac{1}{h}\sum_{i=1}^{\tau_0} X_i K\left(\frac{X_{i-1}-x}{h}\right) & \text{when } k=0 \\ \frac{1}{h}\sum_{i=\tau_{k-1}+1}^{\tau_k} X_i K\left(\frac{X_{i-1}-x}{h}\right) & \text{for } 1\leq k<n \\ \frac{1}{h}\sum_{\tau_{T_n}+1}^{n} X_i K\left(\frac{X_{i-1}-x}{h}\right) & \text{for } k=n \end{cases} .$$

Proposition 2: Under Assumptions 1 and 2,

$$\sup_{x\in\mathcal{C},\, h\in H_n}\left|\frac{1}{T_n}\sum_{k=1}^{T_n}\overline{U}_{k,x,h} - E\left(\overline{U}_{k,x,h}\right)\right| = o_{a.s.}(1) \tag{39}$$

and

$$\sup_{x\in\mathcal{C},\, h\in H_n}\left|\frac{1}{T_n(\mathcal{C})}\sum_{k=1}^{T_n(\mathcal{C})}\overline{U}_{k,x,h} - E\left(\overline{U}_{k,x,h}\right)\right| = o_{a.s.}(1). \tag{40}$$

Proof. The proof of this proposition is very similar to the one of Proposition 1. By the same argument as in Example (4.3) of Pollard (1986), $\overline{\mathcal{F}}$ is Euclidean for the envelope $\overline{F}$. Therefore, there is a collection of $\varepsilon\underline{\tilde{h}}_{a_n,n}$-brackets $\overline{B}(\varepsilon\underline{\tilde{h}}_{a_n,n}, \overline{\mathcal{F}}_{a_n,n}, L^1(\pi_s))$ of cardinality

$$N_{[]}(\varepsilon\underline{\tilde{h}}_{a_n,n}, \overline{\mathcal{F}}_{a_n,n}, L^1(\pi_s)) \leq N_{[]}(\varepsilon\underline{\tilde{h}}_{a_n,n}, \mathcal{F}_{a_n,n}, L^1(\mu)) \leq N_{[]}(\varepsilon\underline{\tilde{h}}_{a_n,n}, \mathcal{F}, L^1(\mu)),$$

where μ is the measure that has density $|x|$ with respect to π_s and

$$\overline{\mathcal{F}}_{a_n,n} := \{f:\mathbb{R}^2\to\mathbb{R}:\ f(x_1,x_2) = x_2 h K_{x,h}(x_1),\ x\in\mathcal{C},\ h\in\tilde{H}_{a_n,n}\}.$$

The truncation and bracketing argument in the proof of Lemma 1 therefore applies here as well. Q.E.D.

Lemma 1b:

Under Assumptions 1 an 2:

$$\sup_{x\in\mathcal{C},\, h\in H_n}|\widehat{\mu}_h(x) - \mu(x)| = o_{a.s.}(1).$$

Proof. The statement follows from Lemma 1 and a similar calculation as in Eq. (38) to show that the bias is negligible. Q.E.D.

Lemma 2:

Proof. Since

$$\widehat{\mu}_h(x) - \mu(x) = (\widehat{\mu}_h(x) - \mu(x))\frac{\widehat{p}_h(x)}{p_s(x)} - (\widehat{\mu}_h(x) - \mu(x))\left(\frac{\widehat{p}_h(x) - p_s(x)}{p_s(x)}\right),$$

we have

$$\begin{aligned}
& d_{A,h}(\widehat{\mu}_h, \mu) \\
= & \frac{1}{T_n(\mathcal{C})}\sum_{j=1}^{n}\left((\widehat{\mu}_h(X_{j-1}) - \mu(X_{j-1}))\frac{\widehat{p}_h(X_{j-1})}{p_s(X_{j-1})}\right)^2 1\{X_{j-1}\in\mathcal{C}\} \\
& + \frac{1}{T_n(\mathcal{C})}\sum_{j=1}^{n}\left((\widehat{\mu}_h(X_{j-1}) - \mu(X_{j-1}))\frac{(\widehat{p}_h(X_{j-1}) - p_s(X_{j-1}))}{p_s(X_{j-1})}\right)^2 1\{X_{j-1}\in\mathcal{C}\} \\
& + \text{cross} \\
= & I_{n,h} + II_{n,h} + \text{cross}. \qquad (41)
\end{aligned}$$

Letting $\widetilde{\mu}_h(x) = \widehat{\mu}_h(x)\frac{\widehat{p}_h(x)}{p_s(x)}$ and $\mu^*(x) = \mu(x)\frac{\widehat{p}_h(x)}{p_s(x)}$, the term $I_{n,h}$ writes

$$\begin{aligned}
I_{n,h} = & \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}(\widetilde{\mu}_h(X_{j-1})))^2 1\{X_{j-1}\in\mathcal{C}\} \\
& + \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}(\mathrm{E}(\widetilde{\mu}_h(X_{j-1})) - \mu^*(X_{j-1})))^2 1\{X_{j-1}\in\mathcal{C}\} \\
& + \frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}(\mathrm{E}(\widetilde{\mu}_h(X_{j-1})) - \mu^*(X_{j-1}))(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}(\widetilde{\mu}_h(X_{j-1})))1\{X_{j-1}\in\mathcal{C}\} \\
= & I^A_{n,h} + I^B_{n,h} + I^C_{n,h}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I^A_{n,h} = & \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\mathrm{E}(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}(\widetilde{\mu}_h(X_{j-1})))^2 1\{X_{j-1}\in\mathcal{C}\} \\
& + \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\Big((\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}(\widetilde{\mu}_h(X_{j-1})))^2 1\{X_{j-1}\in\mathcal{C}\} \\
& - \mathrm{E}(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}(\widetilde{\mu}_h(X_{j-1})))^2 1\{X_{j-1}\in\mathcal{C}\}\Big) \\
= & I^{A1}_{n,h} + I^{A2}_{n,h}.
\end{aligned}$$

Define the set $\Omega_{1,n,h} = \{\omega : \ n^{\beta-\epsilon} << T_n << n^{\beta+\epsilon}\}$, where the symbol "<<" is used to denote smaller order. By Lemma 3.4 in Karlsen and Tjøstheim (2001), we have

$$\mathrm{P}\left(\lim_{n\to\infty}\Omega_{1,n,h}\right) = 1.$$

Now, working conditionally on $\Omega_{1,n,h}$, we have

$$\begin{aligned} I_{n,h}^{A1} &\geq \inf_{x\in\mathcal{C}} \mathrm{E}\left((\widetilde{\mu}_h(x) - \mathrm{E}(\widetilde{\mu}_h(x)))^2\right) \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} 1\left\{X_{j-1} \in \mathcal{C}\right\} \\ &= \inf_{x\in\mathcal{C}} \mathrm{E}\left((\widetilde{\mu}_h(x) - \mathrm{E}(\widetilde{\mu}_h(x)))^2\right) \\ &\geq \inf_{x\in\mathcal{C}} \sigma^2(x) \frac{1}{\sup_{x\in\mathcal{C}} p_s^2(x)} \frac{1}{n^{\beta+\epsilon}h} \int K^2(u)du \left(1 + o(1)\right) \\ &= O\left(\frac{1}{n^{\beta+\epsilon}h}\right), \end{aligned} \tag{42}$$

where the look of the variance term derives from Eq. (3) and the final order (driven by $T_n(\mathcal{C})$) derives from the fact that T_n and $T_n(\mathcal{C})$ are of the same almost-sure order (c.f. Remark 3.5 in Karlsen and Tjøstheim (2001)). We now show that $I_{n,h}^{A2}$ is of smaller order of magnitude than $I_{n,h}^{A1}$. Write

$$\begin{aligned} I_{n,h}^{A2} &= \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \Big(\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^2 1\left\{X_{j-1} \in \mathcal{C}\right\} \\ &\quad - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^2 1\left\{X_{j-1} \in \mathcal{C}\right\} \Big). \\ &= \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \Big(\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^2 \\ &\quad - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^2 \Big). \end{aligned}$$

It suffices to show that $\sqrt{\mathrm{var}\left(I_{n,h}^{A2}\right)} = o\left(\frac{1}{n^{\beta+\epsilon}h}\right)$. We have,

$$\begin{aligned} & \mathrm{var}\left(I_{n,h}^{A2}\right) \\ \sim\ & \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \mathrm{E}\left(\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^2 - \mathrm{var}\left(\widetilde{\mu}_h(X_{j-1})\right) \right)^2 \\ & + \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \sum_{i=2}^{T_n(\mathcal{C})} \mathrm{E}\Big(1\left\{|X_{i-1} - X_{j-1}| \leq h\right\} \left(\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^2 - \mathrm{var}\left(\widetilde{\mu}_h(X_{j-1})\right) \right) \\ & \left(\left(\widetilde{\mu}_h(X_{i-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{i-1})\right)\right)^2 - \mathrm{var}\left(\widetilde{\mu}_h(X_{i-1})\right) \right) \Big) \\ =\ & A_{n,h} + B_{n,h}, \end{aligned}$$

where the symbol "$\sim$" denotes "asymptotic equivalence." Write,

$$A_{n,h} = \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \mathrm{E}\left(\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) \right) \right)^4 \right) - \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \mathrm{var}^2\left(\widetilde{\mu}_h(X_{j-1}) \right).$$

Recalling that $\overline{U}_{k,h,x} = \sum_{j=\tau_{k-1}}^{\tau_k} \frac{1}{h} K\left(\frac{X_{j-1}-x}{h} \right) \varepsilon_j$, we have

$$\begin{aligned} \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) \right) \right)^4 &\leq \tfrac{1}{T_n(\mathcal{C})} \sup_{x \in \mathcal{C}} \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) \right) \right)^4 \\ &\leq \tfrac{1}{T_n(\mathcal{C})} \tfrac{1}{\inf_{x \in \mathcal{C}} p_s^4(x)} \mathrm{E}\left(\tfrac{1}{T_n} \textstyle\sum_{k=1}^{T_n} \overline{U}_{k,h,x} \right)^4. \end{aligned} \tag{43}$$

Working now with the same subsequence $\{a_n : n \geq 1\}$ as in Proposition 1 and defining h, again, on $\tilde{H}_{a_n,n} := \left\{ h \in \mathbb{R} : \underline{\tilde{h}}_{a_n,n} \leq h \leq \overline{\tilde{h}}_{a_n,n} \right\}$ with $\underline{\tilde{h}}_{a_n,n} := \underline{c} a_n^{-\underline{\eta}}$ and $\overline{\tilde{h}}_{a_n,n} := \overline{c} a_n^{-\overline{\eta}}$, we have

$$\begin{aligned} \frac{1}{a_n^4} \sum_{k=1}^{a_n} \mathrm{E}\left(\overline{U}_{k,h,x}^4 \right) + \frac{1}{a_n^4} \sum_{k=1}^{a_n} \sum_{i=1}^{a_n} \mathrm{E}\left(\overline{U}_{k,h,x}^2 \right) \mathrm{E}\left(\overline{U}_{i,h,x}^2 \right) &= O\left(\tfrac{1}{a_n^3 h^3} \right) + O\left(\tfrac{1}{a_n^2 h^2} \right) \\ &= O\left(\tfrac{1}{a_n^2 h^2} \right) (1 + o(1)), \end{aligned} \tag{44}$$

with the order terms in the last two lines deriving from page 935 in Gao et al. (2015). Combining now Eq. (43) and Eq. (44), and using the fact that $a_n >> n^{\beta-\epsilon}$ and $T_n(\mathcal{C}) >> n^{\beta-\epsilon}$, we have

$$\frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) \right) \right)^4 = O\left(\frac{1}{n^{3(\beta-\epsilon)} h^2} \right) (1 + o(1)).$$

Similarly, it is now immediate to see that

$$\frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \mathrm{var}^2\left(\widetilde{\mu}_h(X_{j-1}) \right) \leq \frac{1}{T_n(\mathcal{C})} \sup_{x \in \mathcal{C}} \mathrm{var}^2\left(\widetilde{\mu}_h(x) \right) = O\left(\frac{1}{n^{3(\beta-\epsilon)} h^2} \right)$$

and, thus, $A_{n,h} = O\left(\frac{1}{n^{3(\beta-\epsilon)} h^2} \right)$. Turning to $B_{n,h}$, write

$$\begin{aligned} & B_{n,h} \\ = \; & \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \sum_{i=2}^{T_n(\mathcal{C})} \mathrm{E}\left(1\left\{ |X_{i-1} - X_{j-1}| \leq h \right\} \left(\widetilde{\mu}_h(X_{j-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{j-1}) \right) \right)^2 \left(\widetilde{\mu}_h(X_{i-1}) - \mathrm{E}\left(\widetilde{\mu}_h(X_{i-1}) \right) \right)^2 \right) \\ & + \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \sum_{i=2}^{T_n(\mathcal{C})} \mathrm{E}\left(1\left\{ |X_{i-1} - X_{j-1}| \leq h \right\} \right) \mathrm{var}\left(\widetilde{\mu}_h(X_{j-1}) \right) \mathrm{var}\left(\widetilde{\mu}_h(X_{i-1}) \right) + \text{cross terms}. \end{aligned} \tag{45}$$

Now, using the same methods as for Eq. (43), we obtain

$$\begin{aligned}
& \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \sum_{i=1}^{T_n(\mathcal{C})} \mathrm{E}\left(1\left\{|X_{i-1}-X_{j-1}|\,1_C \le h\right\}\left(\widetilde{\mu}_h(X_{j-1})-\mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^2\left(\widetilde{\mu}_h(X_{i-1})-\mathrm{E}\left(\widetilde{\mu}_h(X_{i-1})\right)\right)^2\right) \\
\le & \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \sum_{i=1}^{T_n(\mathcal{C})} \sqrt{\mathrm{E}\left(1\left\{|X_{i-1}-X_{j-1}|\,1_C \le h\right\}\right)} \times \\
& \times \sqrt{\mathrm{E}\left(\left(\widetilde{\mu}_h(X_{j-1})-\mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^4\left(\widetilde{\mu}_h(X_{i-1})-\mathrm{E}\left(\widetilde{\mu}_h(X_{i-1})\right)\right)^4\right)} \\
\le & \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \sum_{i=1}^{T_n(\mathcal{C})} \sqrt{\mathrm{E}\left(1\left\{|X_{i-1}-X_{j-1}|\,1_C \le h\right\}\right)} \times \\
& \times \mathrm{E}\left(\left(\widetilde{\mu}_h(X_{j-1})-\mathrm{E}\left(\widetilde{\mu}_h(X_{j-1})\right)\right)^8\right)^{1/4} \times \left(\mathrm{E}\left(\widetilde{\mu}_h(X_{i-1})-\mathrm{E}\left(\widetilde{\mu}_h(X_{i-1})\right)\right)^8\right)^{1/4} \\
= & O\left(\frac{\sqrt{h}}{n^{2(\beta-\epsilon)}h^2}\right),
\end{aligned}$$

since

$$\mathrm{E}\left(1\left\{|X_{i-1}-X_{j-1}| \le h\right\}1_C\right) = O(h).$$

Also, as for Eq. (42), we have

$$\begin{aligned}
& \frac{1}{T_n^2(\mathcal{C})} \sum_{j=2}^{T_n(\mathcal{C})} \sum_{i=2}^{T_n(\mathcal{C})} \mathrm{E}\left(1\left\{|X_{i-1}-X_{j-1}| \le h\right\}\right) \mathrm{var}\left(\widetilde{\mu}_h(X_{j-1})\right) \mathrm{var}\left(\widetilde{\mu}_h(X_{i-1})\right) \\
\le & Ch \sup_{x_1,x_2 \in C} \mathrm{var}\left(\widetilde{\mu}_h(x_1)\right) \mathrm{var}\left(\widetilde{\mu}_h(x_2)\right) = O\left(\frac{h}{n^{2(\beta-\epsilon)}h^2}\right).
\end{aligned}$$

Therefore, $B_{n,h} = O\left(\frac{\sqrt{h}}{n^{2(\beta-\epsilon)}h^2}\right)$. Finally,

$$\begin{aligned}
I_{n,h}^{A2} &= \max\left\{O\left(\frac{h^{1/4}}{n^{(\beta-\epsilon)}h}\right), O\left(\frac{1}{n^{\frac{3}{2}(\beta-\epsilon)}h}\right)\right\} \\
&= o\left(\frac{1}{n^{(\beta+\epsilon)}h}\right).
\end{aligned}$$

if $\beta > 5\epsilon$ and if

$$\frac{\frac{1}{n^{\beta-\epsilon}h^{3/4}}}{\frac{1}{n^{\beta+\epsilon}h}} = n^{2\epsilon}h^{1/4} \to 0.$$

The latter condition becomes

$$n^{2\epsilon} n^{-\frac{1}{4}(\beta-\epsilon)\overline{\eta}} \to 0$$

which, for uniformity, ought to be satisfied in the worst case scenario, i.e., $\overline{\eta} = \frac{1}{5}$. We have that

$$n^{2\epsilon} n^{-\frac{1}{20}(\beta-\epsilon)} \to 0$$

if $\beta > 41\epsilon$, which is, of course, stronger than $\beta > 5\epsilon$, but it always satisfied for a β bounded away from zero. Now, notice that

$$I^B_{n,h} \geq \inf_{x \in \mathcal{C}} \left(\mathrm{E}(\widetilde{\mu}_h(x) - \mu^*(x)) \right).$$

By a similar argument as that in Appendix 1 in Gao et al. (2015), for any $x \in \mathcal{C}$,

$$\begin{aligned}
& \mathrm{E}(\widetilde{\mu}_h(x) - \mu^*(x)) \\
= \; & \mathrm{E}\left(\frac{1}{p_s(x)} \left(\frac{1}{T_n h} \sum_{\iota=1}^{n} K\left(\frac{X_{j-1} - x}{h} \right) (\mu(X_{j-1}) - \mu(x)) \right) \right) \\
= \; & \mathrm{E}_{\upsilon}\left(\frac{1}{p_s(x)} \left(\frac{1}{h} \sum_{j=0}^{\tau_0} K\left(\frac{X_{j-1} - x}{h} \right) (\mu(X_{j-1}) - \mu(x)) \right) \right) \\
= \; & \frac{1}{p_s(x)} \int K\left(\frac{u - x}{h} \right) (\mu(x - hu) - \mu(x)) \, \upsilon G_{s,\upsilon} \mathrm{d}u \\
= \; & \frac{1}{p_s(x)} \int K(u) \, (\mu(x - hu) - \mu(x)) \, p_s(x - hu) \mathrm{d}u.
\end{aligned}$$

Thus, uniformly over $h \in H_n$,

$$I^B_{n,h} \geq \inf_{x \in \mathcal{C}} \frac{1}{p_s^2(x)} \left(\int K(u) \, (\mu(x - hu) - \mu(x)) \, p_s(x - hu) \mathrm{d}u \right)^2.$$

Now, because $I_{n,h}$ is positive, the cross product is either positive (in which case the lower bound is simply given by the sum of $I^A_{n,h}$ and $I^B_{n,h}$) or, if negative, it is bounded by the sum of the other two terms. Hence, there is an α with $0 < \alpha \leq 1$, so that

$$\lim_{n \to \infty} \Pr\left\{ \Omega^c_{n,h} \right\} = 0,$$

with

$$\Omega_{n,h} = \left\{ \varpi : I_{n,h} \geq \alpha (I^A_{n,h} + I^B_{n,h}) \right\}.$$

Now, we turn to $II_{n,h}$. Given Lemma 1, it is clear that $II_{n,h} = o_p\left(I_{n,h}\right)$. Finally, consider the cross-product term. By Cauchy-Schwartz's inequality

$$
\begin{aligned}
\text{cross} &= \frac{1}{T_n(\mathcal{C})}\sum_{j=1}^{n}\left(\left(\widehat{\mu}_h\left(X_{j-1}\right)-\mu(X_{j-1})\right)\frac{\left(\widehat{p}_h(X_{j-1})-p_s(X_{j-1})\right)}{p_s(X_{j-1})}\right)\times \\
&\quad \times\left(\left(\widehat{\mu}_h\left(X_{j-1}\right)-\mu(X_{j-1})\right)\frac{\widehat{p}_h(X_{j-1})}{p_s(X_{j-1})}\right)1\left\{X_{j-1}\in\mathcal{C}\right\} \\
&\leq \sqrt{\frac{1}{T_n(\mathcal{C})}\sum_{j=1}^{n}\left(\left(\widehat{\mu}_h\left(X_{j-1}\right)-\mu(X_{j-1})\right)\frac{\left(\widehat{p}_h(X_{j-1})-p_s(X_{j-1})\right)}{p_s(X_{j-1})}\right)^2 1\left\{X_{j-1}\in\mathcal{C}\right\}}\times \\
&\quad \times\sqrt{\frac{1}{T_n(\mathcal{C})}\sum_{j=1}^{n}\left(\left(\widehat{\mu}_h\left(X_{j-1}\right)-\mu(X_{j-1})\right)\frac{\widehat{p}_h(X_{j-1})}{p_s(X_{j-1})}\right)^2 1\left\{X_{j-1}\in\mathcal{C}\right\}} \\
&\leq \sqrt{I_{n,h}}\sqrt{II_{n,h}} \\
&= o_p(I_{n,h}).
\end{aligned}
$$

Q.E.D.

.

Lemma 3:

Proof. By simple arithmetic,

$$
\begin{aligned}
&\overline{d}_{A,h}\left(\widehat{\mu},\mu\right) \\
&= d_{A,h}\left(\widehat{\mu},\mu\right)+\underbrace{\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{h,j}(X_{j-1})-\widehat{\mu}_h(X_{j-1})\right)^2 1\{X_{j-1}\in\mathcal{C}\}}_{A_n} \\
&\quad \underbrace{+2\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{h,j}(X_{j-1})-\widehat{\mu}_h(X_{j-1})\right)\left(\widehat{\mu}_h(X_{j-1})-\mu(X_{j-1})\right)1\{X_{j-1}\in\mathcal{C}\}}_{B_n}. \qquad (46a)
\end{aligned}
$$

Consider A_n first. Define

$$
\widetilde{\mu}_{h,j}(X_{j-1})=\frac{\frac{1}{T_n}\sum_{i=2,i\neq j}^{n}X_iK_h\left(X_{i-1}-X_{j-1}\right)}{\frac{1}{T_n}\sum_{i=2}^{n}K_h\left(X_{i-1}-X_{j-1}\right)}=\widehat{\mu}_{h,j}(X_{j-1})\frac{\widehat{p}_{h,j}(X_{j-1})}{\widehat{p}_h(X_{j-1})},
$$

with $\widehat{p}_{h,j}(x)=\frac{1}{T_n}\sum_{i=2,i\neq j}^{n}K_h\left(X_{i-1}-x\right)$. Note that

$$
\begin{aligned}
& \left(\widehat{\mu}_{h,j}(X_{j-1}) - \widehat{\mu}_h(X_{j-1})\right)^2 \\
= \; & \left(\widehat{\mu}_{h,j}(X_{j-1}) - \widetilde{\mu}_{h,j}(X_{j-1})\right)^2 \\
& + \left(\widetilde{\mu}_{h,j}(X_{j-1}) - \widehat{\mu}_h(X_{j-1})\right)^2 \\
& + 2\left(\widehat{\mu}_{h,j}(X_{j-1}) - \widetilde{\mu}_{h,j}(X_{j-1})\right)\left(\widetilde{\mu}_{h,j}(X_{j-1}) - \widehat{\mu}_h(X_{j-1})\right).
\end{aligned}
$$

We first show that

$$
\sup_{h \in H_n} \left| \frac{\frac{1}{T_n(\mathcal{C})} \sum_{j=2}^n \left(\widehat{\mu}_{h,j}(X_{j-1}) - \widetilde{\mu}_{h,j}(X_{j-1})\right)^2 w(X_{j-1})}{d_{A,h}\left(\widehat{\mu}, \mu\right)} \right| = o_p(1). \tag{47}
$$

Write,

$$
\begin{aligned}
& \widehat{\mu}_{h,j}(X_{j-1}) - \widetilde{\mu}_{h,j}(X_{j-1}) \\
= \; & \frac{1}{T_n} \sum_{i=2, i \neq j}^n X_i K_h\left(X_{i-1} - X_{j-1}\right) \left(\frac{\widehat{p}_h(X_{j-1}) - \widehat{p}_{h,j}(X_{j-1})}{\widehat{p}_{h,j}(X_{j-1}) \widehat{p}_h(X_{j-1})} \right) \\
= \; & \frac{1}{T_n h} K(0) \frac{1}{T_n} \sum_{i=2, i \neq j}^n \frac{X_i K_h\left(X_{i-1} - X_{j-1}\right)}{\widehat{p}_{h,j}(X_{j-1}) \widehat{p}_h(X_{j-1})}
\end{aligned}
$$

and given that, by Lemma 1 and Assumption 1.2, $\sup_{x \in \mathcal{C}} \widehat{p}_{h,j}(X_j) > 0$ and $\sup_{x \in \mathcal{C}} \widehat{p}_h(X_j) > 0$, we have

$$
\begin{aligned}
& \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^n \left(\widehat{\mu}_{h,j}(X_{j-1}) - \widetilde{\mu}_{h,j}(X_{j-1})\right)^2 1\{X_{j-1} \in \mathcal{C}\} \\
= \; & \frac{1}{T_n^2 h^2} K^2(0) \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^n \left(\frac{1}{T_n} \sum_{i=2, i \neq j}^n \frac{X_i K_h\left(X_{i-1} - X_{j-1}\right)}{\widehat{p}_{h,j}(X_{j-1}) \widehat{p}_h(X_{j-1})} \right)^2 1\{X_{j-1} \in \mathcal{C}\} \\
\leq \; & \frac{1}{T_n^2 h^2} K^2(0) \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^n \left(\frac{\mu(X_{j-1})}{p_s(X_j)} \right)^2 1\{X_{j-1} \in \mathcal{C}\} \\
& + \frac{1}{T_n^2 h^2} K^2(0) \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^n \left(\left(\frac{1}{T_n} \sum_{i=2, i \neq j}^n \frac{X_i K_h\left(X_{i-1} - X_{j-1}\right)}{\widehat{p}_{h,j}(X_{j-1}) \widehat{p}_h(X_{j-1})} \right)^2 - \left(\frac{\mu(X_{j-1})}{p_s(X_j)} \right)^2 \right) 1\{X_{j-1} \in \mathcal{C}\} \\
= \; & \frac{1}{T_n^2 h^2} K^2(0) \left(O_p(1) + o_p(1)\right).
\end{aligned}
$$

Eq. (47) follows since, by Lemma 2, $d_{A,h}\left(\widehat{\mu}, \mu\right)$ is of larger probability order than $1/(T_n h)^2$, uniformly in $h \in H_n$. Now,

$$
\widetilde{\mu}_{h,j}(X_{j-1}) - \widehat{\mu}_h(X_{j-1}) = \frac{1}{\widehat{p}_h(X_{j-1})} K(0) \frac{1}{T_n h} X_j \tag{48}
$$

and so

$$\begin{aligned}
&\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{h,j}(X_{j-1})-\widetilde{\mu}_{h,j}(X_{j-1})\right)\left(\widetilde{\mu}_{h,j}(X_{j-1})-\widehat{\mu}_{h}(X_{j-1})\right)1\{X_{j-1}\in\mathcal{C}\}\\
=\;& K^2(0)\frac{1}{T_n^2h^2}\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\frac{1}{T_n}\sum_{i=2,i\neq j}^{n}\frac{X_iK_h\left(X_{i-1}-X_{j-1}\right)}{\widehat{p}_{h,j}(X_{j-1})\widehat{p}_h(X_{j-1})}\right)\frac{X_j}{\widehat{p}_h(X_{j-1})}1\{X_{j-1}\in\mathcal{C}\}\\
=\;& o_p\left(d_{A,h}\left(\widehat{\mu},\mu\right)\right),
\end{aligned}$$

uniformly in $h\in H_n$, because of, again, Lemma 2 and the definition of H_n. From Eq. (48) it is now immediate to see that

$$\begin{aligned}
&\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widetilde{\mu}_{h,j}(X_{j-1})-\widehat{\mu}_{h}(X_{j-1})\right)^2 1\{X_{j-1}\in\mathcal{C}\}\\
=\;& K^2(0)\frac{1}{T_n^2h^2}\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\frac{X_j^2}{\widehat{p}_h^2(X_{j-1})}1\{X_{j-1}\in\mathcal{C}\}\\
=\;& o_p\left(d_{A,h}\left(\widehat{\mu},\mu\right)\right)
\end{aligned}$$

uniformly in $h\in H_n$. Hence, A_n in Eq. (46a) is $o_p\left(d_{A,h}\left(\widehat{\mu},\mu\right)\right)$ uniformly in $h\in H_n$, because of Assumption 2.1 and Lemma 2. As for B_n, by Cauchy-Schwartz's inequality,

$$\begin{aligned}
&\left|\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{h,j}(X_{j-1})-\widehat{\mu}_{h}(X_{j-1})\right)\left(\widehat{\mu}_{h}(X_{j-1})-\mu(X_{j-1})\right)1\{X_{j-1}\in\mathcal{C}\}\right|\\
\leq\;& \left(\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{h}(X_{j-1})-\mu(X_{j-1})\right)^2 1\{X_{j-1}\in\mathcal{C}\}\right)^{1/2}\\
&\times\left(\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{h,j}(X_{j-1})-\widehat{\mu}_{h}(X_{j-1})\right)^2 1\{X_{j-1}\in\mathcal{C}\}\right)^{1/2}\\
=\;& \sqrt{d_{A,h}\left(\widehat{\mu},\mu\right)}o_p\left(\sqrt{d_{A,h}\left(\widehat{\mu},\mu\right)}\right)=o_p\left(d_{A,h}\left(\widehat{\mu},\mu\right)\right).
\end{aligned}$$

Q.E.D.

Lemma 4:

Proof. For Cross(h) defined as in Eq. (14) and u_j defined as in Eq. (1), write

$$\begin{aligned}
&\text{Cross}(h)\\
=\;& \frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{h,j}(X_{j-1})-\mu(X_{j-1})\right)\sigma(X_{j-1})u_j 1\{X_{j-1}\in\mathcal{C}\}\\
=\;& \frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\frac{1}{T_n}\sum_{i=2,i\neq j}^{n}K_h\left(X_{i-1}-X_{j-1}\right)\sigma(X_{i-1})u_i\right)\sigma(X_{j-1})u_j\frac{1\{X_{j-1}\in\mathcal{C}\}}{\widehat{p}_{h,j}(X_{j-1})}\\
&+\frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\frac{1}{T_n}\sum_{i=2,i\neq j}^{n}K_h\left(X_{i-1}-X_{j-1}\right)\left(\mu(X_{i-1})-\mu(X_{j-1})\right)\right)\sigma(X_{j-1})u_j\frac{1\{X_{j-1}\in\mathcal{C}\}}{\widehat{p}_{h,j}(X_{j-1})}\\
=\;& A_{n,h}+B_{n,h}.
\end{aligned}$$

Let $\widetilde{A}_{n,h}$ and $\widetilde{B}_{n,h}$ be defined as $A_{n,h}$ and $B_{n,h}$ but with $\widehat{p}_{h,j}(X_{j-1})$ replaced by $p_s(X_{j-1})$. Given Lemma 1, it is enough to show that

$$\sup_{h\in H_n}\left|\frac{\widetilde{A}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|=o_p(1) \tag{49}$$

and

$$\sup_{h\in H_n}\left|\frac{\widetilde{B}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|=o_p(1), \tag{50}$$

where $h\in H_n$. Using the split chain decomposition,

$$\begin{aligned}
&\widetilde{A}_{n,h}\\
=\;& \frac{T_n}{T_n(\mathcal{C})}\frac{1}{T_n^2}\sum_{k=1}^{T_n}\sum_{k'=1}^{T_n}\left(\frac{1}{h}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1,i_{k'}\neq j_k}^{\tau_{k'}}K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\right.\\
&\left.\sigma(X_{i_{k'}-1})u_{i_{k'}}\sigma(X_{j_k-1})u_{j_k}\frac{1\{X_{j_{k-1}}\in\mathcal{C}\}}{p_s(X_{j_{k-1}})}\right)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{B}_{n,h} \;=\;& \frac{T_n}{T_n(\mathcal{C})}\frac{1}{T_n^2}\sum_{k=1}^{T_n}\sum_{k'=1}^{T_n}\left(\frac{1}{h}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1,i_{k'}\neq j_k}^{\tau_{k'}}K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\right.\\
&\left.\left(\mu(X_{i_{k'}-1})-\mu(X_{j_k-1})\right)\sigma(X_{j_k-1})u_{j_k}\frac{1\{X_{j_{k-1}}\in\mathcal{C}\}}{p_s(X_{j_{k-1}})}\right).
\end{aligned}$$

We introduce the same subsequence $\{a_n:\,n\geq 1\}$ as in Proposition 1 and define h, again, on $\tilde{H}_{a_n,n}:=\left\{h\in\mathbb{R}:\,\underline{\tilde{h}}_{a_n,n}\leq h\leq\overline{\tilde{h}}_{a_n,n}\right\}$ with $\underline{\tilde{h}}_{a_n,n}:=\underline{c}a_n^{-\underline{\eta}}$ and $\overline{\tilde{h}}_{a_n,n}:=\overline{c}a_n^{-\overline{\eta}}$. Define a_n

bandwidths h^* partitioning the space $\tilde{H}_{a_n,n}$ so that $|h^* - h| \leq a_n^{-1}(a_n^{-\overline{\eta}} - a_n^{-\underline{\eta}})$, for every h and at least one h^*. The partition defines balls $H_{a_n,j}$ so that $\tilde{H}_{a_n,n} = \cup_{j=1}^{a_n} H_{a_n,j}$.
We begin by evaluating $\widetilde{A}_{n,h}$. By Boole's and Markov's inequality,

$$\begin{aligned}
&\Pr\left(\sup_{h\in\tilde{H}_{a_n,n}}\left|\frac{\widetilde{A}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|>\zeta\right) \leq \Pr\left(\max_{1\leq j\leq a_n}\sup_{h\in H_{a_n,j}}\left|\frac{\widetilde{A}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|>\zeta\right)\\
\leq\ & a_n \Pr\left(\sup_{h\in H_{a_n,j}}\left|\frac{\widetilde{A}_{n,h_j}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|>\zeta\right) \leq \zeta^{-2\kappa} a_n \mathrm{E}\left(\sup_{h\in H_{a_n,j}}\left|\frac{\widetilde{A}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|\right)^{2\kappa}. \qquad (51)
\end{aligned}$$

Conditioning now on the ω's in $\Omega_{n,h}$, where $\Omega_{n,h}$ was defined in the statement of Lemma 2, we have

$$\begin{aligned}
&\mathrm{E}\left(\sup_{h\in H_{a_n,j}}\left|\frac{\widetilde{A}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|\right)^{2\kappa}\\
=\ & \mathrm{E}\left(\sup_{h\in H_{a_n,j}}\left|\widetilde{A}_{n,h}\right|\frac{1}{d_{A,h}\left(\widehat{\mu},\mu\right)}1\left\{\omega\in\Omega_{n,h}\right\}\right)^{2\kappa}\Pr\left(\Omega_{n,h}\right)\\
&+\mathrm{E}\left(\sup_{h\in H_{a_n,j}}\left|\widetilde{A}_{n,h}\right|\frac{1}{d_{A,h}\left(\widehat{\mu},\mu\right)}1\left\{\omega\in\Omega_{n,h}^c\right\}\right)^{2\kappa}\Pr\left(\Omega_{n,h}^c\right)\\
=\ & \mathrm{E}\left(\sup_{h\in H_{a_n,j}}\left|\widetilde{A}_{n,h}\right|\frac{1}{d_{A,h}\left(\widehat{\mu},\mu\right)}1\left\{\omega\in\Omega_{n,h}\right\}\right)^{2\kappa}\Pr\left(\Omega_{n,h}\right)\left(1+o(1)\right).
\end{aligned}$$

Let $l_{n,h} = \frac{c}{hn^{\beta+\varepsilon}} + \inf_{x\in\mathcal{C}}|b_{\mathcal{C}}^2(x,h)|$, as in Lemma 2. Thus,

$$\begin{aligned}
&\mathrm{E}\left(\sup_{h\in H_{n,j}}\left|\widetilde{A}_{n,h}\right|\frac{1}{d_{A,h}\left(\widehat{\mu},\mu\right)}1\left\{\omega\in\Omega_{n,h}\right\}\right)^{2\kappa}\Pr\left(\Omega_{n,h}\right)\\
\leq\ & \sup_{h\in H_{n,j}}\left(\frac{1}{l_{n,h}^{2\kappa}}\right)\mathrm{E}\left(\sup_{h\in H_{n,j}}\left|\widetilde{A}_{n,h}\right|\right)^{2\kappa}(1+o(1)).
\end{aligned}$$

Letting $u_{i_{k'}}^{\sigma} = \sigma(X_{i_{k'}-1})u_{i_{k'}}$ and $u_{j_k}^{\sigma} = \sigma(X_{j_k-1})u_{j_k}$, write

$$\widetilde{A}_{n,h} = \frac{1}{a_n^2}\sum_{k=1}^{a_n}\sum_{k'=1}^{a_n}\left(\frac{1}{h}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1,i'_k\neq j_k}^{\tau_{k'}} K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)u_{i_{k'}}^{\sigma}u_{j_k}^{\sigma}\frac{1\{X_{j_{k-1}}\in\mathcal{C}\}}{p_s(X_{j_{k-1}})}\right).$$

As in the proof of Lemma 4 in Härdle and Marron (1985), we employ Theorem 2 of Whittle (1960):

$$
\begin{aligned}
& \mathrm{E}\left(\left|\widetilde{A}_{n,h}\right|\right)^{2\kappa} \\
= \;& \mathrm{E}\left(\mathrm{E}\left(\left|\widetilde{A}_{n,h}\right|\right)^{2\kappa} \left| X_{i_{k'}}, X_{j_k-1}, \tau_k, \tau_{k'}, k, k' \leq a_n \right.\right) \\
\leq \;& C\mathrm{E}\left(\frac{1}{a_n^4}\sum_{k=1}^{a_n}\sum_{k'=1}^{a_n}\right. \\
& \left.\left(\frac{1}{h}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1, i'_k\neq j_k}^{\tau_{k'}} K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\sigma(X_{i_{k'}-1})\sigma(X_{j_k-1})\frac{1\{X_{j_{k-1}}\in\mathcal{C}\}}{p_s(X_{j_{k-1}})}\right)^2\right)^{\kappa} \\
\leq \;& Ca_n^{-4\kappa}h^{-2\kappa}a_n^{2\kappa}\left(\mathrm{E}\left(\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1, i'_k\neq j_k}^{\tau_{k'}} K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\sigma(X_{i_{k'}-1})\sigma(X_{j_k-1})\frac{1\{X_{j_{k-1}}\in\mathcal{C}\}}{p_s(X_{j_{k-1}})}\right)^2\right)^{\kappa} \\
\leq \;& Ca_n^{-4\kappa}h^{-2\kappa}a_n^{2\kappa}h^{\kappa} \\
\leq \;& Ca_n^{-2\kappa}h^{-\kappa},
\end{aligned}
$$

where the second to last inequality derives from the fact that

$$
\begin{aligned}
& \left(\mathrm{E}\left(\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1, i'_k\neq j_k}^{\tau_{k'}} K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\sigma(X_{i_{k'}-1})\sigma(X_{j_k-1})\frac{1\{X_{j_{k-1}}\in\mathcal{C}\}}{p_s(X_{j_{k-1}})}\right)^2\right)^{\kappa} \\
= \;& h^{2\kappa}\left(\mathrm{E}\left(\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1, i'_k\neq j_k}^{\tau_{k'}} h^{-1}K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\sigma(X_{i_{k'}-1})\sigma(X_{j_k-1})\frac{1\{X_{j_{k-1}}\in\mathcal{C}\}}{p_s(X_{j_{k-1}})}\right)^2\right)^{\kappa} \\
\leq \;& Ch^{\kappa},
\end{aligned}
$$

since, by Lemma B1 in Gao et al. (2011), $\mathrm{E}\left(\sum_{i_{k'}=\tau_{k'-1}+1, i'_k\neq j_k}^{\tau_{k'}} h^{-1}K\left(\frac{X_{i_{k'}-1}-x}{h}\right)\right)^2 \leq \widetilde{C}h^{-1}$ where $\widetilde{C}$ does not depend on either x or h and since the number of terms in the sum $\sum_{j_k=\tau_{k-1}+1}^{\tau_k}$ is finite almost surely. We also use the fact that $\sigma(x)$ is bounded for $x \in \mathcal{C}$ given Assumption 1.5. Thus,

$$
\begin{aligned}
& \Pr\left(\sup_{h\in H_{a_n,j}}\left|\frac{\widetilde{A}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|>\zeta\right) \\
\leq\ & \zeta^{-2\kappa}a_n\mathrm{E}\left(\sup_{h\in H_{n,j}}\left|\frac{\widetilde{A}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|\right)^{2\kappa} \\
\leq\ & \zeta^{-2\kappa}a_n\frac{a_n^{-2\kappa}h^{-\kappa}}{h^{-2\kappa}n^{-2k(\beta+\epsilon)}} \\
\leq\ & \zeta^{-2\kappa}\frac{a_n^{1-2\kappa}h^{\kappa}}{n^{-2\kappa(\beta+\epsilon)}}.
\end{aligned}
$$

Now, using the fact that $a_n >> n^{\beta-\epsilon}$, we have that the bound becomes

$$
\zeta^{-2\kappa}\frac{a_n^{1-2\kappa}h^{\kappa}}{n^{-2\kappa(\beta+\epsilon)}} \leq \zeta^{-2\kappa}\frac{n^{(1-2\kappa)(\beta-\epsilon)}\overline{h}^{\kappa}}{n^{-2\kappa(\beta+\epsilon)}} \leq \zeta^{-2\kappa}n^{\beta+(4\kappa-1)\epsilon}\overline{h}^{\kappa} \leq \zeta^{-2\kappa}n^{\beta+(4\kappa-1)\epsilon}n^{-\kappa(\beta-\epsilon)\overline{\eta}} \to 0
$$

for $\kappa > \frac{\beta-\epsilon}{(\beta-\varepsilon)\overline{\eta}-4\varepsilon}$ (which we assume in Assumption 2.2) and $\overline{\eta}$ defined in Eq. (16). The statement in Eq. (49) then follows.

We now turn to $\widetilde{B}_{n,h}$. It suffices to show that

$$
\sup_{h\in H_n}\left|\frac{\widetilde{\widetilde{B}}_{n,h}}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right| = o_p(1),
$$

where

$$
\begin{aligned}
\widetilde{\widetilde{B}}_{n,h} \ =\ & \frac{1}{a_n^2}\sum_{k=1}^{a_n}\sum_{k'=1}^{a_n}\left(\frac{1}{h}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1,i'_k\neq j_k}^{\tau_{k'}}K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\right. \\
& \left.\left(\mu(X_{i_{k'}-1})-\mu(X_{j_k-1})\right)\sigma(X_{j_k-1})u_{j_k}1\{X_{j_k}\in\mathcal{C}\}\right).
\end{aligned}
$$

Using, again, Theorem 2 of Whittle (1960), we have

$$
\begin{aligned}
& \mathrm{E}\left(\left|\widetilde{\widetilde{B}}_{n,h}\right|\right)^{2\kappa} \\
= \;& \mathrm{E}\left(\mathrm{E}\left(\left|\widetilde{\widetilde{B}}_{n,h}\right|\right)^{2\kappa} \left| X_{i_{k'}}, X_{j_k-1}, \tau_k, \tau_{k'}, k, k' \leq a_n \right.\right) \\
\leq \;& C\mathrm{E}\left(\frac{1}{a_n^4}\sum_{k=1}^{a_n}\sum_{k'=1}^{a_n}\left(\frac{1}{h}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1, i'_k\neq j_k}^{\tau_{k'}} \right.\right. \\
& \left.\left. K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\left(\mu(X_{i_{k'}-1})-\mu(X_{j_k-1})\right)\sigma(X_{j_k-1})1\{X_{j_k}\in\mathcal{C}\}\right)^2\right)^{\kappa}. \qquad (52)
\end{aligned}
$$

The right hand-side of the inequality in Eq. (52) is majorized by

$$
\begin{aligned}
& a_n^{-4\kappa}h^{-2\kappa}\sum_{l=2}^{2\kappa}a_n^l\mathrm{E}\left(\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1, i'_k\neq j_k}^{\tau_{k'}} K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\right. \\
& \left.\left(\mu(X_{i_{k'}-1})-\mu(X_{j_k-1})\right)\sigma(X_{j_k-1})1\{X_{j_k}\in\mathcal{C}\}\right)^l \\
\leq \;& Ca_n^{-2\kappa}h^{-2\kappa} \\
& \left(\mathrm{E}\left(\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1, i_{k'}\neq j_k}^{\tau_{k'}} K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{h}\right)\sigma(X_{j_k-1})\frac{1\{X_{j_{k-1}}\in\mathcal{C}\}}{p_s(X_{j_{k-1}})}\right)^2\right)^{\kappa},
\end{aligned}
$$

where the last inequality follows from the fact that, for $X_{j_k-1}\in\mathcal{C}$, both $\sigma(X_{j_k-1})$ and $\left(\mu(X_{i_{k'}-1})-\mu(X_{j_k-1})\right)^l$ are bounded. The statement in Eq. (50) derives from the same argument used to prove Eq. (49).

Q.E.D.

Theorem 1:

Proof. Define

$$
\begin{aligned}
\widehat{h} &= \arg\min_h \left[CV(h)\right], \\
\overline{h} &= \arg\min_h \left[\overline{CV}(h)\right] = \arg\min_h \left[CV(h) + \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(X_j-\mu(X_{j-1})\right)^2 1\left\{X_{j-1}\in\mathcal{C}\right\}\right].
\end{aligned}
$$

It follows that $\overline{h} \overset{a.s.}{=} \widehat{h}$, since the second term in the argmin does not depend on the bandwidth. Now, write

$$\begin{aligned}
&\left|\frac{\overline{CV}(\overline{h})}{\inf_{h\in H_n} d_{A,h}\left(\widehat{\mu},\mu\right)} - 1\right| \\
= &\sup_{h\in H_n}\left|\frac{\overline{d}_{A,\overline{h}}\left(\widehat{\mu},\mu\right) - \text{Cross}(\overline{h})}{d_{A,h}\left(\widehat{\mu},\mu\right)} - 1\right| \\
\leq &\sup_{h\in H_n}\left|\frac{\overline{d}_{A,\overline{h}}\left(\widehat{\mu},\mu\right)}{d_{A,h}\left(\widehat{\mu},\mu\right)} - 1\right| + \sup_{h\in H_n}\left|\frac{\text{Cross}(\overline{h})}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right| \\
= &\ o_p(1), \qquad (53)
\end{aligned}$$

by Lemma 3 and 4. Finally,

$$\begin{aligned}
&\left|\frac{d_{A,\overline{h}}\left(\widehat{\mu},\mu\right)}{\inf_{h\in H_n} d_{A,h}\left(\widehat{\mu},\mu\right)} - 1\right| \\
= &\left|\frac{\overline{CV}(\overline{h}) + \text{Cross}(\overline{h}) - \left(\overline{d}_{A,\overline{h}}\left(\widehat{\mu},\mu\right) - d_{A,\overline{h}}\left(\widehat{\mu},\mu\right)\right)}{\inf_{h\in H_n} d_{A,h}\left(\widehat{\mu},\mu\right)} - 1\right| \\
\leq &\left|\frac{\overline{CV}(\overline{h})}{\inf_{h\in H_n} d_{A,h}\left(\widehat{\mu},\mu\right)} - 1\right| + \sup_{h\in H_n}\left|\frac{\text{Cross}(\overline{h})}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right| \\
&+ \sup_{h\in H_n}\left|\frac{\overline{d}_{A,\overline{h}}\left(\widehat{\mu},\mu\right) - d_{A,\overline{h}}\left(\widehat{\mu},\mu\right)}{d_{A,h}\left(\widehat{\mu},\mu\right)}\right|
\end{aligned}$$

by Eq. (53), Lemma 3, and Lemma 4.

Q.E.D.

A.2 Proofs of Section 4

Lemma 5:

Proof. Because of Lemma 1, we can write

$$\sup_{x,\xi\in\Delta_n}\left|\widetilde{\sigma}^2_\xi(x)\frac{\widehat{p}_\xi(x)}{p_s(x)} - \widetilde{\sigma}^2_\xi(x)\right| = o_{a.s.}(1)$$

and, thus, we can prove the statement in the lemma by replacing $\widetilde{\sigma}_{\xi}^{2}(x)$ with $\widetilde{\widetilde{\sigma}}_{\xi}^{2}(x) = \widetilde{\sigma}_{\xi}^{2}(x)\frac{\widehat{p}_{\xi}(x)}{p_s(x)}$ and $\sigma^2(x)$ with $\sigma^{2*}(x) = \sigma^2(x)\frac{\widehat{p}_{\xi}(x)}{p_s(x)}$. The result then follows from the same argument as in the proof of Lemma 2.

Q.E.D.

Proof of Lemma 6:

Proof. Let

$$\begin{aligned} d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right) &= \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widetilde{\sigma}_{\xi}^{2}(X_{j-1}) - \sigma^2(X_{j-1})\right) 1\left\{X_{j-1} \in \mathcal{C}\right\}, \\ \overline{d}_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right) &= \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widetilde{\sigma}_{\xi,j}^{2}(X_{j-1}) - \sigma^2(X_{j-1})\right) 1\left\{X_{j-1} \in \mathcal{C}\right\}. \end{aligned}$$

By the same argument as in Lemma 3,

$$\sup_{\xi\in\Xi_n}\left|\frac{d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right) - \overline{d}_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right)}{d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right)}\right| = o_p(1). \tag{54}$$

Now,

$$\begin{aligned} &\widetilde{\mathrm{CV}}\left(\xi\right) \\ = \;& \overline{d}_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right) - \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})\left(u_j^2 - 1\right)\right)^2 1\left\{X_{j-1} \in \mathcal{C}\right\} \\ & -\frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widetilde{\sigma}_{j,\xi}^{2}(X_{j-1}) - \sigma^2(X_{j-1})\right)\left(\sigma^2(X_{j-1})\left(u_j^2 - 1\right)\right) 1\left\{X_{j-1} \in \mathcal{C}\right\} \\ = \;& \overline{d}_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right) - \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})\left(u_j^2 - 1\right)\right)^2 1\left\{X_{j-1} \in \mathcal{C}\right\} + \widetilde{\mathrm{Cross}}(\xi), \end{aligned} \tag{55}$$

and, by the same argument as in the proof of Lemma 4,

$$\sup_{\xi\in\Xi_n}\left|\frac{\widetilde{\mathrm{Cross}}(\xi)}{d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2\right)}\right| = o_p(1).$$

The statement then follows from the proof of Theorem 1. Q.E.D.

Lemma 7

Proof. By triangle inequality,

$$\sup_{x\in\mathcal{C}}|\widehat{\mu}_{\widehat{h}_n}(x)-\mu(x)|^4 \leq C\left(\sup_{x\in\mathcal{C}}|\widehat{\mu}_{\widehat{h}_n}(x)-\mathrm{E}\left(\widehat{\mu}_{\widehat{h}_n}(x)\right)|^4+\sup_{x\in\mathcal{C}}|\mathrm{E}\left(\widehat{\mu}_{\widehat{h}_n}(x)\right)-\mu(x)|^4\right).$$

Following the proof of Lemma 2, we have

$$\begin{aligned} & |\mathrm{E}\left(\widehat{\mu}_{\widehat{h}_n}(x)-\mu(x)\right)| \\ = & \left|\frac{1}{p_s(x)}\int K(u)\left(\mu(x-\widehat{h}_n u)-\mu(x)\right)p_s(x-\widehat{h}_n u)\mathrm{d}u\right|. \end{aligned}$$

Finally, using the limiting orders in Lemma 3.4 of Karlsen and Tjøstheim (2001), uniformly in $x\in\mathcal{C}$,

$$\sup_{x\in\mathcal{C}}|\widehat{\mu}_{\widehat{h}_n}(x)-\mathrm{E}\left(\widehat{\mu}_{\widehat{h}_n}(x)\right)|^4 = O_p\left(\frac{1}{n^{2(\beta-\varepsilon)}\widehat{h}_n^2}\right).$$

Then, by Lemma 2, we have $\sup_{x\in\mathcal{C}}|\widehat{\mu}_{\widehat{h}_n}(x)-\mu(x)|^4 = O\left(\sup_{x\in\mathcal{C}} b_{\mathcal{C}}^4(x,\widehat{h}_n)\right)+O_p\left(\frac{1}{n^{2(\beta-\varepsilon)}\widehat{h}_n^2}\right) = o_p\left(d_{A,\widehat{h}_n}\left(\widehat{\mu},\mu\right)\right)$.

Q.E.D.

Lemma 8:

Proof. We need to show that Eqs. (18)-(24) are $o_p\left(d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)\right)$ uniformly in ξ. We begin with Eq. (18). Because of Lemma 1, we replace the density estimator in the denominators with the true density. Hence, uniformly in $\xi\in\Xi_n$, we have

$$\begin{aligned} & \widehat{\sigma}_{j,\xi}^2(X_{j-1})-\widetilde{\sigma}_{j,\xi}^2(X_{j-1}) \\ = & \left(\frac{1}{T_n\xi}\sum_{i\neq j}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1})-\mu(X_{i-1})\right)^2\right. \\ & \left.-\frac{2}{T_n\xi}\sum_{i\neq j}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\sigma(X_{i-1})u_i\left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1})-\mu(X_{i-1})\right)\right)(1+o_{a.s.}(1)) \end{aligned} \tag{156}$$

Neglecting the smaller order term,

$$
\begin{aligned}
& \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\sigma}_{j,\xi}^2\left(X_{j-1}\right)-\widetilde{\sigma}_{j,\xi}^2(X_{j-1})\right)^2 1_{\mathcal{C}}(X_{j-1}) \\
= & \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\frac{1}{T_n\xi}\sum_{i\neq j}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1})-\mu(X_{i-1})\right)^2\right)^2 1_{\mathcal{C}}(X_{j-1}) \\
& +\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\frac{2}{T_n\xi}\sum_{i\neq j}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\sigma(X_{i-1})u_i\left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1})-\mu(X_{i-1})\right)\right)^2 1_{\mathcal{C}}(X_{j-1}) \\
& -\frac{2}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\frac{1}{T_n\xi}\sum_{i\neq j}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1})-\mu(X_{i-1})\right)^2\right. \\
& \left.\times\frac{2}{T_n\xi}\sum_{i\neq j}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\sigma(X_{i-1})u_i\left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1})-\mu(X_{i-1})\right)\right) \\
= & A1_{n,\xi}+B1_{n,\xi}+C1_{n,\xi}.
\end{aligned}
$$

Now, write

$$
\begin{aligned}
A1_{n,\xi} &\leq \sup_{x\in\mathcal{C}}\left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1})-\mu(X_{i-1})\right)^4 O_p(1) \\
&= o_p\left(d_{A,\widehat{h}_n}\left(\widehat{\mu},\mu\right)\right)=o_p\left(\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)\right),
\end{aligned}
$$

where the first equality follows from Lemma 7 and the last equality follows from Lemma 5. As for $B1_{n,\xi}$, write

$$
\begin{aligned}
& B1_{n,\xi} \\
= & \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\frac{2}{T_n\xi}\sum_{i\neq j}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\sigma(X_{i-1})u_i\right. \\
& \left.\times\frac{1}{T_n\widehat{h}_n}\sum_{k=1}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{k-1}-X_{i-1}}{\widehat{h}_n}\right)\sigma(X_{k-1})u_k\right)^2 1_{\mathcal{C}}(X_{j-1}) \\
& +\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\frac{2}{T_n\xi}\sum_{i\neq j}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\sigma(X_{i-1})u_i\right. \\
& \left.\times\frac{1}{T_n\widehat{h}_n}\sum_{k=1}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{k-1}-X_{i-1}}{\widehat{h}_n}\right)\left(\mu(X_{k-1})-\mu(X_{i-1})\right)\right)^2 1_{\mathcal{C}}(X_{j-1}) \\
& +\text{cross term}.
\end{aligned}
$$

Now, $B1_{n,\xi}$ is of smaller probability order than Cross(h) as defined at the beginning of the proof of Lemma 4. Hence, it is $o_p\left(\inf_{h\in H_n} d_{A,h}\left(\widehat{\mu},\mu\right)\right) = o_p\left(\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)\right)$. Also, $C1_{n,\xi} = o_p\left(\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)\right)$ by Cauchy-Schwartz inequality. This proves that Eq. (18) is $o_p\left(\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)\right)$. As for Eq. (19), given Lemma 7, we have

$$\begin{aligned}
&\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\left(\sigma^2(X_{j-1})-\widetilde{\sigma}^2_{j,\xi}(X_{j-1})\right)\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)^2\right)1_{\mathcal{C}}(X_{j-1})\\
\leq\ &\left(\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})-\widetilde{\sigma}^2_{j,\xi}(X_{j-1})\right)^2 1_{\mathcal{C}}(X_{j-1})\right)^{1/2}\\
&\times\left(\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\mu}_{\widehat{h}_n}(X_{j-1})-\mu(X_{j-1})\right)^4 1_{\mathcal{C}}(X_{j-1})\right)^{1/2}\\
=\ &O_p\left(\sqrt{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}\right)\sup_{x\in\mathcal{C}}\left(\widehat{\mu}_{\widehat{h}_n}(x)-\mu(x)\right)^2\\
=\ &o_p\left(d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)\right).
\end{aligned}$$

Turning to Eq. (21),

$$\begin{aligned}
&\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\left(\sigma^2(X_{j-1})-\widetilde{\sigma}^2_{j,\xi}(X_{j-1})\right)\left(\widehat{\sigma}^2_{j,\xi}-\widetilde{\sigma}^2_{j,\xi}\right)\right)1_{\mathcal{C}}(X_{j-1})\\
\leq\ &\left(\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})-\widetilde{\sigma}^2_{j,\xi}(X_{j-1})\right)^2 1_{\mathcal{C}}(X_{j-1})\right)^{1/2}\\
&\times\left(\frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\widehat{\sigma}^2_{j,\xi}-\widetilde{\sigma}^2_{j,\xi}\right)^2 1_{\mathcal{C}}(X_{j-1})\right)^{1/2}\\
=\ &\left(\overline{d}_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)\right)^{1/2} o_p\left(\sqrt{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}\right)\\
=\ &O_p\left(d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)^{1/2}\right) o_p\left(\sqrt{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}\right),
\end{aligned}$$

where the last equality derives from Eq. (54). As for Eq. (23), recalling Lemma 7, we obtain

$$
\begin{aligned}
& \frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\left(\widehat{\sigma}^2_{j,\xi} - \widetilde{\sigma}^2_{j,\xi} \right) \left(\widehat{\mu}_{j,\widehat{h}_n}(X_{j-1}) - \mu(X_{j-1}) \right)^2 \right) 1_{\mathcal{C}}(X_{j-1}) \\
\leq & \left(\frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\widehat{\sigma}^2_{j,\xi} - \widetilde{\sigma}^2_{j,\xi} \right)^2 1_{\mathcal{C}}(X_{j-1}) \right)^{1/2} \\
& \times \left(\frac{1}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\widehat{\mu}_{j,\widehat{h}_n}(X_{j-1}) - \mu(X_{j-1}) \right)^4 1_{\mathcal{C}}(X_{j-1}) \right)^{1/2} \\
= & o_p\left(\sqrt{d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2 \right)} \right) o_p\left(\sqrt{d_{A,\widehat{h}_n}\left(\widehat{\mu}, \mu \right)} \right) \\
= & o_p\left(\sqrt{d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2 \right)} \right) o_p\left(\sqrt{\inf_{\xi \in \Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2 \right)} \right) \\
= & o_p\left(d_{A,\xi}\left(\widetilde{\sigma}^2, \sigma^2 \right) \right).
\end{aligned}
$$

We now turn to Eq. (22), which, given (56), writes as

$$
\begin{aligned}
& \frac{2}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\sigma^2(X_{j-1})(u_j^2 - 1) \left(\widehat{\sigma}^2_{j,\xi}(X_{j-1}) - \widetilde{\sigma}^2_{j,\xi}(X_{j-1}) \right) \right) 1_{\mathcal{C}}(X_{j-1}) \\
= & \frac{2}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\sigma^2(X_{j-1})(u_j^2 - 1) \right. \\
& \left. \left(\frac{1}{T_n \xi} \sum_{i \neq j}^{n} \frac{1}{p_s(X_{j-1})} K\left(\frac{X_{i-1} - X_{j-1}}{\xi} \right) \left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1}) - \mu(X_{i-1}) \right)^2 \right) \right) 1_{\mathcal{C}}(X_{j-1}) \\
& + \frac{2}{T_n(\mathcal{C})} \sum_{j=2}^{n} \left(\sigma^2(X_{j-1})(u_j^2 - 1) \frac{1}{T_n \xi} \sum_{i \neq j}^{n} \frac{1}{p_s(X_{j-1})} K\left(\frac{X_{i-1} - X_{j-1}}{\xi} \right) \right. \\
& \left. \sigma(X_{i-1}) u_i \left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1}) - \mu(X_{i-1}) \right) \right) 1_{\mathcal{C}}(X_{j-1}) \\
= & I_{n,\xi} + II_{n,\xi}.
\end{aligned}
$$

Now, using the split chain decomposition, we have

$$\begin{aligned} II_{n,\xi} &= \frac{2}{T_n(\mathcal{C})}\sum_{i\neq j}^{n}\left(\frac{1}{T_n\xi}\sum_{j=2}^{n}\frac{1}{p_s(X_{j-1})}K\left(\frac{X_{i-1}-X_{j-1}}{\xi}\right)\right. \\ &\quad \left.\sigma^2(X_{j-1})(u_j^2-1)\sigma(X_{i-1})u_i\left(\widehat{\mu}_{\widehat{h}_n}(X_{i-1})-\mu(X_{i-1})\right)1_{\mathcal{C}}(X_{j-1})\right) \\ &= \frac{2T_n}{T_n(\mathcal{C})}\frac{1}{T_n^2}\sum_{k=1}^{T_n}\sum_{k'=1}^{T_n}\left(\frac{1}{h_2}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1,i_{k'}\neq j_k}^{\tau_{k'}}\frac{1}{p_s(X_{j_k-1})}K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{\xi}\right)\right. \\ &\quad \left.\sigma^2(X_{j_k})(u_{j_k}^2-1)\sigma(X_{i_{k'}-1})u_{i_{k'}}\left(\widehat{\mu}_{\widehat{h}_n}\left(X_{i_{k'}-1}\right)-\mu\left(X_{i_{k'}-1}\right)\right)1_{\mathcal{C}}(X_{j_k-1})\right) \end{aligned}$$

and

$$\begin{aligned} &\sup_{\xi\in\Xi_n}\mathrm{E}\left(\left(\frac{II_{n,\xi}}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}\right)^{2\kappa}\right) \\ \leq\ &\sup_{\xi\in\Xi_n}\mathrm{E}\left(\frac{\sup_{x\in\mathcal{C}}\left(\widehat{\mu}_{\widehat{h}_n}(x)-\mu(x)\right)}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}\left(\frac{2T_n}{T_n(\mathcal{C})}\frac{1}{T_n^2}\sum_{k=1}^{T_n}\sum_{k'=1}^{T_n}\left(\frac{1}{\xi}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1,i'_k\neq j_k}^{\tau_{k'}}\frac{1}{p_s(X_{j_k-1})}\right.\right.\right. \\ &\left.\left.\left.K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{\xi}\right)\sigma^2(X_{j-1})(u_j^2-1)\sigma(X_{i-1})u_i\right)\right)1_{\mathcal{C}}(X_{j-1})\right)^{2\kappa} \\ =\ &o_p(1), \end{aligned}$$

since the term

$$\begin{aligned} &\frac{2T_n}{T_n(\mathcal{C})}\frac{1}{T_n^2}\sum_{k=1}^{T_n}\sum_{k'=1}^{T_n}\frac{1}{\xi}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1,i'_k\neq j_k}^{\tau_{k'}}\frac{1}{p_s(X_{j_k-1})}K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{\xi}\right)\times \\ &\times\sigma^2(X_{j-1})(u_j^2-1)\sigma(X_{i-1})u_i \end{aligned}$$

is of the same order as $A_{n,h}$ in the proof of that Lemma 4 and $\sup_{x\in\mathcal{C}}\left(\widehat{\mu}_{\widehat{h}_n}(x)-\mu(x)\right) = o_p\left(\left(d_{A,\widehat{h}_n}(\widehat{\mu},\mu)\right)^{1/4}\right)$ by Lemma 7. As for $I_{n,\xi}$, we have

$$\begin{aligned} I_{n,\xi} &= \frac{2T_n}{T_n(\mathcal{C})}\frac{1}{T_n^2}\sum_{k=1}^{T_n}\sum_{k'=1}^{T_n}\left(\frac{1}{\xi}\sum_{j_k=\tau_{k-1}+1}^{\tau_k}\sum_{i_{k'}=\tau_{k'-1}+1,i'_k\neq j_k}^{\tau_{k'}}\frac{1}{p_s(X_{j_k-1})}K\left(\frac{X_{i_{k'}-1}-X_{j_k-1}}{\xi}\right)\right. \\ &\quad \left.\sigma^2(X_{j_k})(u_{j_k}^2-1)\left(\widehat{\mu}_{\widehat{h}_n}(X_{i_{k'}-1})-\mu(X_{i_{k'}-1})\right)^2 1_{\mathcal{C}}(X_{j_k-1})\right) = o_p(II_{n,\xi}) \end{aligned}$$

because of Lemma 7. The terms in Eq. (20) and in Eq. (24) are $o_p\left(d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)\right)$ by a similar argument.

Q.E.D.

Theorem 2

Proof.

$$\begin{aligned}
\overline{CV}(\xi) &= \widetilde{CV}(\xi) + \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})\left(u_j^2-1\right)\right)^2 1\left\{X_{j-1}\in\mathcal{C}\right\} \\
&= CV(\xi) + \widehat{\text{Error}}(\xi) + \widehat{\text{Error}} + \frac{1}{T_n(\mathcal{C})}\sum_{j=2}^{n}\left(\sigma^2(X_{j-1})\left(u_j^2-1\right)\right)^2 1\left\{X_{j-1}\in\mathcal{C}\right\} \\
&= \overline{\overline{CV}}(\xi) + \widehat{\text{Error}}(\xi),
\end{aligned}$$

where $\widehat{\text{Error}}$ denotes the first five terms in Eq. (17). We can write

$$\overline{\xi} = \arg\min_{\xi}\left[\overline{\overline{CV}}(\xi)\right] \overset{a.s.}{=} \arg\min_{\xi}\left[CV(\xi)\right].$$

For $\widetilde{\text{Cross}}(\xi)$ defined as in Eq. (55), we have

$$\begin{aligned}
&\left|\frac{\overline{\overline{CV}}(\overline{\xi})}{\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} - 1\right| \\
= &\sup_{\xi\in\Xi_n}\left|\frac{\overline{d}_{A,\overline{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right) + \widehat{\text{Error}}(\overline{\xi}) - \widetilde{\text{Cross}}(\overline{\xi})}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} - 1\right| \\
\leq &\sup_{\xi\in\Xi_n}\left|\frac{\overline{d}_{A,\overline{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right)}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} - 1\right| + \sup_{\xi\in\Xi_n}\left|\frac{\widetilde{\text{Cross}}(\overline{\xi})}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}\right| + \sup_{\xi\in\Xi_n}\left|\frac{\widehat{\text{Error}}(\overline{\xi})}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)}\right| \\
= &\, o_p(1)
\end{aligned} \tag{57}$$

by the results in the proofs of Lemma 6 and Lemma 8. Now, write

$$
\begin{aligned}
& \left| \frac{d_{A,\bar{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right)}{\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} - 1 \right| \\
= & \left| \frac{\overline{CV}(\bar{\xi}) + \widetilde{\text{Cross}}(\bar{\xi}) - \left(\overline{d}_{A,\bar{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right) - d_{A,\bar{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right)\right)}{\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} - 1 \right| \\
= & \left| \frac{\overline{\overline{CV}}(\bar{\xi}) - \widehat{\text{Error}}(\bar{\xi}) + \widetilde{\text{Cross}}(\bar{\xi}) - \left(\overline{d}_{A,\bar{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right) - d_{A,\bar{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right)\right)}{\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} - 1 \right| \\
\leq & \left| \frac{\overline{\overline{CV}}(\bar{\xi})}{\inf_{\xi\in\Xi_n} d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} - 1 \right| + \sup_{\xi\in\Xi_n} \left| \frac{\widehat{\text{Error}}(\bar{\xi})}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} \right| \\
& + \sup_{\xi\in\Xi_n} \left| \frac{\widetilde{\text{Cross}}(\bar{\xi})}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} \right| + \sup_{\xi\in\Xi_n} \left| \frac{\overline{d}_{A,\bar{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right) - d_{A,\bar{\xi}}\left(\widetilde{\sigma}^2,\sigma^2\right)}{d_{A,\xi}\left(\widetilde{\sigma}^2,\sigma^2\right)} \right| \\
= & o_p(1),
\end{aligned}
$$

by Eq. (57).

Q.E.D.

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		linear autoregression									
		nonparametric			OLS						
T	ρ	bias	stdev	RMSE	bias	stdev	RMSE				
200	0.1	0.161	0.249	0.203	0.014	0.174	0.121				
	0.5	0.215	0.323	0.264	0.003	0.153	0.109				
	0.9	0.157	0.297	0.199	0.009	0.088	0.054				
	0.99	0.104	0.822	0.267	0.012	0.050	0.021				
	1	0.150	1.185	0.372	0.010	0.045	0.015	500	0.1	0.126	0.186
500	0.1	0.126	0.186	0.151	0.001	0.113	0.078				
	0.5	0.150	0.227	0.186	0.001	0.096	0.068				
	0.9	0.095	0.159	0.118	0.005	0.051	0.033				
	0.99	0.048	0.387	0.156	0.004	0.022	0.011				
	1	0.072	0.980	0.252	0.004	0.016	0.006	1,000	0.1	0.095	0.142
1,000	0.1	0.095	0.142	0.118	0.004	0.081	0.057				
	0.5	0.108	0.174	0.137	0.001	0.070	0.047				
	0.9	0.065	0.114	0.086	0.004	0.035	0.023				
	0.99	0.025	0.167	0.105	0.002	0.013	0.008				
	1	0.046	0.842	0.200	0.002	0.008	0.003				

Table 1: Linear autoregression: bias, standard deviation ("stdev"), and root mean-square error ("RMSE") of estimates of $\mu(x)$, averaged over a grid x-values. "CV" denotes the nonparametric estimator using the cross-validated bandwidth and "OLS" the standard linear least-squares estimator.

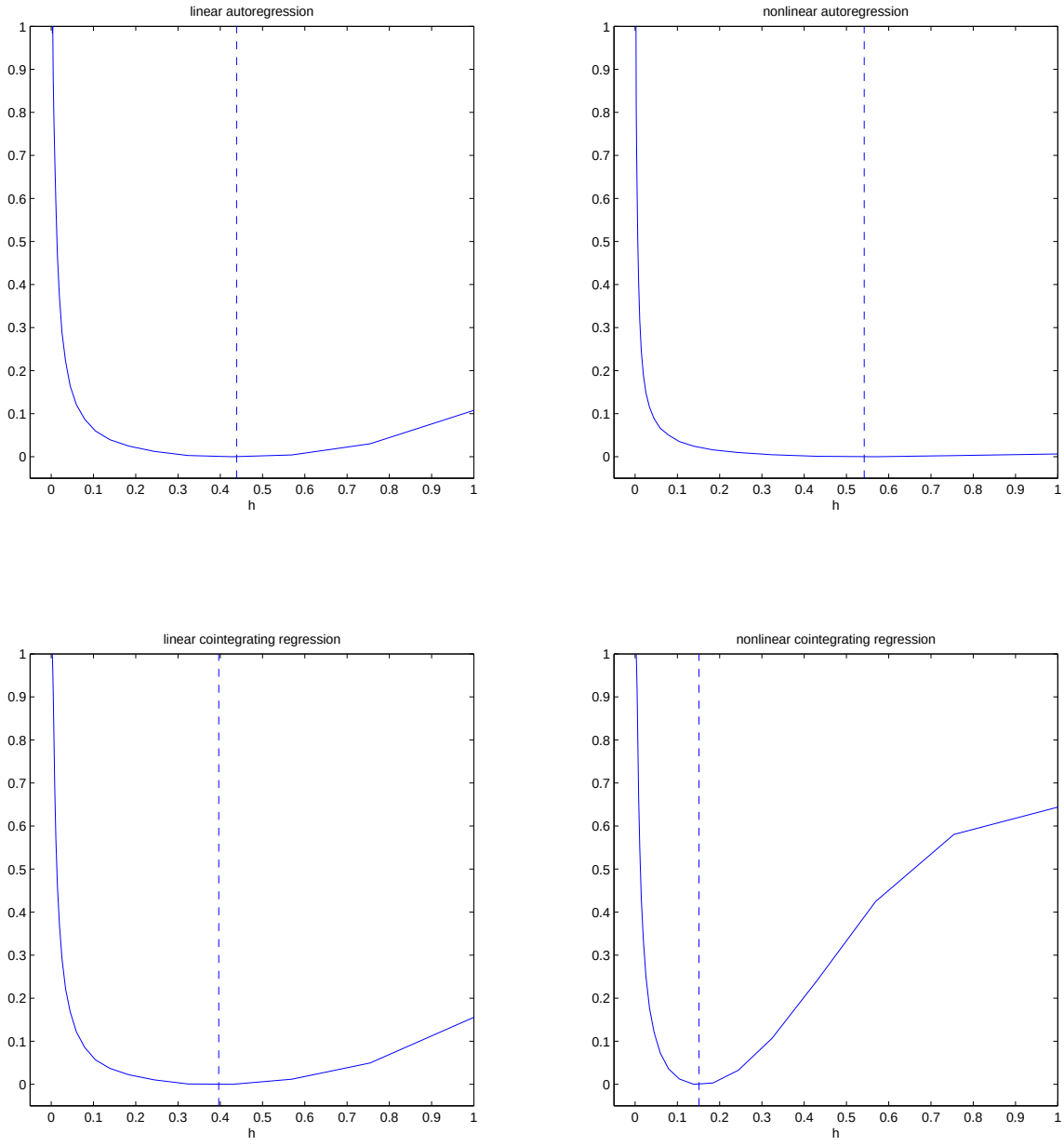

Figure 1: Cross-validation objective function (solid line) together with the selected bandwidth (dashed line), both averaged over all Monte Carlo samples, for $T = 500$ and $\rho = 0.9$.

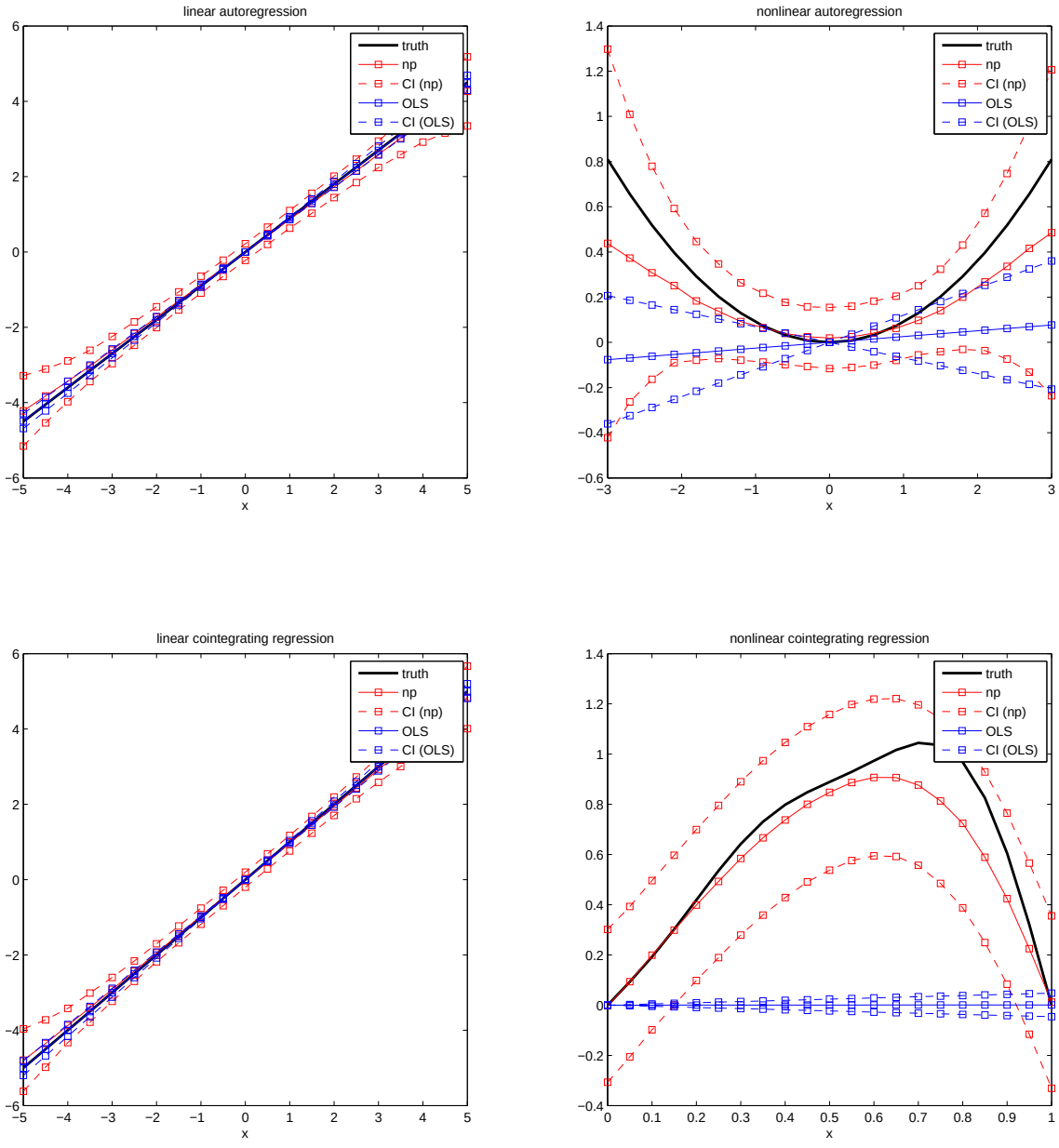

Figure 2: Estimated regression functions together with pointwise empirical confidence intervals. "np" refers to the nonparametric estimator with the cross-validated bandwidth and "OLS" to the linear least-squares estimator, each averaged over all Monte Carlo samples, for $T = 500$ and $\rho = 0.9$.

		nonlinear autoregression									
		nonparametric			OLS						
T	ρ	bias	stdev	RMSE	bias	stdev	RMSE				
200	0.1	0.016	0.114	0.066	0.022	0.105	0.075				
	0.5	0.059	0.127	0.087	0.129	0.106	0.130				
	0.9	0.095	0.139	0.116	0.243	0.111	0.243				
	0.99	0.101	0.142	0.123	0.263	0.113	0.263				
	1	0.102	0.143	0.123	0.265	0.113	0.265	500	0.1	0.012	0.074
500	0.1	0.012	0.074	0.044	0.023	0.068	0.050				
	0.5	0.048	0.088	0.067	0.123	0.069	0.123				
	0.9	0.065	0.107	0.088	0.241	0.072	0.241				
	0.99	0.066	0.111	0.088	0.267	0.073	0.267				
	1	0.066	0.111	0.088	0.269	0.073	0.269	1,000	0.1	0.014	0.051
1,000	0.1	0.014	0.051	0.030	0.024	0.049	0.038				
	0.5	0.044	0.066	0.057	0.124	0.049	0.124				
	0.9	0.056	0.089	0.069	0.241	0.052	0.241				
	0.99	0.057	0.086	0.070	0.266	0.052	0.266				
	1	0.057	0.086	0.070	0.268	0.052	0.268				

Table 2: Nonlinear autoregression: bias, standard deviation ("stdev"), and root mean-square error ("RMSE") of estimates of $\mu(x)$, averaged over a grid x-values. "CV" denotes the nonparametric estimator using the cross-validated bandwidth and "OLS" the standard linear least-squares estimator.

		linear cointegrating regression									
		nonparametric			OLS						
T	ρ	bias	stdev	RMSE	bias	stdev	RMSE				
200	0.1	0.269	0.534	0.397	0.005	0.180	0.128				
	0.5	0.219	0.424	0.311	0.002	0.155	0.110				
	0.9	0.107	0.242	0.165	0.001	0.084	0.054				
	0.99	0.058	0.603	0.197	0.000	0.038	0.023				
	1	0.104	0.823	0.262	0.000	0.030	0.016	500	0.1	0.200	0.347
500	0.1	0.200	0.347	0.281	0.001	0.114	0.079				
	0.5	0.143	0.271	0.205	0.000	0.101	0.070				
	0.9	0.070	0.147	0.104	0.003	0.050	0.032				
	0.99	0.031	0.272	0.123	0.001	0.020	0.012				
	1	0.059	0.703	0.180	0.000	0.011	0.006	1,000	0.1	0.147	0.271
1,000	0.1	0.147	0.271	0.209	0.005	0.080	0.055				
	0.5	0.127	0.197	0.166	0.003	0.067	0.046				
	0.9	0.058	0.103	0.082	0.002	0.034	0.024				
	0.99	0.017	0.128	0.083	0.002	0.012	0.007				
	1	0.033	0.610	0.149	0.000	0.006	0.003				

Table 3: Linear cointegrating regression: bias, standard deviation ("stdev"), and root mean-square error ("RMSE") of estimates of $\mu(x)$, averaged over a grid x-values. "CV" denotes the nonparametric estimator using the cross-validated bandwidth and "OLS" the standard linear least-squares estimator.

		nonlinear cointegrating regression									
		nonparametric			OLS						
T	ρ	bias	stdev	RMSE	bias	stdev	RMSE				
200	0.1	0.069	0.178	0.132	0.724	0.044	0.724				
	0.5	0.070	0.190	0.133	0.728	0.039	0.728				
	0.9	0.114	0.234	0.174	0.731	0.021	0.731				
	0.99	0.154	0.416	0.263	0.731	0.010	0.731				
	1	0.193	0.521	0.324	0.731	0.007	0.731	500	0.1	0.046	0.118
500	0.1	0.046	0.118	0.084	0.722	0.029	0.722				
	0.5	0.051	0.122	0.093	0.729	0.025	0.729				
	0.9	0.058	0.158	0.112	0.730	0.012	0.730				
	0.99	0.099	0.262	0.181	0.731	0.005	0.731				
	1	0.141	0.461	0.257	0.731	0.003	0.731	1,000	0.1	0.029	0.091
1,000	0.1	0.029	0.091	0.065	0.724	0.019	0.724				
	0.5	0.036	0.095	0.070	0.730	0.017	0.730				
	0.9	0.046	0.121	0.091	0.731	0.008	0.731				
	0.99	0.064	0.194	0.138	0.731	0.003	0.731				
	1	0.108	0.418	0.213	0.731	0.001	0.731				

Table 4: Nonlinear cointegrating regression: bias, standard deviation ("stdev"), and root mean-square error ("RMSE") of estimates of $\mu(x)$, averaged over a grid x-values. "CV" denotes the nonparametric estimator using the cross-validated bandwidth and "OLS" the standard linear least-squares estimator.

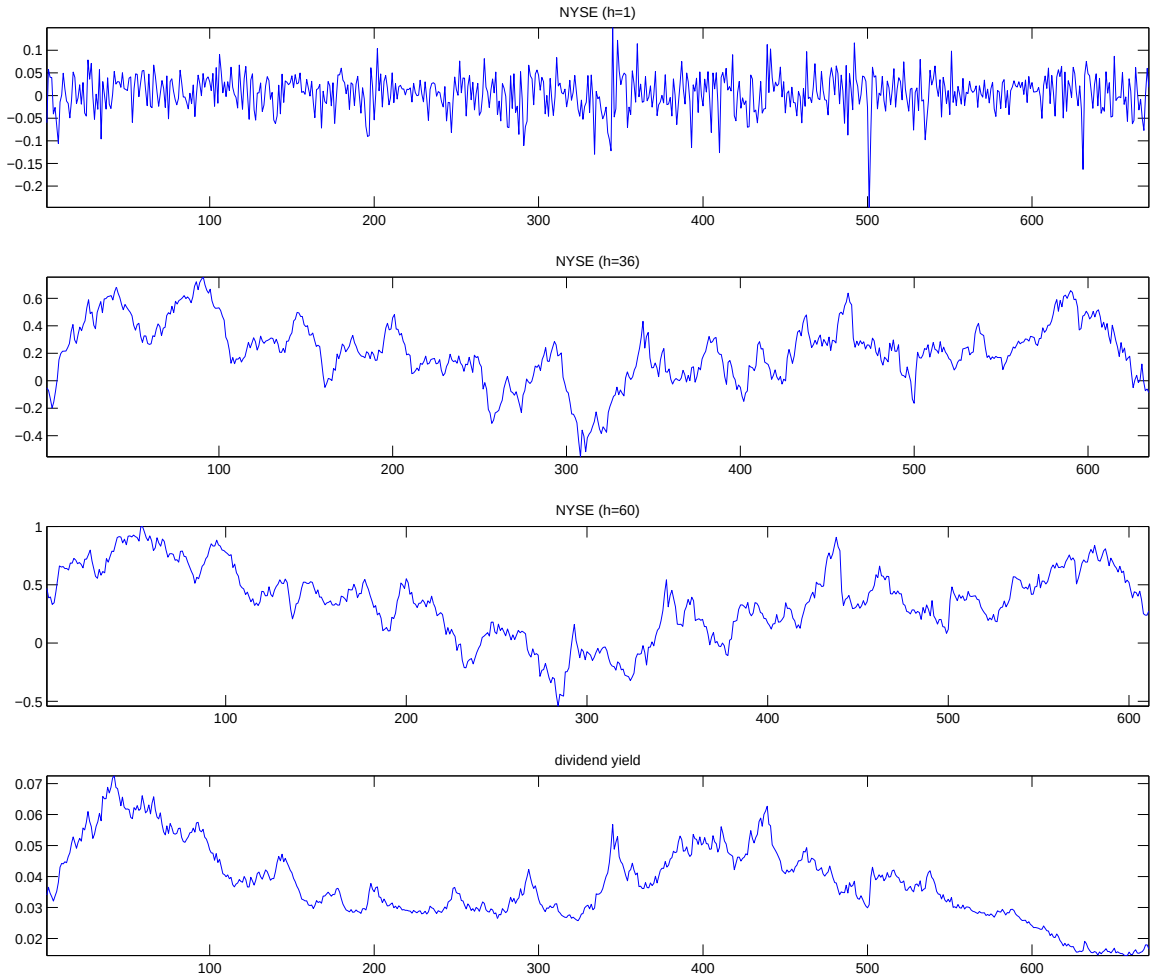

Figure 3: NYSE and dividend yield sample.

	$\gamma = 1$	$\gamma = 5$	$\gamma = 10$
$h = 1$	0.0021	0.0012	0.0011
$h = 36$	0.0619	0.0343	0.0309
$h = 60$	0.0422	0.0179	0.0148

Table 5: Annualized fee Δ.

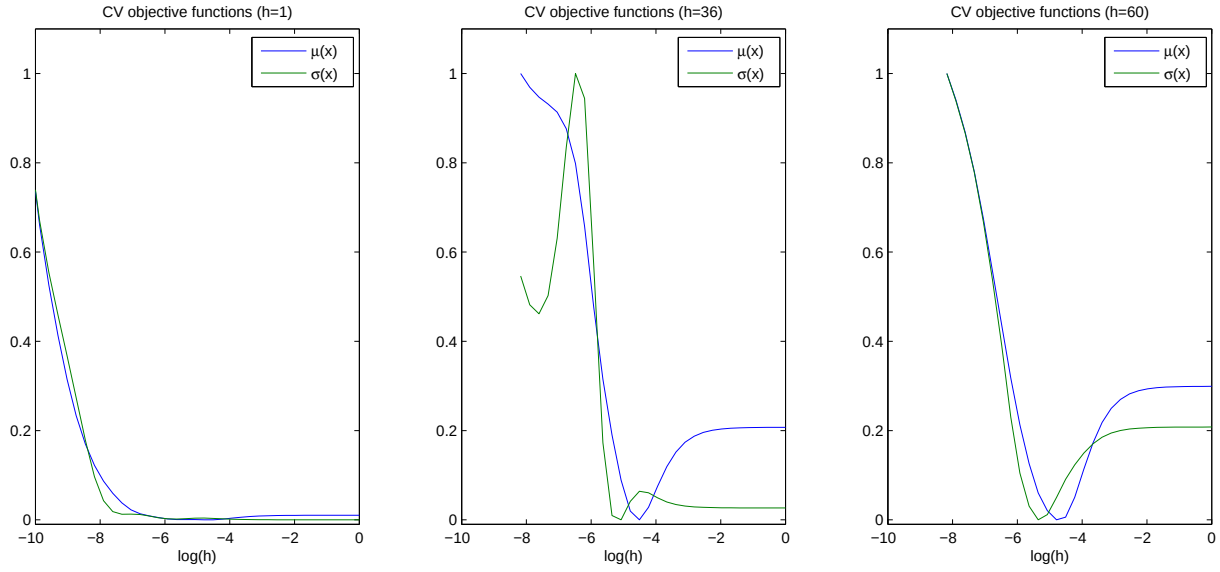

Figure 4: CV criterion functions.

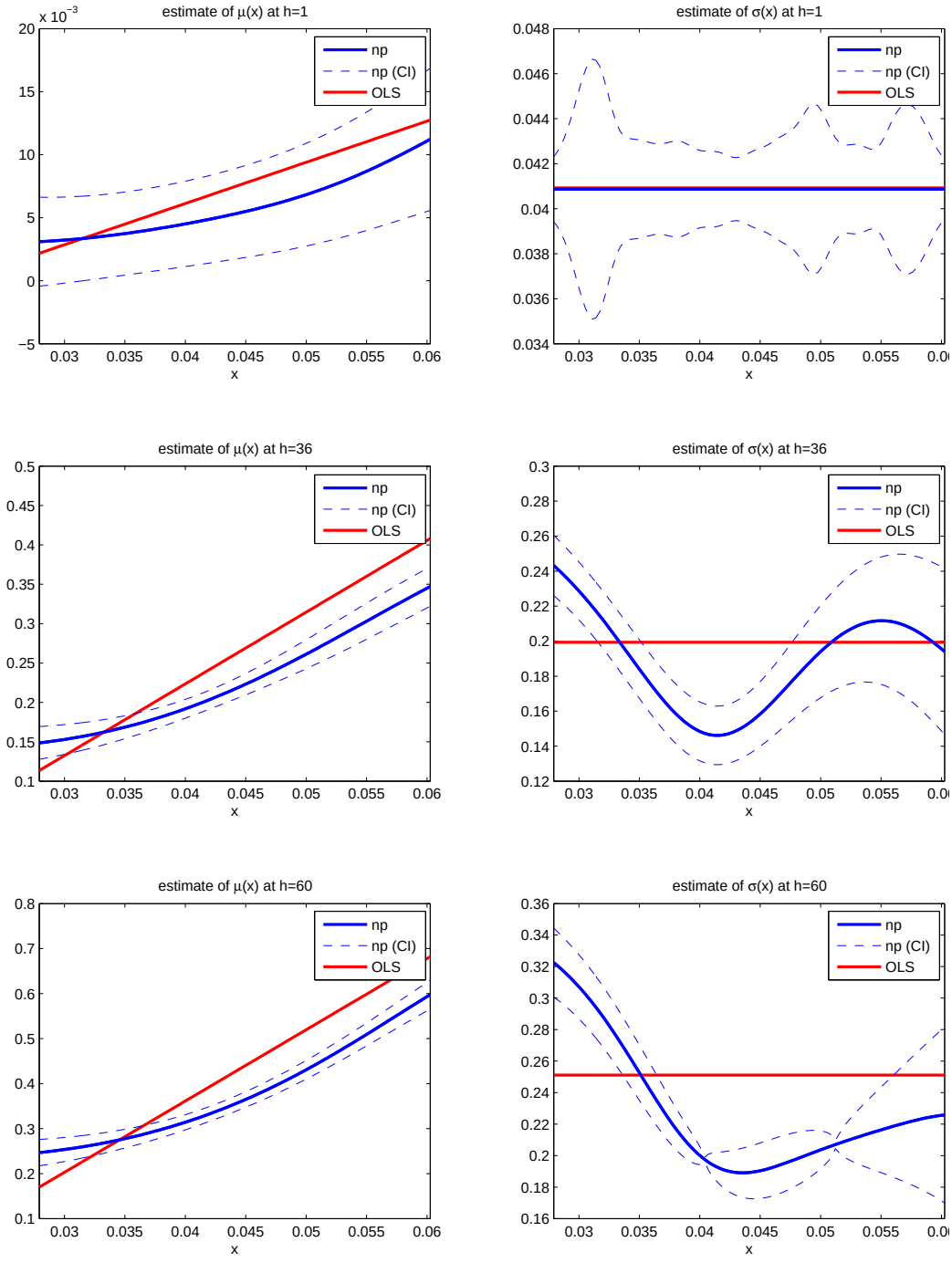

Figure 5: Nonparametric ("np") estimates $\hat{\mu}_{\hat{h}_{1,n}}(x)$ and $\hat{\sigma}_{\hat{h}_{2,n}}(x)$ of $\mu(x)$ and $\sigma(x)$ compared to linear least-squares ("OLS") estimates.

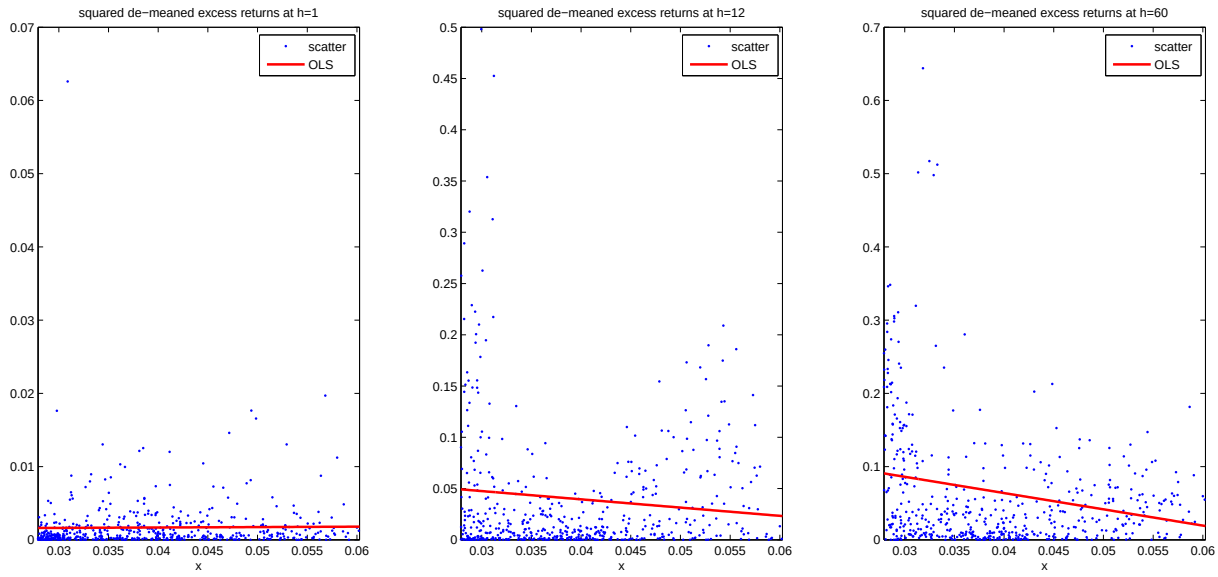

Figure 6: Scatter plots of squared de-meaned excess returns against the dividend-price ratio X_t, together with OLS regression fit.

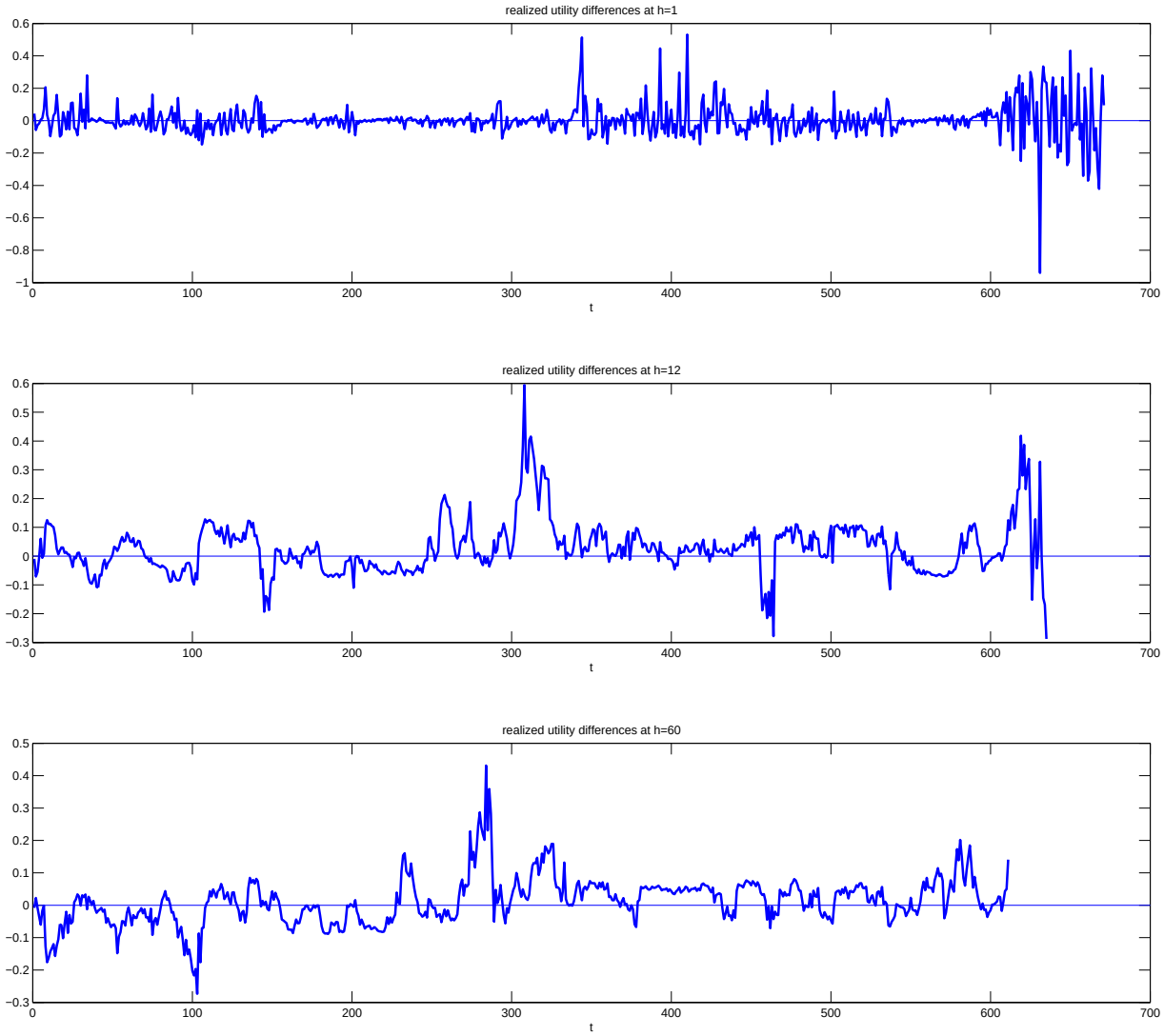

Figure 7: Difference of nonlinear and linear investors' realized utilities over time for $\gamma = 10$.