Cross-sectional Dependence in Idiosyncratic Volatility*

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Abstract

This paper introduces a framework for analysis of cross-sectional dependence in the idiosyncratic volatilities of assets using high frequency data. We first consider the estimation of standard measures of dependence in the idiosyncratic volatilities such as covariances and correlations. Next, we study an idiosyncratic volatility factor model, in which we decompose the co-movements in idiosyncratic volatilities into two parts: those related to factors such as the market volatility, and the residual co-movements. When using high frequency data, naive estimators of all of the above measures are biased due to the estimation errors in idiosyncratic volatility. We provide bias-corrected estimators and establish their asymptotic properties. We apply our estimators to high-frequency data on 27 individual stocks from nine sectors, and document strong cross-sectional dependence in their idiosyncratic volatilities. We also find that on average, 49% of this dependence can be explained by the market volatility.

Keywords: high frequency data; idiosyncratic volatility; errors-in-variables; cross-sectional returns.

JEL Codes: C22, C14.

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1 Introduction

Idiosyncratic Volatility (IV) of returns of an asset or a portfolio is the subject of many recent papers in empirical finance. The IV is usually defined with respect to some popular empirical asset pricing model such as the Fama and French (1993) model, so that IV is the volatility of the risk-adjusted returns. Even if the idiosyncratic returns are not correlated in the cross-section, their volatilities may well be. In fact, cross-sectional correlation of IVs has emerged as a stylized fact, see, e.g., Herskovic, Kelly, Lustig, and Nieuwerburgh (2014) and Duarte, Kamara, Siegel, and Sun (2014). The current paper develops the tools to formally study this empirical phenomenon.

We provide a flexible framework for studying the cross-sectional dependencies in IVs using high-frequency data. Our framework incorporates important stylized facts in asset returns and volatilities and offers a solution to the measurement error problem in estimated IVs.

First, we study the behavior of standard measures of cross-sectional dependence in IVs using high-frequency data. We show that the naive estimators of these measures are biased, and provide bias-corrected estimators. We then obtain the relevant asymptotic distributions, which allow us to perform statistical tests.

Second, we study an idiosyncratic volatility factor model (IV-FM).[1] The IV-FM decomposes the cross-sectional dependence in IVs into two components. The first component is the cross-sectional dependence due to popular factors. The IV factors can include the volatility of the return factors, non-linear transforms of the spot covariance matrices such as correlations, as well as the average variance factor of Chen and Petkova (2012). The second component in the IV-FM is residual dependence in IVs not explained by the IV factors. Again, the standard estimators of this decomposition are biased due to the latency in volatility. We provide bias-corrected estimators, and derive their asymptotic distributions. We build a test for whether the IV-FM can fully account for the dependence between the IVs.

We apply our estimators to high-frequency data on 27 individual US stocks from nine different sectors. We study idiosyncratic volatilities with respect to two models for asset prices, CAPM and the three-factor Fama-French model. In both cases, the average correlation between the idiosyncratic volatilities is above 0.55. Moreover, the average correlation between the IVs is on average the same among those pairs of stocks, which have close to zero correlations between their idiosyncratic returns. In other words, the dependencies in IVs cannot be explained by a missing return factor. This is in line with the recent findings of Herskovic, Kelly, Lustig, and Nieuwerburgh (2014) who use daily and monthly return data. We then consider the IV-FM with market volatility as the IV factor. We find that on average, the systematic component of IV that arises due to IV exposure to market volatility, accounts for 74% of the cross-sectional dependence in the IVs. We find that in 110 out of 351 pairs of stocks analyzed, this idiosyncratic volatility factor model fully

[1] Throughout the paper, we use the term "factor model" to denote a regression model, e.g., we call the Fama and French (1993) model a factor model.

accounts for the cross-sectional dependence in IVs, so that their non-systematic components are no longer significantly correlated.

The importance of accounting for estimation errors in volatilities has been demonstrated in other contexts. Recently, Aït-Sahalia, Fan, and Li (2013) show that failure to account for the latency of volatility drives the leverage effect puzzle.[2] An important aspect of our methods is that we fully account for the latency of IV.

Our paper is related to several strands of the literature. Our inference theory extends the results on estimation of the integrated one-dimensional (total) volatility of volatility (Vetter (2012), Aït-Sahalia and Jacod (2014)). The (total) leverage effect is also a quantity, for which the naive nonparametric estimators are inconsistent due to the measurement errors in volatilities, see Wang and Mykland (2014), Kalnina and Xiu (2015), Aït-Sahalia, Fan, Laeven, Wang, and Yang (2013) and Aït-Sahalia, Fan, and Li (2013) for one-dimensional results. Due to the decomposition of total returns into systematic and idiosyncratic part, our estimators involve aggregation of non-linear functionals of the return volatility matrix, hence our bias-correction terms are related to the general theory developed in Jacod and Rosenbaum (2012) and Jacod and Rosenbaum (2013). We define the IV with respect to a continuous-time factor model for asset returns with observable return factors. This framework was originally studied in Mykland and Zhang (2006) in the case of one factor and in the absence of jumps. It was extended to multiple factors and jumps in Aït-Sahalia, Kalnina, and Xiu (2014).

The remainder of the paper is organized as follows. Section 2 introduces the model and describes quantities of interest. Section 3 describes the identification and estimation of these quantities of interest. Section 4 presents the asymptotic properties of our estimators. Section 5 investigates their finite sample properties. Section 6 uses high-frequency stock return data to study the cross-sectional dependence in IVs using our framework. Section 7 concludes. The Appendix contains the proofs.

2 Model and Quantities of Interest

We first describe a general factor model for the returns, in which the idiosyncratic volatility is defined. We then introduce the idiosyncratic volatility factor model (IV-FM).

Suppose we have (log) prices on d_S assets such as stocks and on d_F observable factors. These factors serve as the observable factors in the model P-FM below. We stack them into the d-dimensional process $Y_t = (S_{1,t}, \ldots, S_{d_S,t}, F_{1,t}, \ldots, F_{d_F,t})^\top$ where $d = d_S + d_F$. We assume Y_t follows an Itô semimartingale,

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t,$$

[2] See Wang and Mykland (2014), Kalnina and Xiu (2015), and Aït-Sahalia, Fan, Laeven, Wang, and Yang (2013) for related results on the leverage effect.

where W is a d'-dimensional Brownian motion ($d' \geq d$), σ_s is a $d \times d'$ stochastic volatility process, and J_t denotes a finite variation jump process. We assume also that the spot variance matrix process $C_t = \sigma_t \sigma_t^\top$ of Y_t is a continuous Itô semimartingale,[3]

$$C_t = C_0 + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s, \tag{1}$$

see Section 4 for the full list of assumptions. We denote $C_t = (C_{ab,t})_{1 \leq a,b \leq d}$. For convenience, we also employ the alternative notation $C_{UV,t}$ to refer to the spot covariance between two elements U and V of Y.

We assume a standard continuous-time factor model for the (log) prices of the assets:

Definition (Factor Model for Prices, P-FM). *For for all $0 \leq t \leq T$ and $j = 1, \ldots, d_S$,*[4]

$$\begin{aligned} dS_{j,t}^c &= \beta_{j,t}^\top dF_t^c + dZ_{j,t}^c \quad \textit{with} \\ [Z_j^c, F^c]_t &= 0, \end{aligned} \tag{2}$$

We do not need the return factors F_t to be the same across assets to identify the model, but without loss of generality, we keep this structure because it is standard in empirical finance. These return factors are assumed to be observable. For example, in the empirical application, we use two sets of return factors: the market portfolio and the three Fama-French factors.[5]

The process $Z_{j,t}$ in the P-FM is the idiosyncratic component of the price of the j^{th} stock with respect to the return factors. We use the superscript c to emphasize that the P-FM only involves the continuous martingale parts of the two observable processes $Y_{j,t}$ and F_t. The jump parts of these processes are left unrestricted. For $j = 1, \ldots, d_S$, the factor loadings $\beta_{j,t}$ is a $\mathbb{R}^{d_F}$-valued process which represents the continuous beta.[6] The k-th component of $\beta_{j,t}$ corresponds to the time-varying loading of the continuous part of the return on stock j to the continuous part of the return on the k-th factor. We set $\beta_t = (\beta_{1,t}, \ldots, \beta_{d_S,t})^\top$ and $Z_t = (Z_{1,t}, \ldots, Z_{d_S,t})^\top$. This framework was originally studied in Mykland and Zhang (2006) in the case of one factor and in the absence of jumps. It was extended to multiple factors and jumps in Aït-Sahalia, Kalnina, and Xiu (2014). See also Li, Todorov, and Tauchen (2013), Fan, Furger, and Xiu (2015), and

[3]Note that assuming that Y and C are driven by the same d'-dimensional Brownian motion W is without loss of generality provided that d' is large enough, see, e.g., equation (8.12) of Aït-Sahalia and Jacod (2014).

[4]If X and Y are two vector-valued Itô semimartingales, their quadratic covariation over the time span $[0, T]$ is defined

$$[X, Y]_T = p - \lim_{M \to \infty} \sum_{j=0}^{M-1} (X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j})^\top,$$

for any sequence $t_0 < t_1 < \ldots < t_M = T$ with $\sup_j \{t_{j+1} - t_j\} \to 0$ as $M \to \infty$, where p-lim stands for the probability limit. Barndorff-Nielsen and Shephard (2004) discuss its estimation when both X and Y are observed.

[5]High-frequency observations on the Fama-French factors are constructed in Aït-Sahalia, Kalnina, and Xiu (2014).

[6]Interestingly, it is possible to define a discontinuous beta, see, e.g., Bollerslev and Todorov (2010) and Li, Todorov, and Tauchen (2014).

Reiß, Todorov, and Tauchen (2015). Our framework can be potentially extended to use principal components instead of observable return factors as in Ait-Sahalia and Xiu (2015).

Idiosyncratic Volatility of stock j is the spot volatility of the residual process Z_j, and we denote it by C_{ZjZj}. Notice that the factor loadings, as well as IV, in (2) are functions of the total spot variance matrix C_t. First, the vector of factor loadings satisfies

$$\beta_{jt} = (C_{FF,t})^{-1} C_{FS_j,t}, \tag{3}$$

for $j = 1, \ldots, d_S$, where $C_{FF,t}$ denotes the spot covariance of the factors F, which is the lower $d_F \times d_F$ sub-matrix of C_t, and $C_{FSj,t}$ denotes the covariance of the factors and the j^{th} stock, which are the last d_F elements of the j^{th} column of C_t. Second, the IV of stock j satisfies

$$C_{ZjZj,t} = C_{YjYj,t} - (C_{FSj,t})^\top (C_{FF,t})^{-1} C_{FSj,t}. \tag{4}$$

Since factor loadings and the IV are functions of the total spot variance matrix, by the Itô lemma, they are also Itô semimartingales with their characteristics related to those of C_t.

We take the following quadratic-covariation based quantity as the natural measure of dependence between the IV shocks of stocks i and j,

$$\rho_{Zi,Zj} = \frac{[C_{ZiZi}, C_{ZjZj}]_T}{\sqrt{[C_{ZiZi}, C_{ZiZi}]_T}\sqrt{[C_{ZjZj}, C_{ZjZj}]_T}}. \tag{5}$$

Alternatively, one can consider the quadratic covariation $[C_{ZiZi}, C_{ZjZj}]_T$. In Section 4.4, we use the estimator of the latter quantity to test for the presence of cross-sectional dependence in IVs.

We now introduce the Idiosyncratic Volatility Factor model (IV-FM). We assume that the cross-sectional dependence in the IVs can be potentially explained by the IV factors. We assume the IV factors are known functions of the matrix C_t. In the empirical application, we use the market volatility as the IV factor; we discuss other possibilities below. We allow the IV factors to be any known function of C_t as long as it satisfies a certain polynomial growth condition in the sense of being in the class $\mathcal{G}(p)$ below,

$$\begin{aligned} \mathcal{G}(p) = \{H : H &\text{ is three-times continuously differentiable and for some } K > 0, \\ &\|\partial^j H(x)\| \le K(1 + \|x\|)^{p-j}, j = 0, 1, 2, 3\}, \quad \text{for some } \ p \ge 3. \end{aligned}$$

Definition (Idiosyncratic Volatility Factor Model, IV-FM). *For all* $0 \le t \le T$ *and* $j = 1, \ldots, d_S$*, the idiosyncratic volatility* C_{ZjZj} *follows,*

$$dC_{ZjZj,t} = b_{Zj}^\top d\Pi_t + dC^{NS}_{ZjZj,t} \quad \textit{with} \tag{6}$$

$$[C^{NS}_{ZjZj}, \Pi]_t = 0. \tag{7}$$

where $\Pi_t = (\Pi_{1t}, \ldots, \Pi_{d_\Pi t})$ *is a* $\mathbb{R}^{d_\Pi}$*-valued vector of IV factors, which satisfy* $\Pi_{kt} = \Pi_k(C_t)$ *with the function* $\Pi_k(.)$ *belonging to* $\mathcal{G}(p)$ *for* $k = 1, \ldots, d_\Pi$.

We call the residual term $C^{NS}_{ZjZj,t}$ the non-systematic idiosyncratic volatility of asset j, and we abbreviate it as NS-IV. We refer to b_{Zj} as the IV beta. The IV beta is time-invariant. Our assumptions imply that the components of the IV-FM, $C_{ZjZj,t}, \Pi_t$ and $C^{NS}_{ZjZj,t}$, are continuous Itô semimartingales. We remark that both the regressand and the regressors in our IV-FM are not directly observable and have to be estimated. As will see in Section 3, this preliminary estimation implies that the naive estimators of all the quantities of interest in the IV-FM are biased. One of the contributions of this paper is to quantify this bias and propose bias-corrected estimators for all the quantities of interest.

The class of IV factors permitted by our theory is rather wide as it includes general non-linear transforms of the spot variance process C_t. For example, IV factors can be linear combinations of the total variances of stocks, see, e.g., the average variance factor of Chen and Petkova (2012). Other examples of IV factors are linear combinations of the IVs, such as the equally-weighted average of the IVs, the "CIV", of Herskovic, Kelly, Lustig, and Nieuwerburgh (2014). The IV factors can also be the volatilities of any other observable processes.

To measure the residual cross-sectional dependence between two IVs after accounting for the effect of the IV factors, we use a quadratic-covariation based correlation measure between NS-IVs,

$$\rho^{NS}_{Zi,Zj} = \frac{[C^{NS}_{ZiZi}, C^{NS}_{ZjZj}]_T}{\sqrt{[C^{NS}_{ZiZi}, C^{NS}_{ZiZi}]_T}\sqrt{[C^{NS}_{ZjZj}, C^{NS}_{ZjZj}]_T}}. \tag{8}$$

When testing for the presence of residual correlation between NS-IVs, we use the quadratic covariation $[C^{NS}_{ZiZi}, C^{NS}_{ZjZj}]_T$ without normalization.

We want to capture how well the IV factors explain the time variation of j^{th} IV. For this purpose, we use the quadratic-covariation based analog of the coefficient of determination. For $j = 1, \ldots, d_S$,

$$R^{2,IV\text{-}FM}_{Zj} = \frac{b^\top_{Zj}[\Pi, \Pi]_T b_{Zj}}{[C_{ZjZj}, C_{ZjZj}]_T}. \tag{9}$$

It is interesting to compare the correlation measure between IVs in equation (5) with the correlation between the non-systematic parts of IVs in (8). We consider their difference,

$$\rho_{Zi,Zj} - \rho^{NS}_{Zi,Zj}, \tag{10}$$

to see how much of the dependence between IVs can be attributed to the IV factors. In practice, if we compare assets that are known to have positive covolatilities (typically, stocks have that property), another useful measure of the systematic part in the overall covariation between IVs is the following quantity,

$$Q^{IV\text{-}FM}_{Zi,Zj} = \frac{b^\top_{Zi}[\Pi, \Pi]_T b_{Zj}}{[C_{ZiZi}, C_{ZjZj}]_T}. \tag{11}$$

This measure is bounded by 1 if the covariations between NS-IVs are nonnegative and smaller than the covariations between IVs, which is what we find for every pair in our empirical application with high-frequency observations on stock returns.

The next section outlines identification and estimation of the above key quantities. It also presents the asymptotic distributions, which can be used to conduct statistical tests. We conduct three tests. First, we test whether the total cross-correlation in IVs is nonzero for a given pair of assets, which corresponds to the hypothesis $[C_{ZiZi}, C_{ZjZj}]_T = 0$. Second, we test whether the IV factors contribute to the cross-correlation in IVs by considering the null hypothesis $[C_{ZjZj}, \Pi]_T = 0$. Third, we test the hypothesis of whether IV-FM can explain all the cross-sectional IV dependence, i.e., $[C^{NS}_{ZiZi}, C^{NS}_{ZjZj}]_T = 0$.

It is interesting to compare our framework with the following null hypothesis studied in Li, Todorov, and Tauchen (2013), $H_0: \ C_{ZjZj,t} = a_{Zj} + b^{\top}_{Zj}\Pi_t, \ 0 \leq t \leq T$. This H_0 implies that the IV is a deterministic function of the factors, which does not allow for a non-systematic error term. In particular, this null hypothesis implies $R^{2,IV\text{-}FM}_{Zj} = 1$.

3 Estimation

We now discuss the estimation of our main quantities of interest introduced in Section 2,

$$[C_{ZiZi}, C_{ZjZj}]_T, \ \rho_{Zi,Zj}, \ [C^{NS}_{ZjZj}, \ C^{NS}_{ZjZj}]_T, \ \rho^{NS}_{Zi,Zj}, \ Q^{IV\text{-}FM}_{Zi,Zj}, \text{ and } R^{2,IV\text{-}FM}_{Zi}, \tag{12}$$

for $i, j = 1, \ldots, d_S$. We first show that each of them can be written as

$$\varphi\left([H_1(C), G_1(C)]_T, \ldots, [H_\kappa(C), G_\kappa(C)]_T\right),$$

where φ as well as H_r and G_r, for $r = 1, \ldots, \kappa$, are known real-valued functions. Each element in this expression is of the form $[H(C), G(C)]_T$, i.e., it is a quadratic covariation between functions of C_t. We then show how to estimate $[H(C), G(C)]_T$.

First, consider the quadratic covariation between i^{th} and j^{th} IV, $[C_{ZiZi}, C_{ZjZj}]_T$. It can be written as $[H(C), G(C)]_T$ if we choose $H(C_t) = C_{ZiZi,t}$ and $G(C_t) = C_{ZjZj,t}$. By (4), both $C_{ZiZi,t}$ and $C_{ZjZj,t}$ are smooth functions of C_t. Next, consider the correlation $\rho_{Zi,Zj}$ defined in (5). By the argument above, its numerator and each of the two components in the denominator can be written as $[H(C), G(C)]_T$ for different functions H and G. Therefore, $\rho_{Zi,Zj}$ is itself a known smooth function of three objects of the form $[H(C), G(C)]_T$.

To show that the remaining quantities in (12) can also be expressed in terms of objects of the form $[H(C), G(C)]_T$, note that the IV-FM implies

$$b_{Zj} = \left([\Pi, \Pi]_T\right)^{-1}[\Pi, C_{ZjZj}]_T \ \text{ and } \ [C^{NS}_{ZiZi}, C^{NS}_{ZjZj}]_T = [C_{ZiZi}, C_{ZjZj}]_T - b^{\top}_{Zi}[\Pi, \Pi]_T b_{Zj},$$

for $i, j = 1, \ldots, d_S$. Since $C_{ZiZi,t}$, $C_{ZjZj,t}$ and every element in Π_t are real-valued functions of C_t, the above equalities imply that all quantities of interest in (12) can be written as real-valued, known smooth functions of a finite number of quantities of the form $[H(C), G(C)]_T$.

To estimate $[H(C), G(C)]_T$, suppose we have discrete observations on Y_t over an interval $[0, T]$.

Denote by Δ_n the distance between observations. Note that we can estimate the spot covariance matrix C_t at time $(i-1)\Delta_n$ with a local truncated realized volatility estimator (Mancini (2001)),

$$\widehat{C}_{i\Delta_n} = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n-1} (\Delta_{i+j}^n Y)(\Delta_{i+j}^n Y)^\top 1_{\{\|\Delta_{i+j}^n Y\| \leq \chi \Delta_n^{\varpi}\}}, \tag{13}$$

where $\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$ and where k_n is the number of observations in a local window.[7] Throughout the paper we set $\widehat{C}_{i\Delta_n} = (\widehat{C}_{ab,i\Delta_n})_{1\leq a,b\leq d}$.

We propose two estimators for the general quantity $[H(C), G(C)]_T$.[8] The first is based on the analog of the definition of quadratic covariation between two Itô processes,

$$\begin{aligned} \widehat{[H(C), G(C)]}_T^{AN} = \frac{3}{2k_n} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \Bigg(\Big(H(\widehat{C}_{(i+k_n)\Delta_n}) - H(\widehat{C}_{i\Delta_n})\Big)\Big(G(\widehat{C}_{(i+k_n)\Delta_n}) - G(\widehat{C}_{i\Delta_n})\Big) \\ - \frac{2}{k_n} \sum_{g,h,a,b=1}^{d} (\partial_{gh} H \partial_{ab} G)(\widehat{C}_{i\Delta_n}) \Big(\widehat{C}_{ga,i\Delta_n}\widehat{C}_{gb,i\Delta_n} + \widehat{C}_{gb,i\Delta_n}\widehat{C}_{ha,i\Delta_n}\Big)\Bigg), \end{aligned} \tag{14}$$

where the factor 3/2 and last term correct for the biases arising due to the estimation of volatility C_t. The increments used in the above expression are computed over overlapping blocks, which results in a smaller asymptotic variance compared to the version using non-overlapping blocks.

Our second estimator is based on the following equality, which follows by the Itô lemma,

$$[H(C), G(C)]_T = \sum_{g,h,a,b=1}^{d} \int_0^T \big(\partial_{gh} H \partial_{ab} G\big)(C_t) \overline{C}_t^{gh,ab} dt, \tag{15}$$

where $\overline{C}_t^{gh,ab}$ denotes the covariation between the volatility processes $C_{gh,t}$ and $C_{ab,t}$. The quantity is thus a non-linear functional of the spot covariance and spot volatility of volatility matrices. Our second estimator is based on this "linearized" expression,

$$\begin{aligned} \widehat{[H(C), G(C)]}_T^{LIN} = \frac{3}{2k_n} \sum_{g,h,a,b=1}^{d} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} (\partial_{gh} H \partial_{ab} G)(\widehat{C}_{i\Delta_n}) \times \\ \Big((\widehat{C}_{gh,(i+k_n)\Delta_n} - \widehat{C}_{gh,i\Delta_n})(\widehat{C}_{ab,(i+k_n)\Delta_n} - \widehat{C}_{ab,i\Delta_n}) - \frac{2}{k_n}(\widehat{C}_{ga,i\Delta_n}\widehat{C}_{gb,i\Delta_n} + \widehat{C}_{gb,i\Delta_n}\widehat{C}_{ha,i\Delta_n})\Big). \end{aligned} \tag{16}$$

Consistency for a similar estimator has been established by Jacod and Rosenbaum (2012).[9] We go beyond their result by deriving the asymptotic distribution and proposing a consistent estimator

[7] It is also possible to define kernel-based estimators as in Kristensen (2010).

[8] As evident from their formulas, the computation time required for the calculation of the two estimators is increasing with the number of stocks and factors d. To ease the implementation of the procedure, we compute all the quantities of interest for pairs of stocks which means practically one needs only to set $d_S = 2$ so that $d = d_F + 2$.

[9] Jacod and Rosenbaum (2012) derive the probability limit of the following estimator:

$$\frac{3}{2k_n} \sum_{g,h,a,b=1}^{d} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} (\partial^2_{gh,ab} H)(\widehat{C}_{i\Delta_n}) \Big((\widehat{C}_{(i+k_n)\Delta_n} - \widehat{C}_{i\Delta_n})(\widehat{C}_{(i+k_n)\Delta_n} - \widehat{C}_{i\Delta_n}) - \frac{2}{k_n}(\widehat{C}_{ga,i\Delta_n}\widehat{C}_{gb,i\Delta_n} + \widehat{C}_{gb,i\Delta_n}\widehat{C}_{ha,i\Delta_n})\Big).$$

of its asymptotic variance.

Note that the same additive bias-correcting term,

$$-\frac{3}{k_n^2}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\left(\sum_{g,h,a,b=1}^{d}(\partial_{gh}H\partial_{ab}G)(\widehat{C}_{i\Delta_n})\Big(\widehat{C}_{ga,i\Delta_n}\widehat{C}_{gb,i\Delta_n}+\widehat{C}_{gb,i\Delta_n}\widehat{C}_{ha,i\Delta_n}\Big)\right), \quad (17)$$

is used for the two estimators. This term is (up to a scale factor) an estimator of the asymptotic covariance between the sampling errors embedded in estimators of $\int_0^T H(C_t)dt$ and $\int_0^T G(C_t)dt$ defined in Jacod and Rosenbaum (2013).

The two estimators are identical when H and G are linear, for example, when estimating the covariation between two volatility processes. In the univariate case $d=1$, when $H(C)=G(C)=C$, our estimator coincides with the volatility of volatility estimator of Vetter (2012) , which was extended to allow for jumps in Jacod and Rosenbaum (2012). Our contribution is the extension of this theory to the multivariate $d>1$ case with nonlinear functionals.

4 Asymptotic Properties

We start by outlining the full list of assumptions for our asymptotic results. We then state the asymptotic distribution for the general functionals introduced in the previous section, and develop estimators for the asymptotic variance. Finally, we outline three statistical tests of interest that follow from our theoretical results.

4.1 Assumptions

Recall that the d-dimensional process Y_t represents the (log) prices of stocks and factors.

Assumption 1. *Suppose Y is an Itô semimartingale on a filtered space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$,*

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t\int_E \delta(s,z)\mu(ds,dz),$$

where W is a d'-dimensional Brownian motion ($d'\geq d$) and μ is a Poisson random measure on $\mathbb{R}_+\times E$, with E an auxiliary Polish space with intensity measure $\nu(dt,dz)=dt\otimes\lambda(dz)$ for some σ-finite measure λ on E. The process b_t is $\mathbb{R}^d$-valued optional, σ_t is $\mathbb{R}^d\times\mathbb{R}^{d'}$-valued, and $\delta=\delta(w,t,z)$ is a predictable $\mathbb{R}^d$ -valued function on $\Omega\times\mathbb{R}_+\times E$. Moreover, $\|\delta(w,t\wedge\tau_m(w),z)\|\wedge 1\leq\Gamma_m(z)$, for all (w,t,z), where (τ_m) is a localizing sequence of stopping times and, for some $r\in[0,1]$, the function Γ_m on E satisfies $\int_E\Gamma_m(z)^r\lambda(dz)<\infty$. The spot volatility matrix of Y is then defined as $C_t=\sigma_t\sigma_t^\top$. We assume that C_t is a continuous Itô semimartingale,[10]

$$C_t = C_0 + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s. \quad (18)$$

[10]Note that $\widetilde{\sigma}_s=(\widetilde{\sigma}_s^{gh,m})$ is $(d\times d\times d')$-dimensional and $\widetilde{\sigma}_s dW_s$ is $(d\times d)$-dimensional with $(\widetilde{\sigma}_s dW_s)^{gh}=\sum_{m=1}^{d'}\widetilde{\sigma}_s^{gh,m}dW_s^m$.

where $\widetilde{b}$ is $\mathbb{R}^d \times \mathbb{R}^d$-valued optional.

With the above notation, the elements of the spot volatility of volatility matrix and spot covariation of the continuous martingale parts of X and c are defined as follows,

$$\overline{C}_t^{gh,ab} = \sum_{m=1}^{d'} \widetilde{\sigma}_t^{gh,m} \widetilde{\sigma}_t^{ab,m}, \quad \overline{C}_t'^{g,ab} = \sum_{m=1}^{d'} \sigma_t^{gm} \widetilde{\sigma}_t^{ab,m}. \tag{19}$$

The process $\widetilde{\sigma}_t$ is restricted as follows:

Assumption 2. *$\widetilde{\sigma}_t$ is a continuous Itô semimartingale with its characteristics satisfying the same requirements as that of C_t.*

Assumption 1 is very general and nests most of the multivariate continuous-time models used in economics and finance. It allows for potential stochastic volatility and jumps in prices. Assumption 2 is required to obtain the asymptotic distribution of estimators of the quadratic covariation between functionals of the spot covariance matrix C_t. It is not needed to prove consistency. This restriction also appears in Vetter (2012), Kalnina and Xiu (2015) and Wang and Mykland (2014).

4.2 Asymptotic Distribution

We have seen in Section 3 that all quantities of interest in (12) are functions of multiple objects of the form $[H(C), G(C)]_T$. Therefore, if we can obtain a multivariate asymptotic distribution for a vector with elements of the form $[H(C), G(C)]_T$, the asymptotic distributions for all our estimators follow by the delta method. Presenting this asymptotic distribution is the purpose of the current section.

Let $H_1, G_1, \ldots, H_\kappa, G_\kappa$ be some arbitrary elements of $\mathcal{G}(p)$. We are interested in the asymptotic behavior of vectors

$$\left([H_1(\widehat{C), G_1}(C)]_T^{AN}, \ldots, [H_\kappa(\widehat{C), G_\kappa}(C)]_T^{AN}\right)^\top \text{ and } \left([H_1(\widehat{C), G_1}(C)]_T^{LIN}, \ldots, [H_\kappa(\widehat{C), G_\kappa}(C)]_T^{LIN}\right)^\top.$$

The smoothness requirement on the different functions H_j and G_j is useful for obtaining the asymptotic distribution of the bias correcting terms (see for example Jacod and Rosenbaum (2012) and Jacod and Rosenbaum (2013)).The following theorem summarizes the joint asymptotic behavior of the estimators.

Theorem 1. *Let $[H_r(\widehat{C), G_r}(C)]_T$ be either $[H_r(\widehat{C), G_r}(C)]_T^{AN}$ or $[H_r(\widehat{C), G_r}(C)]_T^{LIN}$ defined in (14) and (16), respectively. Suppose Assumption 1 and Assumption 2 hold. Fix $k_n = \theta \Delta_n^{-1/2}$ for some $\theta \in (0, \infty)$ and set $(8p-1)/4(4p-r) \leq \varpi < \frac{1}{2}$. Then, as $\Delta_n \longrightarrow 0$,*

$$\Delta_n^{-1/4} \begin{pmatrix} [H_1(\widehat{C), G_1}(C)]_T - [H_1(C), G_1(C)]_T \\ \vdots \\ [H_\kappa(\widehat{C), G_\kappa}(C)]_T - [H_\kappa(C), G_\kappa(C)]_T \end{pmatrix} \xrightarrow{L-s} MN(0, \Sigma_T), \tag{20}$$

where $\Sigma_T = \left(\Sigma_T^{r,s}\right)_{1\le r,s\le\kappa}$ denotes the asymptotic covariance between the estimators $[H_r(\widehat{C), G_r}(C)]_T$ and $[H_s(\widehat{C), G_s}(C)]_T$. The elements of the matrix Σ_T are

$$\Sigma_T^{r,s} = \Sigma_T^{r,s,(1)} + \Sigma_T^{r,s,(2)} + \Sigma_T^{r,s,(3)},$$

$$\Sigma_T^{r,s,(1)} = \frac{6}{\theta^3} \sum_{g,h,a,b=1}^{d} \sum_{j,k,l,m=1}^{d} \int_0^T \left(\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(C_s)\right) \Big[C_t(gh,jk)C_t(ab,lm) + C_t(ab,jk)C_t(gh,lm)\Big] dt,$$

$$\Sigma_T^{r,s,(2)} = \frac{151\theta}{140} \sum_{g,h,a,b=1}^{d} \sum_{j,k,l,m=1}^{d} \int_0^T \left(\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(C_t)\right) \Big[\overline{C}_t^{gh,jk}\overline{C}_t^{ab,lm} + \overline{C}_t^{ab,jk}\overline{C}_t^{gh,lm}\Big] dt,$$

$$\Sigma_T^{r,s,(3)} = \frac{3}{2\theta} \sum_{g,h,a,b=1}^{d} \sum_{j,k,l,m=1}^{d} \int_0^T \left(\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(C_t)\right) \Big[C_t(gh,jk)\overline{C}_t^{ab,lm} + C_t(ab,lm)\overline{C}_t^{gh,jk} + C_t(gh,lm)\overline{C}_t^{ab,jk} + C_t(ab,jk)\overline{C}_t^{gh,lm}\Big] dt,$$

with

$$C_t(gh,jk) = C_{gj,t}C_{hk,t} + C_{gk,t}C_{hj,t}.$$

The convergence in Theorem 1 is stable in law (denoted L-s, see for example Aldous and Eagleson (1978) and Jacod and Protter (2012)). The limit is mixed gaussian and the precision of the estimators depends on the paths of the spot covariance and the volatility of volatility process. The rate of convergence $\Delta_n^{-1/4}$ has been shown to be the optimal for volatility of volatility estimation (under the assumption of no volatility jumps).

The asymptotic variance of the estimators depends on the tuning parameter θ whose choice may be crucial for the reliability of the inference. We document the sensitivity of the inference theory to the choice of the parameter θ in a Monte Carlo experiment (see Section 5).

4.3 Estimation of the Asymptotic Covariance Matrix

To provide a consistent estimator for the element $\Sigma_T^{r,s}$ of the asymptotic covariance matrix in Theorem 1, we introduce the following quantities:

$$\widehat{\Omega}_T^{r,s,(1)} = \Delta_n \sum_{g,h,a,b=1}^{d} \sum_{j,k,l,m=1}^{d} \sum_{i=1}^{[T/\Delta_n]-4k_n+1} \left(\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(\widehat{C}_i^n)\right) \Big[\widehat{C}_{i\Delta_n}(gh,jk)\widehat{C}_{i\Delta_n}(ab,lm) + \widehat{C}_{i\Delta_n}(ab,jk)\widehat{C}_{i\Delta_n}(gh,lm)\Big],$$

$$\widehat{\Omega}_T^{r,s,(2)} = \sum_{g,h,a,b=1}^{d} \sum_{j,k,l,m=1}^{d} \sum_{i=1}^{[T/\Delta_n]-4k_n+1} \left(\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(\widehat{C}_i^n)\right) \Big[\frac{1}{2}\widehat{\gamma}_i^{n,gh}\widehat{\gamma}_i^{n,jk}\widehat{\gamma}_{i+2k_n}^{n,ab}\widehat{\gamma}_{i+2k_n}^{n,lm} + \frac{1}{2}\widehat{\gamma}_i^{n,ab}\widehat{\gamma}_i^{n,lm}\widehat{\gamma}_{i+2k_n}^{n,gh}\widehat{\gamma}_{i+2k_n}^{n,jk} + \frac{1}{2}\widehat{\gamma}_i^{n,ab}\widehat{\gamma}_i^{n,jk}\widehat{\gamma}_{i+2k_n}^{n,gh}\widehat{\gamma}_{i+2k_n}^{n,lm} + \frac{1}{2}\widehat{\gamma}_i^{n,gh}\widehat{\gamma}_i^{n,lm}\widehat{\gamma}_{i+2k_n}^{n,ab}\widehat{\gamma}_{i+2k_n}^{n,jk}\Big],$$

$$\widehat{\Omega}_T^{r,s,(3)} = \frac{3}{2k_n} \sum_{g,h,a,b=1}^{d} \sum_{j,k,l,m=1}^{d} \sum_{i=1}^{[T/\Delta_n]-4k_n+1} \left(\partial_{gh} H_r \partial_{ab} G_r \partial_{jk} H_s \partial_{lm} G_s(\widehat{C}_i^n)\right)\times$$
$$\Big[\widehat{C}_{i\Delta_n}(gh,jk)\widehat{\gamma}_i^{n,ab}\widehat{\gamma}_i^{n,lm} + \widehat{C}_{i\Delta_n}(ab,lm)\widehat{\gamma}_i^{n,gh}\widehat{\gamma}_i^{n,jk} + \widehat{C}_{i\Delta_n}(gh,lm)\widehat{\gamma}_i^{n,ab}\widehat{\gamma}_i^{n,jk} + (\widehat{C}_{i\Delta_n}(ab,jk)\widehat{\gamma}_i^{n,gh}\widehat{\gamma}_i^{n,lm}\Big],$$

with $\widehat{\gamma}_i^{n,jk} = \widehat{C}_{i+k_n}^{n,jk} - \widehat{C}_i^{n,jk}$ and $\widehat{C}_{i\Delta_n}(gh,jk) = (\widehat{C}_{gj,i\Delta_n}\widehat{C}_{hk,i\Delta_n} + \widehat{C}_{gk,i\Delta_n}\widehat{C}_{hj,i\Delta_n})$.
The following result holds,

Theorem 2. *Suppose the assumptions of Theorem 1 hold, then, as $\Delta_n \longrightarrow 0$*

$$\begin{aligned}
&\frac{6}{\theta^3}\widehat{\Omega}_T^{r,s,(1)} \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(1)}\\
&\frac{3}{2\theta}[\widehat{\Omega}_T^{r,s,(3)} - \frac{6}{\theta}\widehat{\Omega}_T^{r,s,(1)}] \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(3)}\\
&\frac{151\theta}{140}\frac{9}{4\theta^2}[\widehat{\Omega}_T^{r,s,(2)} + \frac{4}{\theta^2}\widehat{\Omega}_T^{r,s,(1)} - \frac{4}{3}\widehat{\Omega}_T^{r,s,(3)}] \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(2)}.
\end{aligned}$$

The estimated matrix $\widehat{\Sigma}_T$ is symmetric but is not guaranteed to be positive semi-definite. By Theorem 1, $\widehat{\Sigma}_T$ is positive semi-definite in large samples. The form of the asymptotic variance is relatively complicated. Estimating it using subsampling or bootstrap techniques is an interesting research question that is beyond the scope of this paper.

Remark 1: Using results in Jacod and Rosenbaum (2012), it can be shown that the first convergence in Theorem 2 holds at a rate of $\Delta_n^{-1/2}$ while the last convergence rate is $\Delta_n^{-1/4}$ by a straightforward extension of Theorem 1. Our estimator of $\Sigma_T^{r,s,(2)}$ can be shown to have a rate of convergence $\Delta_n^{-1/4}$.

Remark 2: In the one-dimensional case ($d = 1$), much simpler estimators of $\Sigma_T^{r,s,(2)}$ can be constructed using the quantities $\widehat{\gamma}_i^{n,jk}\widehat{\gamma}_i^{n,lm}\widehat{\gamma}_{i+k_n}^{n,gh}\widehat{\gamma}_{i+k_n}^{n,xy}$ or $\widehat{\gamma}_i^{n,jk}\widehat{\gamma}_i^{n,lm}\widehat{\gamma}_i^{n,gh}\widehat{\gamma}_i^{n,xy}$ as in Vetter (2012). However, in the multidimensional case, the latter quantities do not identify separately the quantity $\overline{C_t}^{jk,lm}\overline{C_t}^{gh,xy}$ since the combination $\overline{C_t}^{jk,lm}\overline{C_t}^{gh,xy} + \overline{C_t}^{jk,gh}\overline{C_t}^{lm,xy} + \overline{C_t}^{jk,xy}\overline{C_t}^{gh,lm}$ shows up in a non-trivial way in the limit of the estimator.

Corollary 3. *For $1 \le r \le \kappa$, let $[H_r(\widehat{C), G_r}(C)]_T$ be either $[H_r(\widehat{C), G_r}(C)]_T^{AN}$ or $[H_r(\widehat{C), G_r}(C)]_T^{LIN}$ defined in (16) and (14), respectively. Suppose the assumptions of theorem 1 hold, then we have:*

$$\Delta_n^{-1/4}\widehat{\Sigma}_T^{-1/2}\begin{pmatrix} [H_1(\widehat{C), G_1}(C)]_T - [H_1(C), G_1(C)]_T \\ \vdots \\ [H_\kappa(\widehat{C), G_\kappa}(C)]_T - [H_\kappa(C), G_\kappa(C)]_T \end{pmatrix} \xrightarrow{L} N(0, I_\kappa), \tag{21}$$

In the above, we use the notation L to denote the convergence in distribution and I_κ the identity matrix of order κ. Corollary 3 states the standardized asymptotic distribution, which follows directly from the properties of stable-in-law convergence. Similarly, by the delta method, standardized asymptotic distribution can also be derived for the estimators of the quantities in (12). These standardized distributions allow the construction of confidence intervals for all the latent quantities of the form $[H_r(C), G_r(C)]_T$ and, more generally, functions of these quantities.

4.4 Testing procedure

We now describe the three statistical tests that we are interested in. The test of absence of dependence between the IV of the returns on asset i and j can be formulated as:

$$H_0^1 : [C_{ZiZi}, C_{ZjZj}]_T = 0 \text{ vs } H_1^1 : [C_{ZiZi}, C_{ZjZj}]_T \neq 0.$$

The null hypothesis H_0^1 is rejected whenever,

$$\Delta_n^{-1/4} \frac{\left|\widehat{[C_{ZiZi}, C_{ZjZj}]}_T\right|}{\sqrt{\widehat{AVAR}\Big(C_{ZiZi}, C_{ZjZj}\Big)}} > Z_\alpha.$$

The test of absence of dependence between the IV of stock j and all IV factors Π takes the following form:

$$H_0^2 : [C_{ZjZj}, \Pi]_T = 0 \text{ vs } H_1^2 : [C_{ZjZj}, \Pi]_T \neq 0.$$

Recalling that d_Π denotes the number of IV factors, we reject the above null hypothesis H_0^2 when,

$$\Delta_n^{-1/4} \left(\widehat{[C_{ZjZj}, \Pi]}_T\right)^\top \left(\widehat{AVAR}\Big(C_{ZjZj}, \Pi\Big)\right)^{-1} \widehat{[C_{ZjZj}, \Pi]}_T > \mathcal{X}^2_{d_\Pi, 1-\alpha}.$$

The test of absence of dependence between the NS-IVs can be stated as:

$$H_0^3 : [C^{NS}_{ZiZi}, C^{NS}_{ZjZj}]_T = 0 \text{ vs } H_1^3 : [C^{NS}_{ZiZi}, C^{NS}_{ZjZj}]_T \neq 0,$$

with the null rejected if

$$\Delta_n^{-1/4} \frac{\left|\widehat{[C^{NS}_{ZiZi}, C^{NS}_{ZjZj}]}_T\right|}{\sqrt{\widehat{AVAR}\Big(C^{NS}_{ZiZi}, C^{NS}_{ZjZj}\Big)}} > Z_\alpha.$$

Our inference theory also allows to test more general hypotheses, which are joint across any subset of the panel. In the above statements, $\widehat{[H(C), G(C)]}_T$ can be either $\widehat{[H(C), G(C)]}_T^{AN}$ or $\widehat{[H(C), G(C)]}_T^{LIN}$, $\widehat{AVAR}\Big(H(C), G(C)\Big)$ is an estimate of the asymptotic variance of $\widehat{[H(C), G(C)]}_T$, Z_α stands for the $(1-\alpha)$ quantile of the $N(0,1)$, and $\mathcal{X}^2_{d_q,1-\alpha}$ stands for $(1-\alpha)$ quantile of the $\mathcal{X}^2_{d_q}$ distribution. For the first two tests, the expression for the true asymptotic variance is obtained using Theorem 1 and its estimation follows from Theorem 2. The asymptotic variance of the third test is obtained by an application of the delta method to the convergence result in Theorem 1. The expression of the AVAR for the third test involves some of the latent quantities defined in (12), which can be estimated using either AN- or LIN-type estimators. Therefore in general, we have two tests for each null hypothesis, corresponding to the two type of estimators for $[H(C), G(C)]_T$. Under (2) and the assumptions of Theorem 1, Corollary 3 implies that the asymptotic size of the two types of tests for the null hypotheses H_0^1 and H_0^2 is α, and their power approaches 1. The same properties apply for the tests of the null hypotheses H_0^3 as long as (2) and our IV-FM representation (6) hold.

Theoretically, it is possible to test for absence of dependence in the IVs at the spot level. In this case the null hypothesis is $H_0'^1 : [C_{ZiZi}, C_{ZjZj}]_t = 0 \;\;$ for all $\;\; 0 \leq t \leq T$, which is, in theory, stronger than our $H_0'^1$. In particular, Theorem 1 can be used to set up Kolmogorov-Smirnov type of tests for $H_0'^1$ in the same spirit as Vetter (2012). However, we do not pursue this direction in the current paper for two reasons. First, the testing procedure would be more involved. Second, empirical evidence suggests nonnegative dependence between IVs, which means that in practice, it is not too restrictive to assume $[C_{ZiZi}, C_{ZjZj}]_t \geq 0 \; \forall t$, under which H_0^1 and $H_0'^1$ are equivalent.

5 Monte Carlo

This section investigates the finite sample properties of our estimators and tests. The data generating process (DGP) is constructed as follows. Denote by $(Y_j)_{1 \leq j \leq 27}$ the log-prices of 27 individual stocks, and by X the log-price of the market portfolio. The number of stocks matches the one we use in the empirical application. Recall that the superscript c indicates the continuous part of a process. We assume

$$dX_t = dX_t^c + dJ_t, \quad dX_t^c = \sqrt{C_{XX,t}} dW_t,$$

and, for $j = 1, \ldots, 27$,

$$dY_{j,t} = \beta_{j,t} dX_t^c + d\widetilde{Y}_{j,t}^c + dJ_{j,t}, \quad d\widetilde{Y}_{j,t}^c = \sqrt{C_{ZjZj,t}} d\widetilde{W}_{j,t}.$$

In the above, C_{XX} is the spot volatility of the market portfolio, $(\widetilde{W}_j)_{1 \leq j \leq 27}$, are Brownian motions with $\text{Corr}(d\widetilde{W}_{j,t}, d\widetilde{W}) = 0.2\omega_j$ where $(\omega_j)_{1 \leq j \leq 27}$ are independent variables uniformly distributed over [-1,1]; W and $\widetilde{W}$ are two independent Brownian motions; J and $(J_j)_{1 \leq j \leq 27}$ are independent compound Poisson processes with intensity equal to 2 jumps per year and jump size distribution N(0,0.02^2). The beta process is time-varying and is specified as $\beta_{j,t} = \pi_j \beta_t$ with the π_j's being independent variables uniformly distributed over [0,1] and $\beta_t = 0.5 + 0.1 \sin(100t)$.

We next specify the volatility processes. The market volatility process $C_{XX,t}$ is generated as follows,

$$dC_{XX,t} = 5(0.09 - C_{XX,t})dt + 0.35\sqrt{C_{XX,t}}(-0.8 dW_t + \sqrt{1 - 0.8^2} dB_t),$$

where B is a standard Brownian Motion, which is independent from the Brownian Motions of the return Factor Model. The common Brownian Motion W_t in the market portfolio price process X_t and its volatility process $C_{XX,t}$ generates a leverage effect for the market portfolio.[11] The value of the leverage effect is -0.8, which is standard in the literature, see Kalnina and Xiu (2015), Aït-Sahalia, Fan, and Li (2013) and Aït-Sahalia, Fan, Laeven, Wang, and Yang (2013). To create a

[11] Due to the factor structure of the stocks returns, the leverage effect of the market portfolio translates into a leverage effect for the individual stocks.

factor structure in the IV processes $C_{ZjZj,t}$, we generate the following processes

$$df_{j,t} = 5(0.09 - f_{j,t})dt + 0.35\sqrt{f_{j,t}}dB_{j,t} \quad , \text{ for } j = 0, \dots, 27,$$

where $(B_j)_{0\leq j\leq 27}$ are independent Browian motions which are independent from existing ones, see Table 1 for the different specifications for the IV processes.[12]

	$C_{ZjZj,t}$
Model 1	$0.67u_j + 0.8f_{j,t}$
Model 2	$0.67u_j + (0.32 + 0.8v_j)C_{XX,t} + 0.8f_{j,t}$
Model 3	$0.67u_j + (0.32 + 0.8v_j)C_{XX,t} + (0.28 + 0.8w_j)f_{0,t} + 0.8f_{j,t}$

Table 1: Different specifications for the Idiosyncratic Volatility processes $C_{ZjZj,t}$ for $j = 1, \dots, 27$. The random variables u_j, v_j and w_j and independent and uniformly distributed over [0,1].

We set the time span T equal 1260 or 2520 days, which correspond approximately to 5 and 10 business years. These values are close to those typically used in the nonparametric leverage effect estimation literature (see Aït-Sahalia, Fan, and Li (2013) and Kalnina and Xiu (2015)), which is related to the problem of volatility of volatility estimation. Each day consists of 6.5 trading hours. We consider two different values for the sampling frequency, $\Delta_n = 1$ minute and $\Delta_n = 5$ minutes. We follow Li, Todorov, and Tauchen (2013) and set the truncation threshold u_n in day t at $3\widehat{\sigma}_t\Delta_n^{0.49}$, where $\widehat{\sigma}_t$ is the squared root of the annualized bipower variation of Barndorff-Nielsen and Shephard (2004). We use 10 000 Monte Carlo replications in all the experiments.

We first investigate the finite sample properties of the estimators under Model 3. The considered estimators include:

- the IV beta of the first stock $\left(b_{Z1}\right)$,
- the contribution of the market volatility to the variation of the IV of the first stock $\left(R_{Z1}^{2,IV\text{-}FM}\right)$,
- the correlation between the idiosyncratic volatilities of stocks 1 and 2 $\left(\rho_{Z1,Z2}\right)$,
- the correlation between non-systematic idiosyncratic volatilities $\left(\rho_{Z1,Z2}^{NS}\right)$,

The interpretation of simulation results is much simpler when the quantities of interest do not change across simulations. To achieve that, we generate once and keep fixed the paths of the processes $C_{XX,t}$ and $(f_{j,t})_{0\leq j\leq 27}$ and replicate several times the other parts of the DGP. In Table 2, we report the bias and the interquartile range (IQR) of the two type of estimators for each quantity using 5 minutes data sampled over 10 years. We choose four different values for the width of the subsamples, which corresponds to $\theta = 1.5, 2, 2.5$ and 3 (recall that the number of observations in a window is $k_n = \theta/\sqrt{\Delta_n}$). It seems that larger values of the parameters produce better results.

[12] The $(f_{j,t})_{0\leq j\leq 27}$ are independent Cox Ingersoll Ross processes which fulfill the Feller property. Therefore they are ensured to take always positive values.

Next, we investigate how these results change when we increase the sampling frequency. In Table 3, we report the results with $\Delta_n = 1$ minute in the same setting. We note a reduction of the bias and IQR at all levels of significance. However, the magnitude of the decrease of the IQR is very small. Finally, we conduct the same experiment using data sampled at one minute over 5 years. Despite using more than twice as many observations than in the first experiment, the precision is not as good. In other words, increasing the time span is more effective for precision gain than increasing the sampling frequency. This result is typical for $\Delta_n^{1/4}$-convergent estimators, see, e.g., Kalnina and Xiu (2015).

	AN				LIN			
θ	1.5	2	2.5	3	1.5	2	2.5	3
				Median Bias				
$\widehat{b}_{Z1}$	-0.047	-0.025	-0.011	-0.003	-0.006	0.001	0.009	0.015
$\widehat{R}_{Z1}^{2,IV\text{-}FM}$	0.176	0.130	0.103	0.085	0.181	0.140	0.112	0.092
$\widehat{\rho}_{Z1,Z2}$	-0.288	-0.212	-0.163	-0.133	-0.249	-0.190	-0.146	-0.120
$\widehat{\rho}_{Z1,Z2}^{NS}$	-0.189	-0.113	-0.064	-0.034	-0.150	-0.091	-0.047	-0.021
				IQR				
$\widehat{b}_{Z1}$	0.222	0.166	0.138	0.121	0.226	0.168	0.139	0.122
$\widehat{R}_{Z1}^{2,IV\text{-}FM}$	0.210	0.188	0.172	0.152	0.181	0.166	0.152	0.140
$\widehat{\rho}_{Z1,Z2}$	0.404	0.325	0.263	0.223	0.338	0.283	0.237	0.205
$\widehat{\rho}_{Z1,Z2}^{NS}$	0.456	0.384	0.315	0.272	0.388	0.337	0.285	0.250

Table 2: Finite sample properties of our estimators using 10 years of data sampled at 5 minutes. The true values are $b_{Z1} = 0.450$, $R_{Z1}^{IV\text{-}FM} = 0.342$, $\rho_{Z1,Z2} = 0.523$, $\rho_{Z1,Z2}^{NS} = 0.424$.

	AN				LIN			
θ	1.5	2	2.5	3	1.5	2	2.5	3
				Median Bias				
$\widehat{b}_{Z1}$	-0.022	-0.012	-0.003	0.004	-0.003	-0.000	0.006	0.012
$\widehat{R}_{Z1}^{IV\text{-}FM}$	0.107	0.091	0.073	0.056	0.113	0.095	0.075	0.058
$\widehat{\rho}_{Z1,Z2}$	-0.147	-0.104	-0.073	-0.048	-0.133	-0.097	-0.067	-0.042
$\widehat{\rho}_{Z1,Z2}^{NS}$	-0.135	-0.086	-0.058	-0.039	-0.119	-0.078	-0.052	-0.032
				IQR				
$\widehat{b}_{Z1}$	0.156	0.112	0.088	0.075	0.157	0.112	0.088	0.075
$\widehat{R}_{Z1}^{IV\text{-}FM}$	0.201	0.146	0.118	0.100	0.184	0.138	0.113	0.096
$\widehat{\rho}_{Z1,Z2}$	0.340	0.238	0.184	0.150	0.309	0.226	0.177	0.145
$\widehat{\rho}_{Z1,Z2}^{NS}$	0.417	0.291	0.228	0.184	0.378	0.274	0.217	0.177

Table 3: Finite sample properties of our estimators using 10 years of data sampled at 1 minute. The true values are $b_{Z1} = 0.450$, $R_{Z1}^{2,IV\text{-}FM} = 0.336$, $\rho_{Z1,Z2} = 0.514$, $\rho_{Z1,Z2}^{NS} = 0.408$.

Next, we study the size and power of the three statistical tests as outlined in Section 4.4. We

θ	AN 1.5	AN 2	AN 2.5	AN 3	LIN 1.5	LIN 2	LIN 2.5	LIN 3
				Median Bias				
$\widehat{b}_{Z1}$	-0.019	-0.011	-0.007	0.000	-0.001	-0.001	0.002	0.008
$\widehat{R}_{Z1}^{2,IV\text{-}FM}$	0.115	0.096	0.081	0.069	0.119	0.100	0.084	0.071
$\widehat{\rho}_{Z1,Z2}$	-0.168	-0.101	-0.064	-0.038	-0.149	-0.092	-0.057	-0.033
$\widehat{\rho}_{Z1,Z2}^{NS}$	-0.141	-0.079	-0.035	-0.007	-0.127	-0.067	-0.029	-0.001
				IQR				
$\widehat{b}_{Z1}$	0.215	0.159	0.128	0.110	0.216	0.158	0.129	0.110
$\widehat{R}_{Z1}^{2,IV\text{-}FM}$	0.282	0.204	0.168	0.144	0.260	0.194	0.161	0.139
$\widehat{\rho}_{Z1,Z2}$	0.472	0.337	0.263	0.213	0.436	0.319	0.252	0.206
$\widehat{\rho}_{Z1,Z2}^{NS}$	0.541	0.412	0.324	0.266	0.510	0.391	0.311	0.256

Table 4: Finite sample properties of our estimators using 5 years of data sampled at 1 minute. The true values are $b_{Z1} = 0.450$, $R_{Z1}^{2,IV\text{-}FM} = 0.35$, $\rho_{Z1,Z2} = 0.517$, $\rho_{Z1,Z2}^{NS} = 0.417$.

use Model 1 to study the size properties of the first two tests: the test of the absence of dependence between the IVs ($H_0^1 : [C_{Z1Z1}, C_{Z2Z2}]_T = 0$), and the absence of dependence between the IV of the first stock and the market volatility ($H_0^2 : [C_{Z1Z1}, C_{XX}]_T = 0$). We use Model 2 to study the size properties of the third test ($H_0^3 : [C_{Z1Z1}^{NS}, C_{Z2Z2}^{NS}]_T = 0$). Finally, we use Model 3 to study power properties of all three tests.

The upper panel Tables 5, 6, and 7 reports the size results while the lower panels shows the results for the power. We present the results for the two sampling frequencies ($\Delta_n = 1$ minute and $\Delta_n = 5$ minutes) and the two type of tests (AN and LIN). We observe that the size of three tests are reasonably close to their nominal levels. The rejection probabilities under the alternatives are rather high, except when the data is sampled at 5 minutes frequency and the nominal level at 1%.[13] We note that the tests based on LIN estimators have better testing power compared to those that build on AN estimators. Increasing the window length induces some size distortions but is very effective for power gain. Consistent with the asymptotic theory, the size of the three tests are closer to the nominal levels and the power is higher at the one minute sampling frequency. Clearly, the test of absence of dependence between IV and the market volatility has the best power, followed by the test of absence of dependence between the two IVs. This ranking is compatible with the notion that the finite sample properties of the tests deteriorate with the degree of latency embedded in each null hypothesis.

[13] We set the nominal level at 5% in the empirical application.

	$\Delta_n = 5$ minutes						$\Delta_n = 1$ minute					
	$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$		$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$	
Type of the test	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN
						Panel A : Size Analysis-Model 1						
$\alpha = 20\%$	19.5	21.0	19.4	20.4	19.4	20.7	20.2	19.6	19.7	19.9	19.8	20.1
$\alpha = 10\%$	9.7	10.6	10.6	12.6	9.7	10.3	10.2	9.7	10.0	10.2	9.8	10.2
$\alpha = 5\%$	4.7	5.1	4.5	5.3	4.8	5.6	5.3	5.3	5.2	5.3	4.9	5.1
$\alpha = 1\%$	0.9	1.1	0.9	1.2	0.9	1.1	1.1	1.1	1.2	1.1	1.0	1.0
						Panel B : Power Analysis-Model 3						
$\alpha = 20\%$	33.5	45.7	50.4	63.1	67.8	78.1	48.5	56.2	77.7	82.3	94.1	95.8
$\alpha = 10\%$	20.5	31.5	35.7	48.3	53.3	65.8	33.9	41.0	65.6	71.6	88.0	91.2
$\alpha = 5\%$	11.9	21.0	23.9	35.76	40.6	53.4	22.3	29.5	52.8	59.8	79.6	84.4
$\alpha = 1\%$	3.3	6.9	8.7	15.6	18.4	28.6	8.9	12.4	28.6	34.5	57.4	64.1

Table 5: Size and Power of the test of absence of dependence between idiosyncratic volatilities for $T =$ 10 years.

6 Empirical Analysis

We apply our methods to study the cross-sectional dependence in IV using high frequency data. We find that stocks' idiosyncratic volatilities co-move strongly with the market volatility. This is a quite surprising finding. It is of course well known that the total volatility of stocks moves with the market volatility. However, we stress that we find that the strong effect is still present when considering the idiosyncratic volatilities.

We use full record transaction prices from NYSE TAQ database for 27 stocks over the time period 2003-2012. After removing the non-trading days, our sample contains 2517 days. The selected stocks have been part of S&P 500 index throughout our sample. Our 27 stocks contain three liquid stocks in each of the nine sectors of the index (Consumer Discretionary, Consumer Staples, Energy, Financial, Health Care, Industrial, Materials, Technology, and Utilities). For each day, we consider data from the regular exchange opening hours from time stamped between 9:30 a.m. until 4 p.m. We clean the data following the procedure suggested by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008), remove the overnight returns and then sample at 5 minutes. This sparse sampling has been widely used in the literature because the effect of the microstructure noise and potential asynchronicity of the data is less important at this frequency, see also Liu, Patton, and Sheppard (2014).

The parameter choices for the estimators are as follows. Guided by our Monte Carlo results, we set the length of window to be approximately one week for the estimators in Section 3 (this corresponds to $\theta = 2.5$ where $k_n = \theta\Delta_n^{-1/2}$ is the number of observations in a window). The truncation threshold for all estimators is set as in the Monte Carlo study ($3\widehat{\sigma}_t\Delta_n^{0.49}$ where $\widehat{\sigma}_t^2$ is the

	$\Delta_n = 5$ minutes						$\Delta_n = 1$ minute					
	$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$		$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$	
Type of test	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN
						Panel A : Size Analysis-Model 1						
$\alpha = 20\%$	22.6	20.1	20.0	21.0	19.8	21.5	21.6	20.6	21.6	21.5	21.1	21.5
$\alpha = 10\%$	12.1	10.2	10.0	10.6	9.8	11.0	11.0	10.4	10.3	10.4	10.4	10.4
$\alpha = 5\%$	6.2	5.0	4.5	5.2	4.6	5.4	5.5	5.4	5.2	5.1	5.2	5.3
$\alpha = 1\%$	1.5	1.0	0.8	1.0	0.9	1.2	1.1	1.1	1.0	0.9	0.8	1.0
						Panel B : Power Analysis-Model 3						
$\alpha = 20\%$	73.1	80.7	91.4	93.9	97.4	98.3	95.8	97.2	99.7	99.8	100	100
$\alpha = 10\%$	60.0	69.0	84.0	88.3	94.6	96.1	91.1	93.3	99.2	99.4	100	100
$\alpha = 5\%$	47.7	57.2	75.0	81.0	89.6	92.6	84.9	88.2	98.2	98.6	100	100
$\alpha = 1\%$	24.1	32.3	52.2	60.1	73.7	78.9	67.7	72.0	93.0	94.5	99.2	99.4

Table 6: Size and Power of the test of absence of dependence between the idiosyncratic volatility and the market volatility for $T = 10$ years.

bipower variation).

We consider two sets of factors in the factor model for the prices: the S&P500 market index and the three Fama-French factors (FF3 henceforth). All factors are sampled at 5 minutes over 2003-2012.[14]

Figures 1 and 2 contain plots of the time series of the estimated R^2_{Yj} of the return regressions, i.e., the estimated monthly contribution of the return factors to the total volatility, for each stock in the two models (CAPM and FF3).[15] In Table 8, we report the average of these monthly statistics over the full sample. As we can see in Table 8, the return factors have a relatively low contribution to the total variation of the returns in the two models which implies a high contribution of the idiosyncratic volatility $(1 - R^2_{Yj})$. Indeed, the minimum value (across all the stocks) for $1 - R^2_{Yj}$ is 61.5% for the one-factor model and 53.5% in the FF3 model. Figures 1 and 2 show that the time series of all stocks follow approximately the same trend with a considerable increase in the contribution around the crisis year 2008, which shows that the systematic risk became relatively more important during this period. Overall our results suggest that IV contributes to more than a half of the total variation for each stock. Therefore, studying the source of variation in IVs is potentially useful. We proceed to investigate the dynamic properties of the panel of idiosyncratic volatilities.

[14] The high-frequency data on the Fama-French factors were obtained from Aït-Sahalia, Kalnina, and Xiu (2014).

[15] For the j^{th} stock, our analog of the coefficient of determination in the return factor model for this stock is $R^2_{Yj} = 1 - \frac{\int_0^T c_{ZjZj,t}dt}{\int_0^T c_{YjYj,t}dt}$. We estimate R^2_{Yj} using the general method of Jacod and Rosenbaum (2013). The resulting estimator of R^2_{Yj} requires a choice of a block size for the spot volatility estimation; we choose two hours in practice (the number of observations in a block, say l_n, has to satisfy $l_n^2\Delta_n \to 0$ and $l_n^3\Delta_n \to \infty$, so it is of smaller order than the number of observations k_n in our estimators of Section 3).

	$\Delta_n = 5$ minutes						$\Delta_n = 1$ minute					
	$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$		$\theta = 1.5$		$\theta = 2.0$		$\theta = 2.5$	
Type of test	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN	AN	LIN
						Panel A : Size Analysis-Model 2						
$\alpha = 20\%$	19.9	23	20.4	23.7	20.2	23.2	19.7	20.5	20.3	21.7	20.0	22.3
$\alpha = 10\%$	10.0	10.1	12.1	10.8	9.9	12.6	10.1	10.3	10.6	11.3	10.1	11.4
$\alpha = 5\%$	5.0	6.3	5.1	6.3	5.1	6.7	5.5	5.5	5.3	5.9	5.2	6.0
$\alpha = 1\%$	1.1	1.5	0.8	1.6	1.1	1.4	1.1	1.2	1.3	1.3	1.3	1.5
						Panel B : Power Analysis-Model 3						
$\alpha = 20\%$	25.1	32.1	29.0	36.5	42.8	51.7	31.0	35	50.0	54.6	68.0	72.3
$\alpha = 10\%$	13.7	19.2	16.8	23.0	28.1	36.9	19.0	22.2	35.0	39.4	53.4	58.3
$\alpha = 5\%$	7.4	11.3	9.3	14.2	18.3	25.2	11.0	13.7	23.9	28.0	40.0	44.9
$\alpha = 1\%$	1.6	3.1	2.3	3.9	6.0	9.5	2.9	4.0	9.3	11.6	18.8	22.2

Table 7: Size and Power of the test of absence of dependence between NS-IVs for $T = 10$ years.

We first investigate the (total) dependence in the idiosyncratic volatilities. We have 351 pairs of stocks available in the panel. For each pair of stocks, we compute the correlation between the IVs, $\rho_{Zi,Zj}$. The upper and lower panels of Table 13 display the correlations estimated using the LIN-type estimators in the CAPM and FF3 as the factor model for prices.[16] The values in parenthesis correspond to the p-values of the test of dependence in the IVs (see Section 4.4 for an expression of the test statistic). The reader should be careful when interpreting these p-values because they are not adjusted for multiple testing. Clearly, there is evidence for strong dependence between the IVs. Indeed, the absolute values of the t-statistics are bigger than 1.96 for 350 pairs over 351. Only the dependence between the IV of the Goldman Sachs (GS) and IBM gives rise to the absolute value of the t-statistic smaller than 1.96. Using the Bonferroni correction, the p-value of the test of absence of dependence in all pairs is less than 0.0001. The estimated correlation is positive for each pair of stocks. We also observe substantial heterogeneity in the correlation with a maximum value of 94.4% (Exxon Mobil (XOM) and Chevron Corp (CVX)) and a minimum value of 24.8% (Duke Energy (DUK) and Avery Dennison Corporation (AVY)). Table 9 is a summary of the results of Table 13; it shows the number of pairs with the estimated correlation greater than a set of thresholds. For example, it shows that the correlation is greater than 50% for more than two thirds of the pairs (265). Interestingly, the results of the test are unchanged for the FF3 model, and the estimated correlations are very close to those obtained in the CAPM. This result is not surprising given the relatively small difference between the values of R^2_{Yj} in the two models.

We next ask the question of whether potential missing factors in the factor model for returns might be responsible for the strong dependence in IVs. Omitted factors in the factor model for

[16]This choice is motivated by our simulation results where LIN type of estimators and tests appear to have better finite sample properties than AN type estimators.

returns induce correlation between the estimated idiosyncratic returns, $\text{Corr}(Z_i, Z_j)$.[17] We report in Table 12 the estimated correlations $\text{Corr}(Z_i, Z_j)$. Table 10 presents a summary of how estimates of $\text{Corr}(Z_i, Z_j)$ in Table 12 are related to the estimates of correlation in IVs, $\rho_{Zi,Zj}$, in Table 13. In particular, different rows in Table 10 display average values of $\rho_{Zi,Zj}$ among those pairs, for which $\text{Corr}(Z_i, Z_j)$ is below some threshold. For example, the last-but-one row in Table 10 indicates that there are 56 pairs of stocks with estimated $\text{Corr}(Z_i, Z_j) < 0.01$, and among those stocks, the average correlation between IVs, $\rho_{Zi,Zj}$, is estimated to be 0.579. This estimate $\rho_{Zi,Zj}$ is virtually the same among pairs of stocks with high $\text{Corr}(Z_i, Z_j)$. Therefore, we know that among 56 pairs of stocks, a missing return factor cannot explain dependence in IVs. Moreover, these results suggest that missing return factor cannot explain dependence in IVs for all considered stocks. These results are in line with recent findings of Herskovic, Kelly, Lustig, and Nieuwerburgh (2014) with daily and monthly returns.

To understand the source of the strong dependence in the IVs, we consider the Idiosyncratic Volatility Factor Model (IV-FM) of Section 2. We use the market volatility as the single IV factor.[18] We start by considering individual stocks separately. In Table 11, we report the estimates of the idiosyncratic volatility beta ($\widehat{b}_{Zi}$) and the contribution of the market volatility to the aggregate variation in IV ($R^{2,IV\text{-}FM}_{Zi}$). The absolute values of the t-statistics based on the covariation between IV and the market volatility are bigger than 1.96 for each stock. For every stock, the estimated IV beta is positive, suggesting that the idiosyncratic volatility co-moves with the market volatility. For 10 stocks out of 27, the NS-IV contributes to more than 50% of the variation in their IV, with the average being 44%.

Next, we turn to the implications of the IV-FM for the cross-section. We conduct inference on dependence in the NS-IVs. Table 14 displays the estimated correlation between the NS-IVs. The residual correlations are smaller than the total IV correlations. There are only 26 pairs of stocks with this correlation higher than 50% in the CAPM model and 27 pairs in the FF3 model. Interestingly, they all remain positive. The t-statistics based on covariation between NS-IVs are larger than 1.96 for 241 pairs in the CAPM and 244 pairs in the FF3 model (see the values in parenthesis of both Tables 14 and 13). From Table 11, each stock has at least eight other stocks with whom it produces a t-statistic bigger than 1.96. Using the Bonferroni correction, the p-value of the test of absence of dependence in all pairs is less than 0.0001. In Table 14 we report, for each pair of stocks, the correlation between the NS-IVs, $\rho^{NS}_{Zi,Zj}$. The values are much smaller than

[17] Our measure of correlation between the idiosyncratic returns Z_i and Z_j is

$$\text{Corr}(Z_i, Z_j) = \frac{\int_0^T C_{ZiZj,t}dt}{\sqrt{\int_0^T C_{ZiZi,t}dt}\sqrt{\int_0^T C_{ZjZj,t}dt}}, \quad i,j = 1, \ldots, d_S, \tag{22}$$

where $C_{ZiZj,t}$ is the spot covariation between the idiosyncratic returns Z_i and Z_j. Similarly to R^2_{Yj}, we estimate $\text{Corr}(Z_i, Z_j)$ using the method of Jacod and Rosenbaum (2013).

[18] We also considered the volatility of size and value Fama-French factors. However, both these factors turned out to have very low volatility of volatility and therefore did not significantly change the results.

the correlations between the total IVs ($\rho_{Zi,Zj}$) in Table 13. We conclude that despite the market volatility explaining most of the cross-sectional dependence in IVs, it does not explain all of it. Additional IV factors may help to explain all the dependence in the idiosyncratic volatilities.

Sector	Stock	Ticker	CAPM	FF3 Model
Financial	American Express Co.	AXP	34.8	43.5
	Goldman Sachs Group	GS	34.5	43.1
	JPMorgan Chase & Co.	JPM	37.0	45.2
Energy	Chevron Corp.	CVX	35.8	44.2
	Schlumberger Ltd.	SLB	26.0	35.7
	Exxon Mobil Corp.	XOM	38.5	46.5
Consumer Staples	Coca Cola Company	KO	24.8	34.6
	Procter & Gamble	PG	25.2	34.9
	Wal-Mart Stores	WMT	26.3	36.0
Industrials	Caterpillar Inc.	CAT	36.8	45.2
	3M Company	MMM	36.6	45.0
	United Technologies	UTX	36.2	44.5
Technology	Cisco Systems	CSCO	34.5	43.1
	International Bus. Machines	IBM	367	44.8
	Intel Corp.	INTC	37.0	45.2
Health Care	Johnson & Johnson	JNJ	24.4	34.3
	Merck & Co.	MRK	21.4	31.5
	Pfizer Inc.	PFE	25.5	33.9
Consumer Discretionary	Home Depot	HD	30.9	39.9
	McDonald's Corp.	MCD	24.2	34.1
	Nike	NKE	25.5	35.3
Utilities	Air Products & Chemicals Inc	APD	33.2	42.0
	Allegheny Technologies Inc	ATI	28.7	38
	Avery Dennison Corp	AVY	29.3	38.5
Material	Duke Energy	DUK	19.1	29.7
	CenterPoint Energy	CNP	17.2	28.0
	Exelon Corp.	EXC	19.2	29.8

Table 8: Average of the monthly contribution of the market volatility to stocks total volatility(R^2_{Yj}) over the period 2003:2012 in percentages. The first column provides information on the sectors, the second the names of the companies and the third their tickers. The fourth and and fifth columns show R^2_{Yj} for the CAPM and FF3 return model.

$\widehat{\rho}_{Zi,Zj}$	CAPM	FF3 Model
> 0.9	2	2
> 0.8	13	13
> 0.7	60	58
> 0.6	158	163
> 0.5	265	265
> 0.4	323	323
> 0.3	350	350
> 0.2	350	350
> 0.1	350	350
$\neq 0$	350	350

Table 9: Number of pairs of stocks with significant dependence between their IVs and the estimated correlation greater than the threshold given in the first column. The second column shows the results for the CAPM model. The results for the FF3 model are reported in the third column.

	CAPM			FF3 Model		
$\lvert\widehat{\mathrm{Corr}}(Z_i, Z_j)\rvert$	Pairs	Avg $\lvert\widehat{\mathrm{Corr}}(Z_i, Z_j)\rvert$	Avg $\widehat{\rho}_{Zi,Zj}$	Pairs	Avg $\lvert\widehat{\mathrm{Corr}}(Z_i, Z_j)\rvert$	Avg $\widehat{\rho}_{Zi,Zj}$
< 0.6	351	0.043	0.585	351	0.043	0.586
< 0.4	350	0.042	0.584	350	0.042	0.585
< 0.3	348	0.040	0.583	348	0.041	0.584
< 0.2	343	0.037	0.583	343	0.038	0.584
< 0.1	323	0.031	0.580	323	0.031	0.581
< 0.075	303	0.028	0.579	304	0.028	0.581
< 0.05	265	0.023	0.570	266	0.023	0.571
< 0.025	152	0.013	0.568	152	0.013	0.566
< 0.01	56	0.005	0.579	56	0.005	0.574
< 0.005	29	0.003	0.580	27	0.003	0.580

Table 10: We report the number of pairs of stocks with the absolute value of the correlation between their idiosyncratic returns smaller than the threshold given in the first column, the average of the absolute value of the idiosyncratic returns correlation for those pairs as well as the average of the IVs correlation for the same pairs. The results are presented both for the CAPM and the FF3 models.

Stock	CAPM $\widehat{b}_z$	CAPM $\widehat{R}_Z^{2,IV\text{-}FM}(\%)$	CAPM p-val	CAPM #	FF3 Model $\widehat{b}_z$	FF3 Model $\widehat{R}_Z^{2,IV\text{-}FM}(\%)$	FF3 Model p-val	FF3 Model #
AXP	1.600	46.9	0.010	13	1.584	46.8	0.011	13
GS	2.313	24.8	0.024	13	2.341	25.5	0.018	13
JPM	1.899	29.1	0.004	8	1.894	29.0	0.004	9
CVX	0.611	51.0	0.008	20	0.603	51.1	0.008	20
SLB	1.064	52.2	0.005	16	1.043	48.4	0.005	15
XOM	0.576	58.0	0.004	21	0.575	51.6	0.004	19
KO	0.327	56.8	0.013	13	0.328	56.7	0.012	14
PG	0.427	61.5	0.002	18	0.424	61.5	0.002	19
WMT	0.445	56.5	0.007	22	0.444	56.7	0.007	22
CAT	0.589	42.1	0.002	13	0.590	42.0	0.003	14
MMM	0.389	53.7	0.000	16	0.386	53.1	0.000	15
UTX	0.501	48.6	0.004	17	0.501	51.3	0.004	17
CSCO	0.571	51.4	0.002	20	0.562	44.1	0.002	21
IBM	0.339	42.3	0.015	9	0.343	43.2	0.011	9
INTC	0.454	44.7	0.003	23	0.451	44.0	0.003	22
JNJ	0.404	67.6	0.007	22	0.401	67.1	0.007	21
MRK	0.535	31.1	0.001	24	0.534	31.1	0.001	24
PFE	0.434	34.1	0.002	22	0.425	24.0	0.001	22
HD	0.500	42.5	0.005	19	0.499	42.5	0.004	16
MCD	0.516	39.5	0.002	15	0.518	39.7	0.002	15
NKE	0.528	44.1	0.000	25	0.526	44.1	0.000	24
APD	0.535	52.4	0.001	24	0.527	51.2	0.001	24
ATI	1.698	33.2	0.001	20	1.726	34.1	0.001	19
AVY	0.315	18.4	0.001	15	0.312	18.3	0.001	14
DUK	0.415	21.3	0.002	19	0.415	21.7	0.002	20
CNP	0.740	29.2	0.001	20	0.737	29.3	0.001	20
EXC	0.927	60.2	0.001	21	0.926	59.9	0.001	21

Table 11: Estimates of the IV beta ($\widehat{b}_Z$) and the contribution of the market volatility to the variation in the IV ($\widehat{R}_Z^{2,IV\text{-}FM}$). We use the market volatility as the IV factor. P-val is the p-value of the test of the absence of dependence between the IV and the market volatility for a given individual stock. In the column with the heading #, we report the number of stocks with their NS-IV having a relatively large covariation with the NS-IV of the stock listed in the first column (in particular, when the t-statistic based on the covariation between the NS-IVs is larger than 1.96 in the absolute value).

7 Conclusion

This paper provides tools for the analysis of cross-sectional dependencies in idiosyncratic volatilities using high frequency data. First, using a factor model in prices, we develop inference theory for covariances and correlations between the idiosyncratic volatilities. Next, we study an idiosyncratic volatility factor model, in which we decompose the co-movements in idiosyncratic volatilities into two parts: those related to factors such as the market volatility, and the residual co-movements. We provide the asymptotic theory for the estimators in the decomposition.

Empirically, we find that our IV Factor Model with market volatility as the only factor explains a large part of the cross-sectional dependence in IVs. However, it is not able to explain all of it. It therefore opens the room for the construction of additional IV factors based on economic theory, for example, along the lines of the heterogeneous agents model of Herskovic, Kelly, Lustig, and Nieuwerburgh (2014).

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Appendix

A Figures and Tables

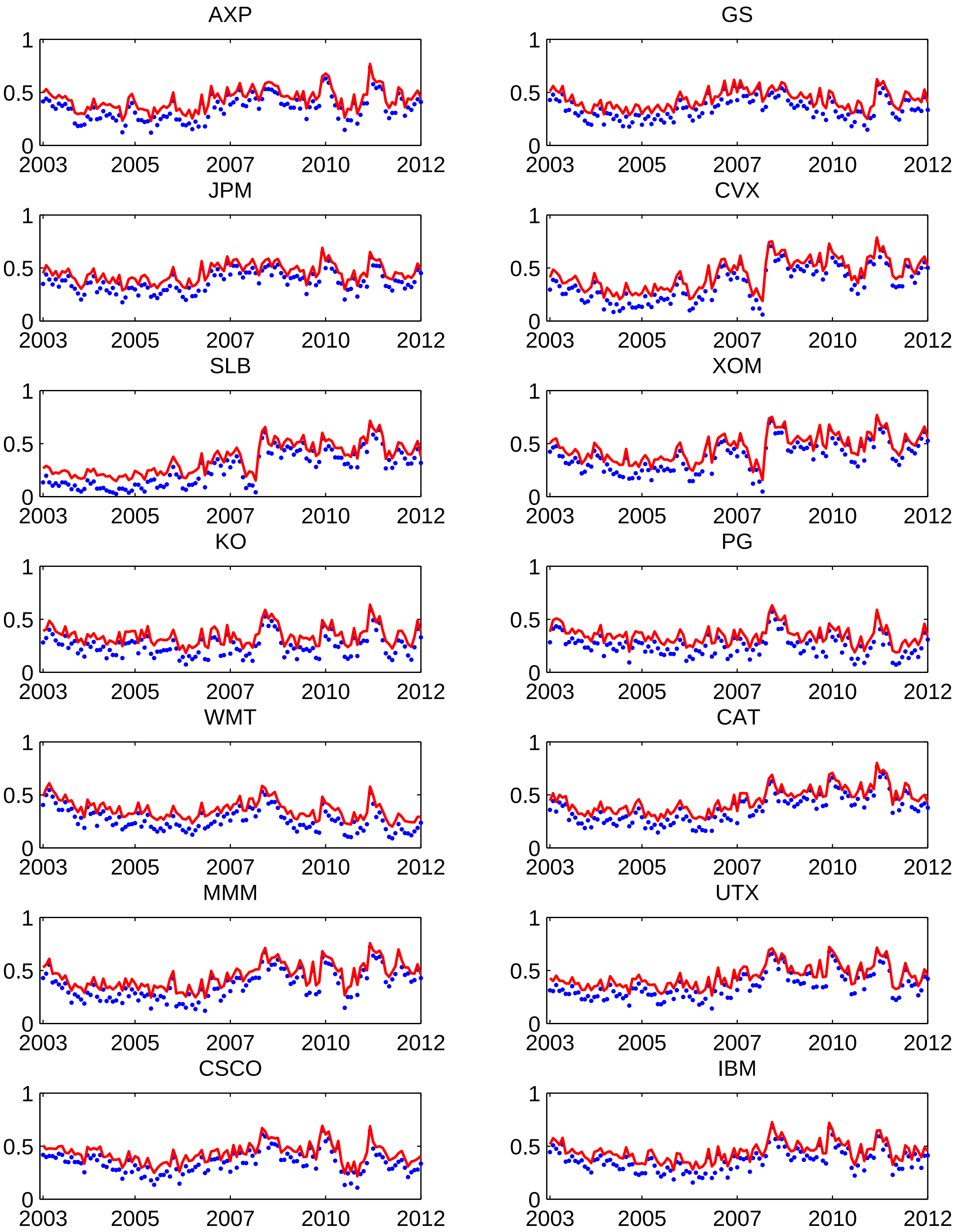

Figure 1: Monthly contribution of the return factors to the total volatility ($\widehat{R}^2_{Y_j}$) over the period 2003:2012. The dotted blue line plots this measure calculated in CAPM model. The solid red line plots the same measure obtained in the FF3 model. We use the ticker of the stocks to label the graphs.

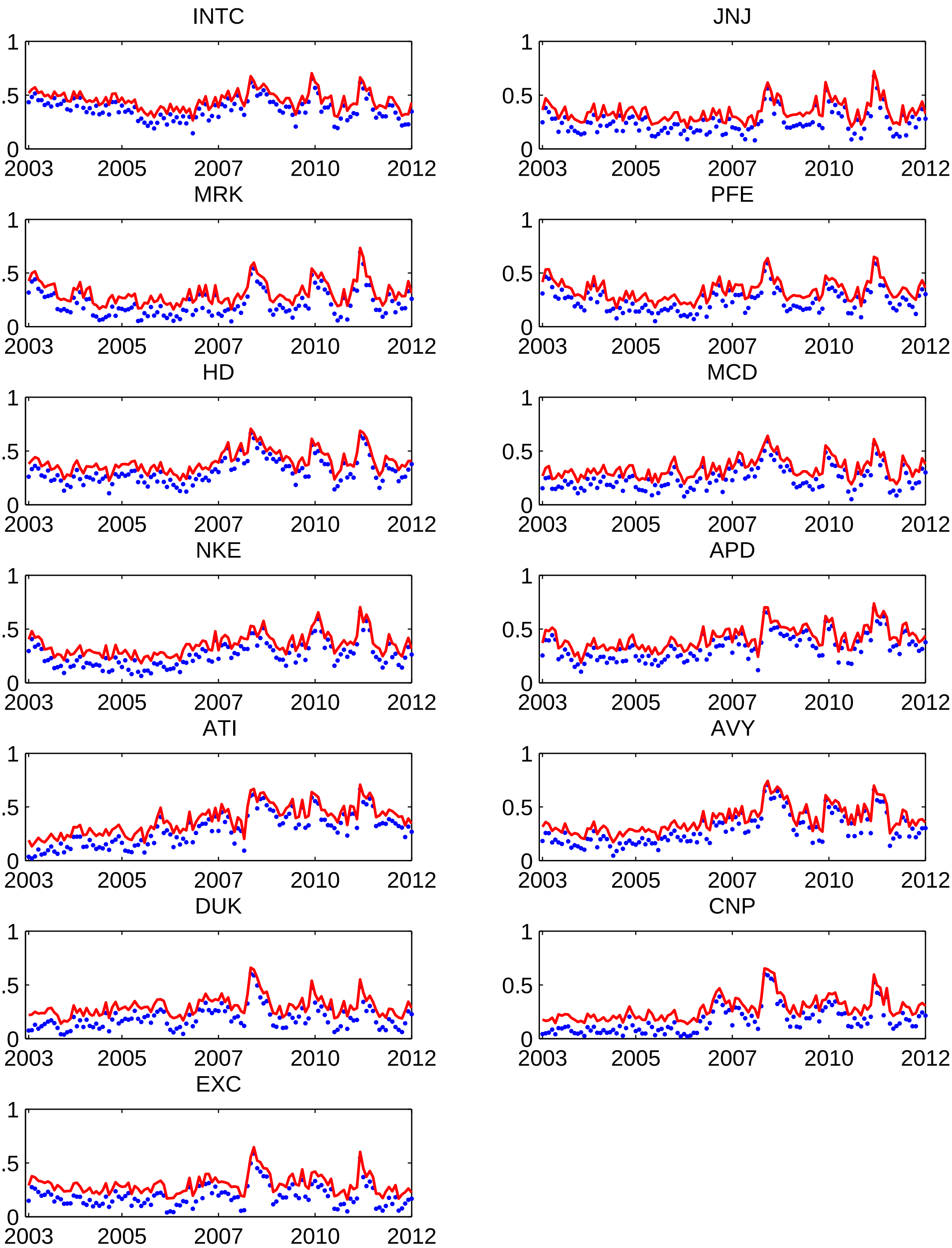

Figure 2: Monthly contribution of the return factors to stocks total volatility $(\widehat{R}^2_{Yj})$ over the period 2003:2012. The dotted blue line plots this measure calculated in CAPM model. The solid red line plots the same measure obtained in the FF3 model. We use the ticker of the stocks to label the graphs.

	AXP	GS	JPM	CVX	SLB	XOM	KO	PG	WMT	CAT	MMM	UTX	CSCO	IBM	INTC	JNJ	MRK	PFE	HD	MCD	NKE	APD	ATI	AVY	DUK	CNP	EXC
AXP		0.146	0.189	-0.032	-0.018	-0.038	-0.003	-0.009	0.028	0.039	0.016	0.023	0.011	0.016	0.013	-0.009	-0.003	0.002	0.052	0.021	0.029	0.007	0.017	0.022	-0.023	-0.003	-0.013
GS	0.148		0.230	-0.022	-0.001	-0.025	-0.016	-0.009	0.013	0.040	0.025	0.007	0.019	0.016	0.017	-0.025	-0.011	-0.006	0.040	0.013	0.020	0.014	0.043	0.018	-0.018	-0.003	-0.027
JPM	0.190	0.230		-0.049	-0.031	-0.044	-0.012	-0.013	0.015	0.006	-0.003	-0.009	-0.002	0.007	0.007	-0.012	-0.008	0.003	0.040	0.020	0.008	-0.007	0.014	0.002	-0.011	-0.015	-0.032
CVX	-0.032	-0.022	-0.052		0.324	0.515	0.008	-0.003	-0.031	0.025	0.016	0.014	-0.007	-0.002	-0.014	0.004	-0.001	-0.004	-0.047	-0.005	-0.001	0.066	0.088	0.011	0.045	0.048	0.053
SLB	-0.017	-0.002	-0.031	0.327		0.293	-0.030	-0.041	-0.044	0.055	0.011	0.011	-0.012	-0.019	-0.011	-0.025	-0.023	-0.027	-0.037	-0.014	0.000	0.055	0.108	0.013	0.008	0.021	0.021
XOM	-0.038	-0.024	-0.045	0.514	0.295		0.017	0.018	-0.025	0.019	0.013	0.010	-0.003	-0.002	-0.017	0.012	-0.003	0.004	-0.042	0.001	-0.013	0.051	0.074	-0.002	0.040	0.036	0.048
KO	-0.004	-0.015	-0.010	0.007	-0.030	0.016		0.167	0.075	0.001	0.058	0.060	0.025	0.059	0.019	0.119	0.084	0.081	0.023	0.066	0.042	0.023	-0.023	0.036	0.070	0.048	0.076
PG	-0.012	-0.010	-0.014	-0.004	-0.042	0.016	0.170		0.080	-0.005	0.056	0.063	0.021	0.053	0.007	0.151	0.095	0.080	0.040	0.077	0.044	0.027	-0.038	0.031	0.076	0.044	0.068
WMT	0.028	0.012	0.015	-0.031	-0.045	-0.027	0.078	0.079		0.017	0.037	0.035	0.037	0.050	0.034	0.057	0.032	0.045	0.191	0.081	0.086	0.028	0.000	0.035	0.032	0.011	0.019
CAT	0.040	0.040	0.007	0.024	0.054	0.018	0.002	-0.006	0.018		0.102	0.114	0.041	0.030	0.039	-0.012	-0.013	-0.006	0.041	0.038	0.053	0.099	0.107	0.087	0.005	0.022	0.014
MMM	0.016	0.026	-0.004	0.016	0.011	0.012	0.057	0.056	0.035	0.100		0.139	0.028	0.068	0.038	0.054	0.037	0.037	0.044	0.059	0.047	0.115	0.055	0.097	0.043	0.047	0.053
UTX	0.024	0.006	-0.009	0.015	0.010	0.011	0.061	0.061	0.034	0.113	0.139		0.042	0.062	0.027	0.048	0.037	0.027	0.036	0.061	0.067	0.089	0.043	0.091	0.040	0.034	0.044
CSCO	0.011	0.020	-0.003	-0.010	-0.011	-0.004	0.026	0.019	0.038	0.041	0.028	0.043		0.103	0.182	0.020	0.018	0.021	0.036	0.037	0.029	0.024	0.021	0.032	0.010	0.003	0.006
IBM	0.017	0.017	0.007	-0.004	-0.021	-0.003	0.061	0.050	0.048	0.032	0.069	0.060	0.102		0.101	0.046	0.030	0.038	0.041	0.058	0.047	0.033	0.008	0.036	0.016	0.014	0.022
INTC	0.015	0.017	0.007	-0.014	-0.012	-0.017	0.018	0.009	0.034	0.039	0.037	0.028	0.182	0.102		0.009	0.019	0.022	0.046	0.030	0.038	0.025	0.028	0.030	-0.007	-0.008	-0.005
JNJ	-0.011	-0.027	-0.014	0.003	-0.027	0.010	0.119	0.149	0.055	-0.012	0.054	0.048	0.020	0.046	0.008		0.170	0.166	0.017	0.056	0.029	0.019	-0.030	0.014	0.066	0.047	0.065
MRK	-0.004	-0.011	-0.009	-0.002	-0.023	-0.004	0.084	0.094	0.032	-0.012	0.036	0.037	0.019	0.030	0.019	0.169		0.203	0.018	0.046	0.026	0.022	-0.014	0.021	0.055	0.041	0.060
PFE	-0.000	-0.008	0.002	-0.003	-0.028	0.003	0.082	0.078	0.045	-0.007	0.035	0.028	0.021	0.039	0.023	0.166	0.203		0.025	0.038	0.022	0.014	-0.020	0.016	0.056	0.046	0.050
HD	0.051	0.040	0.041	-0.045	-0.038	-0.041	0.025	0.041	0.192	0.041	0.045	0.035	0.035	0.041	0.046	0.017	0.016	0.025		0.094	0.107	0.030	0.017	0.049	0.021	0.008	0.007
MCD	0.019	0.014	0.020	-0.006	-0.015	0.000	0.066	0.074	0.081	0.038	0.058	0.059	0.037	0.055	0.031	0.055	0.046	0.036	0.093		0.082	0.044	0.010	0.043	0.037	0.024	0.029
NKE	0.032	0.022	0.008	-0.000	-0.000	-0.011	0.043	0.045	0.087	0.052	0.047	0.067	0.030	0.045	0.039	0.029	0.026	0.022	0.107	0.083		0.064	0.035	0.071	0.023	0.030	0.031
APD	0.008	0.017	-0.008	0.066	0.054	0.051	0.023	0.026	0.028	0.098	0.116	0.089	0.024	0.033	0.027	0.018	0.020	0.014	0.030	0.044	0.065		0.144	0.162	0.041	0.049	0.051
ATI	0.019	0.043	0.014	0.090	0.110	0.075	-0.023	-0.036	-0.000	0.107	0.055	0.043	0.022	0.009	0.027	-0.030	-0.015	-0.020	0.016	0.011	0.034	0.144		0.086	0.008	0.028	0.013
AVY	0.022	0.020	0.003	0.009	0.012	-0.003	0.037	0.031	0.034	0.088	0.097	0.091	0.032	0.036	0.030	0.014	0.021	0.016	0.048	0.042	0.071	0.163	0.086		0.043	0.056	0.046
DUK	-0.024	-0.020	-0.013	0.044	0.008	0.038	0.070	0.077	0.033	0.007	0.042	0.040	0.010	0.016	-0.007	0.066	0.054	0.055	0.020	0.036	0.023	0.042	0.008	0.042		0.231	0.306
CNP	-0.002	-0.005	-0.016	0.049	0.021	0.036	0.048	0.044	0.011	0.022	0.047	0.035	0.003	0.015	-0.007	0.048	0.042	0.047	0.009	0.024	0.029	0.050	0.028	0.055	0.230		0.243
EXC	-0.013	-0.030	-0.037	0.054	0.021	0.048	0.075	0.070	0.020	0.014	0.054	0.044	0.006	0.022	-0.004	0.067	0.059	0.050	0.007	0.031	0.032	0.053	0.012	0.046	0.306	0.242	

Table 12: The correlation between stocks idiosyncratic returns over 2003-2012, $\text{Corr}(Z_i, Z_j)$.
The top panel reports the results for the CAPM, the bottom panel presents the same results for the FF3 model.

	AXP	GS	JPM	CVX	SLB	XOM	KO	PG	WMT	CAT	MMM	UTX	CSCO	IBM	INTC	JNJ	MRK	PFE	HD	MCD	NKE	APD	ATI	AVY	DUK	CNP	EXC
AXP		0.600	0.536	0.798	0.652	0.802	0.651	0.769	0.785	0.527	0.557	0.647	0.568	0.602	0.674	0.699	0.541	0.577	0.660	0.759	0.545	0.607	0.463	0.332	0.419	0.533	0.690
		(0.003)	(0.000)	(0.001)	(0.013)	(0.001)	(0.020)	(0.000)	(0.001)	(0.005)	(0.012)	(0.009)	(0.010)	(0.030)	(0.000)	(0.004)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)	(0.006)	(0.003)	(0.000)	(0.000)	(0.001)
GS	0.598		0.678	0.581	0.447	0.666	0.531	0.580	0.505	0.452	0.430	0.486	0.448	0.421	0.612	0.483	0.402	0.478	0.613	0.409	0.513	0.653	0.372	0.279	0.386	0.444	0.594
	(0.002)		(0.000)	(0.007)	(0.029)	(0.006)	(0.008)	(0.008)	(0.010)	(0.014)	(0.041)	(0.007)	(0.032)	(0.098)	(0.001)	(0.013)	(0.004)	(0.008)	(0.002)	(0.020)	(0.001)	(0.001)	(0.040)	(0.006)	(0.002)	(0.004)	(0.003)
JPM	0.532	0.679		0.529	0.498	0.590	0.552	0.547	0.490	0.477	0.421	0.690	0.506	0.679	0.577	0.535	0.433	0.414	0.454	0.459	0.561	0.609	0.446	0.317	0.351	0.436	0.562
	(0.001)	(0.000)		(0.000)	(0.001)	(0.000)	(0.001)	(0.000)	(0.000)	(0.002)	(0.008)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.003)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)
CVX	0.793	0.587	0.524		0.779	0.944	0.739	0.877	0.831	0.572	0.653	0.756	0.643	0.646	0.701	0.792	0.551	0.549	0.694	0.759	0.622	0.673	0.538	0.388	0.472	0.592	0.757
	(0.001)	(0.007)	(0.000)		(0.005)	(0.000)	(0.015)	(0.000)	(0.001)	(0.005)	(0.002)	(0.003)	(0.005)	(0.030)	(0.001)	(0.003)	(0.000)	(0.001)	(0.001)	(0.000)	(0.000)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
SLB	0.648	0.455	0.495	0.779		0.756	0.735	0.704	0.739	0.671	0.600	0.733	0.634	0.680	0.622	0.710	0.512	0.492	0.680	0.642	0.609	0.714	0.576	0.412	0.399	0.494	0.694
	(0.016)	(0.025)	(0.002)	(0.006)		(0.007)	(0.013)	(0.004)	(0.004)	(0.000)	(0.001)	(0.005)	(0.004)	(0.012)	(0.003)	(0.005)	(0.001)	(0.003)	(0.001)	(0.004)	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)	(0.002)	(0.001)
XOM	0.800	0.675	0.592	0.944	0.753		0.725	0.904	0.852	0.597	0.727	0.758	0.723	0.658	0.725	0.806	0.587	0.632	0.678	0.786	0.669	0.709	0.572	0.427	0.500	0.639	0.812
	(0.002)	(0.006)	(0.000)	(0.000)	(0.007)		(0.019)	(0.000)	(0.001)	(0.006)	(0.001)	(0.003)	(0.003)	(0.036)	(0.001)	(0.003)	(0.000)	(0.001)	(0.001)	(0.000)	(0.000)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
KO	0.651	0.539	0.558	0.743	0.737	0.726		0.712	0.694	0.662	0.625	0.813	0.584	0.681	0.664	0.753	0.576	0.460	0.650	0.566	0.621	0.671	0.444	0.421	0.477	0.552	0.683
	(0.021)	(0.008)	(0.001)	(0.016)	(0.013)	(0.020)		(0.014)	(0.014)	(0.003)	(0.003)	(0.004)	(0.014)	(0.025)	(0.003)	(0.012)	(0.001)	(0.010)	(0.003)	(0.015)	(0.001)	(0.002)	(0.012)	(0.000)	(0.001)	(0.002)	(0.005)
PG	0.761	0.586	0.540	0.874	0.700	0.904	0.712		0.849	0.587	0.728	0.760	0.709	0.670	0.709	0.814	0.577	0.615	0.692	0.794	0.684	0.706	0.568	0.399	0.502	0.644	0.791
	(0.001)	(0.007)	(0.000)	(0.000)	(0.005)	(0.000)	(0.014)		(0.000)	(0.003)	(0.000)	(0.001)	(0.001)	(0.017)	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
WMT	0.782	0.512	0.492	0.826	0.734	0.853	0.693	0.847		0.592	0.702	0.706	0.733	0.651	0.665	0.818	0.574	0.598	0.724	0.802	0.631	0.693	0.538	0.390	0.469	0.582	0.796
	(0.001)	(0.008)	(0.001)	(0.001)	(0.005)	(0.001)	(0.016)	(0.000)		(0.004)	(0.002)	(0.002)	(0.002)	(0.018)	(0.001)	(0.004)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.001)	(0.002)	(0.001)	(0.001)	(0.000)	(0.000)
CAT	0.530	0.455	0.474	0.573	0.674	0.599	0.663	0.592	0.596		0.607	0.608	0.558	0.609	0.623	0.605	0.503	0.498	0.550	0.512	0.600	0.688	0.543	0.403	0.376	0.416	0.597
	(0.008)	(0.015)	(0.002)	(0.007)	(0.001)	(0.006)	(0.003)	(0.004)	(0.006)		(0.000)	(0.001)	(0.002)	(0.002)	(0.000)	(0.006)	(0.000)	(0.001)	(0.001)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.002)	(0.002)
MMM	0.554	0.434	0.425	0.653	0.601	0.725	0.621	0.732	0.705	0.609		0.617	0.645	0.527	0.564	0.730	0.608	0.547	0.556	0.623	0.593	0.702	0.534	0.488	0.446	0.533	0.748
	(0.013)	(0.034)	(0.009)	(0.001)	(0.001)	(0.001)	(0.003)	(0.000)	(0.002)	(0.001)		(0.000)	(0.000)	(0.027)	(0.002)	(0.002)	(0.000)	(0.001)	(0.002)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.001)
UTX	0.647	0.495	0.690	0.758	0.729	0.762	0.816	0.756	0.708	0.609	0.619		0.687	0.735	0.687	0.734	0.565	0.515	0.605	0.665	0.689	0.665	0.548	0.458	0.443	0.581	0.648
	(0.010)	(0.006)	(0.000)	(0.004)	(0.006)	(0.003)	(0.004)	(0.001)	(0.002)	(0.001)	(0.000)		(0.000)	(0.006)	(0.001)	(0.001)	(0.000)	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
CSCO	(0.012)	(0.033)	(0.000)	(0.006)	(0.005)	(0.004)	(0.015)	(0.001)	(0.003)	(0.003)	(0.000)	(0.000)		(0.002)	(0.001)	(0.006)	(0.001)	(0.001)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.002)	(0.002)	(0.001)	(0.001)
	0.565	0.455	0.509	0.640	0.629	0.718	0.587	0.706	0.731	0.563	0.644	0.689		0.699	0.670	0.679	0.506	0.546	0.598	0.613	0.638	0.652	0.567	0.335	0.406	0.502	0.685
IBM	0.606	0.429	0.682	0.647	0.681	0.664	0.686	0.669	0.656	0.613	0.537	0.742	0.710		0.673	0.643	0.477	0.465	0.529	0.596	0.598	0.594	0.543	0.341	0.364	0.450	0.618
	(0.033)	(0.095)	(0.001)	(0.034)	(0.012)	(0.036)	(0.025)	(0.016)	(0.020)	(0.003)	(0.026)	(0.004)	(0.003)		(0.002)	(0.030)	(0.004)	(0.005)	(0.011)	(0.019)	(0.000)	(0.002)	(0.001)	(0.005)	(0.010)	(0.012)	(0.008)
INTC	0.667	0.619	0.581	0.697	0.618	0.717	0.669	0.698	0.656	0.623	0.557	0.683	0.665	0.673		0.636	0.509	0.504	0.657	0.568	0.660	0.671	0.517	0.381	0.389	0.541	0.662
	(0.001)	(0.002)	(0.000)	(0.001)	(0.004)	(0.001)	(0.003)	(0.000)	(0.001)	(0.000)	(0.002)	(0.001)	(0.001)	(0.003)		(0.002)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.001)	(0.000)	(0.000)
JNJ	0.697	0.492	0.533	0.791	0.704	0.806	0.751	0.817	0.817	0.603	0.735	0.732	0.675	0.644	0.628		0.592	0.582	0.631	0.702	0.653	0.699	0.549	0.402	0.576	0.592	0.781
	(0.004)	(0.011)	(0.001)	(0.003)	(0.007)	(0.003)	(0.013)	(0.002)	(0.005)	(0.007)	(0.002)	(0.001)	(0.008)	(0.032)	(0.003)		(0.001)	(0.002)	(0.002)	(0.001)	(0.001)	(0.002)	(0.003)	(0.001)	(0.000)	(0.001)	(0.002)
MRK	0.538	0.402	0.431	0.552	0.511	0.588	0.575	0.577	0.574	0.505	0.611	0.565	0.510	0.482	0.505	0.595		0.548	0.521	0.544	0.479	0.587	0.402	0.347	0.369	0.447	0.652
	(0.000)	(0.004)	(0.000)	(0.000)	(0.001)	(0.000)	(0.001)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.001)	(0.003)	(0.000)	(0.001)		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
PFE	0.563	0.480	0.407	0.536	0.485	0.621	0.463	0.604	0.588	0.504	0.548	0.510	0.537	0.464	0.495	0.576	0.546		0.518	0.566	0.523	0.536	0.456	0.387	0.380	0.447	0.595
	(0.001)	(0.008)	(0.000)	(0.001)	(0.004)	(0.001)	(0.010)	(0.000)	(0.001)	(0.001)	(0.001)	(0.000)	(0.001)	(0.006)	(0.001)	(0.002)	(0.000)		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
HD	0.658	0.621	0.459	0.700	0.683	0.685	0.654	0.693	0.725	0.554	0.559	0.609	0.603	0.535	0.659	0.633	0.522	0.516		0.583	0.565	0.689	0.482	0.363	0.422	0.494	0.668
	(0.000)	(0.002)	(0.003)	(0.001)	(0.001)	(0.001)	(0.003)	(0.000)	(0.000)	(0.001)	(0.003)	(0.002)	(0.001)	(0.012)	(0.000)	(0.002)	(0.000)	(0.000)		(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
MCD	0.759	0.417	0.460	0.757	0.639	0.791	0.567	0.792	0.805	0.517	0.628	0.668	0.612	0.601	0.558	0.706	0.545	0.552	0.587		0.549	0.587	0.532	0.320	0.439	0.524	0.682
	(0.000)	(0.016)	(0.000)	(0.000)	(0.005)	(0.000)	(0.016)	(0.000)	(0.000)	(0.004)	(0.001)	(0.000)	(0.002)	(0.021)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
NKE	0.550	0.521	0.570	0.623	0.608	0.673	0.626	0.685	0.634	0.601	0.599	0.692	0.643	0.606	0.655	0.655	0.483	0.526	0.572	0.553		0.694	0.537	0.432	0.390	0.537	0.687
	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.001)	(0.000)		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
APD	0.609	0.661	0.609	0.677	0.715	0.710	0.669	0.708	0.694	0.687	0.700	0.663	0.651	0.603	0.674	0.695	0.585	0.535	0.693	0.593	0.695		0.628	0.403	0.462	0.522	0.783
	(0.003)	(0.001)	(0.000)	(0.002)	(0.000)	(0.001)	(0.002)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.001)	(0.002)	(0.000)	(0.003)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
ATI	0.469	0.377	0.444	0.538	0.578	0.574	0.448	0.573	0.544	0.547	0.540	0.549	0.568	0.549	0.517	0.555	0.407	0.457	0.490	0.537	0.542	0.630		0.340	0.439	0.422	0.575
	(0.007)	(0.038)	(0.000)	(0.002)	(0.001)	(0.001)	(0.013)	(0.001)	(0.003)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)	(0.003)	(0.001)	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)		(0.000)	(0.001)	(0.001)	(0.000)
AVY	0.331	0.284	0.320	0.387	0.413	0.427	0.424	0.398	0.391	0.406	0.488	0.459	0.337	0.345	0.382	0.399	0.350	0.389	0.366	0.321	0.436	0.402	0.343		0.248	0.310	0.398
	(0.004)	(0.006)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.002)	(0.005)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)		(0.001)	(0.000)	(0.000)
DUK	0.423	0.393	0.353	0.476	0.402	0.506	0.481	0.509	0.474	0.382	0.451	0.446	0.410	0.370	0.390	0.582	0.373	0.381	0.426	0.444	0.396	0.467	0.442	0.251		0.536	0.524
	(0.000)	(0.001)	(0.000)	(0.000)	(0.002)	(0.000)	(0.001)	(0.000)	(0.001)	(0.002)	(0.001)	(0.000)	(0.002)	(0.009)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)		(0.000)	(0.000)
CNP	0.532	0.451	0.435	0.594	0.492	0.640	0.556	0.645	0.585	0.422	0.537	0.581	0.503	0.453	0.537	0.595	0.446	0.446	0.497	0.526	0.537	0.524	0.427	0.312	0.540		0.616
	(0.000)	(0.004)	(0.000)	(0.000)	(0.003)	(0.000)	(0.002)	(0.000)	(0.000)	(0.002)	(0.000)	(0.000)	(0.002)	(0.013)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)		(0.000)
EXC	0.686	0.600	0.563	0.752	0.689	0.811	0.675	0.790	0.796	0.593	0.751	0.642	0.684	0.619	0.652	0.783	0.650	0.591	0.669	0.686	0.685	0.781	0.578	0.398	0.531	0.615	
	(0.001)	(0.002)	(0.000)	(0.000)	(0.001)	(0.000)	(0.006)	(0.000)	(0.000)	(0.003)	(0.001)	(0.000)	(0.001)	(0.006)	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	

Table 13: The correlation between the idiosyncratic volatilities, $\rho_{Zi,Zj}$.
The results for CAPM are in the upper panel; the results for FF3 model are in the bottom panel.
The figures in the parenthesis are the p-values of the LIN test of the absence of dependence in the IVs.
We use the market volatility as IV factor. $\theta = 2.5$, $T = 10$ years, and $\Delta_n = 5$mins.

	AXP	GS	JPM	CVX	SLB	XOM	KO	PG	WMT	CAT	MMM	UTX	CSCO	IBM	INTC	JNJ	MRK	PFE	HD	MCD	NKE	APD	ATI	AVY	DUK	CNP	EXC
AXP		0.410	0.271	0.605	0.312	0.594	0.282	0.513	0.562	0.149	0.112	0.308	0.206	0.283	0.399	0.327	0.263	0.299	0.387	0.581	0.166	0.221	0.115	0.058	0.159	0.266	0.345
		(0.028)	(0.172)	(0.022)	(0.132)	(0.004)	(0.192)	(0.012)	(0.000)	(0.192)	(0.505)	(0.188)	(0.109)	(0.228)	(0.017)	(0.001)	(0.003)	(0.000)	(0.051)	(0.001)	(0.178)	(0.086)	(0.181)	(0.325)	(0.003)	(0.017)	(0.004)
GS	0.402		0.561	0.372	0.146	0.510	0.274	0.353	0.229	0.196	0.110	0.214	0.180	0.148	0.433	0.149	0.173	0.266	0.439	0.143	0.281	0.490	0.121	0.084	0.204	0.240	0.381
	(0.028)		(0.003)	(0.045)	(0.340)	(0.008)	(0.110)	(0.078)	(0.166)	(0.256)	(0.317)	(0.106)	(0.215)	(0.633)	(0.001)	(0.290)	(0.018)	(0.008)	(0.002)	(0.397)	(0.002)	(0.000)	(0.276)	(0.063)	(0.001)	(0.021)	(0.006)
JPM	0.266	0.560		0.244	0.186	0.328	0.263	0.237	0.152	0.198	0.044	0.516	0.234	0.513	0.346	0.191	0.188	0.145	0.159	0.183	0.323	0.376	0.196	0.113	0.136	0.204	0.270
	(0.186)	(0.003)		(0.257)	(0.181)	(0.119)	(0.093)	(0.265)	(0.434)	(0.167)	(0.744)	(0.005)	(0.131)	(0.014)	(0.041)	(0.195)	(0.035)	(0.108)	(0.312)	(0.278)	(0.030)	(0.003)	(0.017)	(0.153)	(0.046)	(0.099)	(0.064)
CVX	0.597	0.375	0.235		0.544	0.882	0.437	0.729	0.637	0.204	0.272	0.501	0.319	0.341	0.430	0.515	0.263	0.233	0.431	0.570	0.282	0.324	0.221	0.129	0.229	0.351	0.459
	(0.025)	(0.040)	(0.277)		(0.024)	(0.000)	(0.071)	(0.001)	(0.000)	(0.179)	(0.045)	(0.082)	(0.031)	(0.263)	(0.018)	(0.000)	(0.001)	(0.002)	(0.032)	(0.005)	(0.029)	(0.025)	(0.026)	(0.078)	(0.000)	(0.005)	(0.002)
SLB	0.309	0.153	0.184	0.546		0.460	0.420	0.321	0.430	0.384	0.150	0.446	0.296	0.401	0.270	0.294	0.189	0.125	0.399	0.350	0.250	0.400	0.284	0.163	0.107	0.178	0.306
	(0.122)	(0.306)	(0.202)	(0.022)		(0.053)	(0.012)	(0.122)	(0.014)	(0.000)	(0.172)	(0.034)	(0.036)	(0.005)	(0.050)	(0.014)	(0.072)	(0.147)	(0.006)	(0.107)	(0.011)	(0.001)	(0.005)	(0.020)	(0.110)	(0.127)	(0.009)
XOM	0.592	0.518	0.334	0.882	0.459		0.354	0.764	0.653	0.208	0.382	0.469	0.445	0.330	0.447	0.487	0.303	0.357	0.368	0.610	0.338	0.353	0.252	0.172	0.259	0.418	0.540
	(0.004)	(0.005)	(0.109)	(0.000)	(0.049)		(0.143)	(0.000)	(0.000)	(0.160)	(0.016)	(0.079)	(0.002)	(0.288)	(0.009)	(0.000)	(0.000)	(0.000)	(0.055)	(0.000)	(0.006)	(0.014)	(0.016)	(0.014)	(0.000)	(0.000)	(0.000)
KO	0.283	0.279	0.276	0.445	0.430	0.361		0.298	0.295	0.347	0.162	0.594	0.166	0.382	0.328	0.358	0.286	0.039	0.318	0.181	0.247	0.278	0.019	0.165	0.223	0.263	0.236
	(0.196)	(0.089)	(0.086)	(0.076)	(0.012)	(0.171)		(0.146)	(0.043)	(0.020)	(0.248)	(0.015)	(0.170)	(0.018)	(0.023)	(0.012)	(0.004)	(0.545)	(0.027)	(0.347)	(0.014)	(0.017)	(0.805)	(0.026)	(0.001)	(0.020)	(0.080)
PG	0.495	0.354	0.226	0.722	0.316	0.764	0.299		0.634	0.166	0.362	0.456	0.402	0.340	0.400	0.478	0.272	0.311	0.384	0.625	0.351	0.322	0.229	0.111	0.255	0.421	0.465
	(0.012)	(0.067)	(0.287)	(0.001)	(0.152)	(0.000)	(0.153)		(0.000)	(0.202)	(0.040)	(0.043)	(0.002)	(0.208)	(0.014)	(0.002)	(0.001)	(0.000)	(0.047)	(0.000)	(0.008)	(0.037)	(0.033)	(0.111)	(0.000)	(0.000)	(0.002)
WMT	0.555	0.231	0.157	0.626	0.422	0.657	0.292	0.628		0.207	0.337	0.362	0.472	0.324	0.331	0.533	0.283	0.297	0.468	0.642	0.267	0.326	0.195	0.113	0.209	0.318	0.511
	(0.000)	(0.160)	(0.418)	(0.001)	(0.014)	(0.000)	(0.061)	(0.000)		(0.028)	(0.037)	(0.052)	(0.000)	(0.103)	(0.013)	(0.000)	(0.000)	(0.000)	(0.007)	(0.000)	(0.011)	(0.002)	(0.005)	(0.027)	(0.000)	(0.000)	(0.000)
CAT	0.155	0.193	0.195	0.205	0.393	0.215	0.349	0.176	0.216		0.255	0.268	0.221	0.324	0.334	0.166	0.223	0.194	0.220	0.177	0.298	0.416	0.272	0.181	0.114	0.103	0.194
	(0.233)	(0.262)	(0.175)	(0.221)	(0.000)	(0.140)	(0.020)	(0.261)	(0.046)		(0.019)	(0.033)	(0.035)	(0.005)	(0.010)	(0.157)	(0.010)	(0.010)	(0.090)	(0.103)	(0.008)	(0.001)	(0.002)	(0.005)	(0.059)	(0.197)	(0.079)
MMM	0.111	0.112	0.057	0.275	0.163	0.384	0.161	0.378	0.347	0.262		0.193	0.308	0.098	0.147	0.330	0.353	0.217	0.152	0.307	0.210	0.366	0.202	0.282	0.178	0.240	0.418
	(0.522)	(0.339)	(0.729)	(0.047)	(0.146)	(0.024)	(0.352)	(0.031)	(0.038)	(0.024)		(0.153)	(0.036)	(0.614)	(0.181)	(0.017)	(0.000)	(0.032)	(0.152)	(0.078)	(0.052)	(0.001)	(0.028)	(0.000)	(0.003)	(0.012)	(0.004)
UTX	0.309	0.221	0.518	0.503	0.442	0.480	0.603	0.449	0.368	0.272	0.203		0.402	0.507	0.400	0.365	0.286	0.170	0.259	0.395	0.408	0.303	0.236	0.239	0.182	0.329	0.208
	(0.208)	(0.094)	(0.005)	(0.095)	(0.035)	(0.074)	(0.014)	(0.055)	(0.047)	(0.028)	(0.155)		(0.015)	(0.022)	(0.024)	(0.004)	(0.005)	(0.017)	(0.088)	(0.088)	(0.005)	(0.028)	(0.033)	(0.002)	(0.007)	(0.024)	(0.097)
CSCO	0.204	0.186	0.241	0.316	0.292	0.439	0.177	0.399	0.470	0.233	0.312	0.409		0.469	0.404	0.310	0.217	0.259	0.289	0.336	0.350	0.330	0.300	0.072	0.149	0.226	0.356
	(0.110)	(0.222)	(0.147)	(0.043)	(0.044)	(0.003)	(0.232)	(0.003)	(0.000)	(0.062)	(0.037)	(0.014)		(0.001)	(0.005)	(0.001)	(0.035)	(0.000)	(0.035)	(0.005)	(0.002)	(0.007)	(0.001)	(0.285)	(0.024)	(0.011)	(0.000)
IBM	0.284	0.149	0.517	0.337	0.399	0.336	0.386	0.329	0.326	0.325	0.112	0.516	0.486		0.422	0.250	0.181	0.138	0.181	0.318	0.293	0.235	0.271	0.090	0.094	0.154	0.236
	(0.242)	(0.634)	(0.013)	(0.267)	(0.005)	(0.281)	(0.018)	(0.245)	(0.088)	(0.006)	(0.577)	(0.012)	(0.002)		(0.016)	(0.296)	(0.078)	(0.120)	(0.318)	(0.193)	(0.009)	(0.125)	(0.007)	(0.198)	(0.269)	(0.280)	(0.140)
INTC	0.390	0.439	0.355	0.425	0.273	0.438	0.343	0.383	0.317	0.338	0.143	0.399	0.401	0.421		0.203	0.221	0.188	0.391	0.256	0.389	0.364	0.218	0.140	0.123	0.288	0.306
	(0.018)	(0.001)	(0.038)	(0.016)	(0.039)	(0.010)	(0.015)	(0.016)	(0.016)	(0.014)	(0.294)	(0.021)	(0.006)	(0.018)		(0.042)	(0.003)	(0.007)	(0.005)	(0.074)	(0.002)	(0.003)	(0.019)	(0.053)	(0.085)	(0.006)	(0.008)
JNJ	0.325	0.158	0.192	0.512	0.289	0.492	0.355	0.491	0.530	0.164	0.352	0.364	0.306	0.244	0.198		0.284	0.220	0.220	0.420	0.252	0.263	0.161	0.096	0.389	0.308	0.398
	(0.000)	(0.233)	(0.187)	(0.000)	(0.029)	(0.000)	(0.018)	(0.001)	(0.000)	(0.152)	(0.012)	(0.004)	(0.005)	(0.309)	(0.031)		(0.000)	(0.000)	(0.074)	(0.000)	(0.001)	(0.008)	(0.038)	(0.045)	(0.000)	(0.003)	(0.004)
MRK	0.259	0.168	0.187	0.263	0.191	0.304	0.284	0.271	0.282	0.227	0.361	0.286	0.225	0.184	0.218	0.290		0.331	0.250	0.300	0.175	0.319	0.119	0.144	0.151	0.208	0.419
	(0.003)	(0.007)	(0.032)	(0.000)	(0.070)	(0.000)	(0.002)	(0.001)	(0.000)	(0.012)	(0.000)	(0.004)	(0.031)	(0.075)	(0.004)	(0.000)		(0.000)	(0.011)	(0.000)	(0.006)	(0.000)	(0.029)	(0.002)	(0.000)	(0.000)	(0.000)
PFE	0.276	0.264	0.136	0.211	0.116	0.337	0.044	0.290	0.279	0.203	0.222	0.163	0.247	0.132	0.179	0.211	0.327		0.223	0.315	0.223	0.202	0.181	0.187	0.154	0.193	0.277
	(0.000)	(0.008)	(0.125)	(0.005)	(0.201)	(0.000)	(0.507)	(0.000)	(0.000)	(0.013)	(0.026)	(0.018)	(0.000)	(0.148)	(0.010)	(0.000)	(0.000)		(0.007)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
HD	0.383	0.446	0.170	0.441	0.408	0.385	0.327	0.386	0.470	0.228	0.162	0.269	0.299	0.186	0.399	0.227	0.252	0.220		0.294	0.233	0.414	0.172	0.121	0.180	0.222	0.337
	(0.050)	(0.001)	(0.333)	(0.025)	(0.004)	(0.044)	(0.018)	(0.040)	(0.006)	(0.099)	(0.205)	(0.077)	(0.031)	(0.390)	(0.002)	(0.050)	(0.010)	(0.004)		(0.094)	(0.031)	(0.001)	(0.053)	(0.027)	(0.005)	(0.036)	(0.009)
MCD	0.578	0.147	0.184	0.565	0.345	0.619	0.181	0.618	0.647	0.184	0.317	0.401	0.334	0.319	0.240	0.426	0.301	0.292	0.299		0.227	0.247	0.268	0.072	0.216	0.283	0.397
	(0.001)	(0.369)	(0.280)	(0.007)	(0.105)	(0.000)	(0.358)	(0.000)	(0.000)	(0.107)	(0.077)	(0.084)	(0.008)	(0.193)	(0.089)	(0.000)	(0.000)	(0.000)	(0.085)		(0.035)	(0.034)	(0.004)	(0.250)	(0.000)	(0.003)	(0.000)
NKE	0.175	0.288	0.338	0.283	0.253	0.345	0.255	0.354	0.272	0.300	0.224	0.414	0.361	0.301	0.383	0.258	0.181	0.228	0.244	0.232		0.414	0.254	0.219	0.127	0.283	0.365
	(0.155)	(0.001)	(0.025)	(0.026)	(0.008)	(0.011)	(0.011)	(0.007)	(0.010)	(0.017)	(0.036)	(0.004)	(0.001)	(0.008)	(0.004)	(0.000)	(0.005)	(0.000)	(0.027)	(0.030)		(0.000)	(0.001)	(0.002)	(0.007)	(0.001)	(0.000)
APD	0.234	0.498	0.380	0.337	0.414	0.366	0.284	0.339	0.338	0.419	0.374	0.310	0.337	0.252	0.381	0.272	0.321	0.208	0.428	0.261	0.421		0.373	0.148	0.210	0.225	0.509
	(0.074)	(0.000)	(0.003)	(0.030)	(0.001)	(0.008)	(0.014)	(0.032)	(0.001)	(0.001)	(0.001)	(0.028)	(0.006)	(0.106)	(0.002)	(0.007)	(0.000)	(0.001)	(0.000)	(0.026)	(0.000)		(0.000)	(0.023)	(0.001)	(0.018)	(0.000)
ATI	0.118	0.117	0.190	0.213	0.282	0.247	0.017	0.229	0.197	0.273	0.206	0.231	0.297	0.270	0.213	0.165	0.122	0.177	0.178	0.268	0.253	0.374		0.126	0.239	0.161	0.249
	(0.192)	(0.307)	(0.018)	(0.034)	(0.006)	(0.019)	(0.983)	(0.040)	(0.011)	(0.003)	(0.040)	(0.036)	(0.002)	(0.009)	(0.021)	(0.038)	(0.029)	(0.001)	(0.044)	(0.007)	(0.001)	(0.000)		(0.086)	(0.064)	(0.145)	(0.000)
AVY	0.058	0.088	0.118	0.129	0.168	0.174	0.171	0.113	0.116	0.187	0.285	0.242	0.078	0.095	0.146	0.095	0.149	0.190	0.127	0.073	0.224	0.153	0.127		0.062	0.103	0.114
	(0.352)	(0.059)	(0.129)	(0.094)	(0.023)	(0.012)	(0.020)	(0.148)	(0.028)	(0.005)	(0.000)	(0.002)	(0.289)	(0.183)	(0.041)	(0.049)	(0.001)	(0.000)	(0.020)	(0.275)	(0.003)	(0.017)	(0.086)		(0.232)	(0.100)	(0.043)
DUK	0.161	0.206	0.137	0.231	0.109	0.265	0.224	0.261	0.212	0.119	0.184	0.182	0.152	0.096	0.122	0.394	0.154	0.153	0.182	0.220	0.130	0.216	0.237	0.065		0.385	0.297
	(0.003)	(0.001)	(0.057)	(0.001)	(0.103)	(0.000)	(0.001)	(0.000)	(0.000)	(0.073)	(0.004)	(0.008)	(0.030)	(0.281)	(0.111)	(0.000)	(0.000)	(0.000)	(0.006)	(0.000)	(0.008)	(0.001)	(0.081)	(0.212)		(0.018)	(0.000)
CNP	0.263	0.244	0.203	0.352	0.177	0.418	0.268	0.422	0.321	0.111	0.248	0.330	0.228	0.154	0.282	0.314	0.207	0.190	0.226	0.284	0.283	0.232	0.163	0.107	0.387		0.371
	(0.022)	(0.019)	(0.102)	(0.008)	(0.173)	(0.000)	(0.019)	(0.001)	(0.000)	(0.157)	(0.009)	(0.039)	(0.029)	(0.285)	(0.007)	(0.002)	(0.000)	(0.000)	(0.032)	(0.005)	(0.003)	(0.013)	(0.149)	(0.126)	(0.022)		(0.000)
EXC	0.340	0.383	0.276	0.450	0.302	0.542	0.223	0.467	0.512	0.189	0.431	0.199	0.359	0.232	0.292	0.410	0.417	0.272	0.342	0.403	0.361	0.514	0.245	0.117	0.304	0.369	
	(0.005)	(0.004)	(0.056)	(0.002)	(0.009)	(0.000)	(0.141)	(0.002)	(0.000)	(0.130)	(0.006)	(0.127)	(0.000)	(0.146)	(0.021)	(0.003)	(0.000)	(0.000)	(0.008)	(0.000)	(0.000)	(0.000)	(0.000)	(0.039)	(0.000)	(0.000)	

Table 14: The correlation between the NS-IVs, $\rho^{NS}_{Zi,Zj}$.
The results for CAPM are in the upper panel; the results for the FF3 model are in the bottom panel.
The figures in the parenthesis are the p-values of the LIN test of the absence of dependence in the NS-IVs.
We use the market volatility as IV factor. $\theta = 2.5$, $T = 10$ years, and $\Delta_n = 5$mins.

B Proofs

Throughout, we denote by K a generic constant, which may change from line to line. When it depends on a parameter p we use the notation K_p instead. We assume by convention $\sum_{i=a}^{a'} = 0$ when $a > a'$.

B.1 Proof of Theorem 1

We prove this theorem in three steps. For simplicity, in the first two steps we focus on the estimation of $[H(C), G(C)]_T$ with $H, G \in \mathcal{G}(p)$. The joint estimation is discussed in Step 3.
By a localization argument (See Lemma 4.4.9 of Jacod and Protter (2012)), there exists a λ-integrable function J on E and a constant such that the stochastic processes in (18) and (19) satisfy

$$\|b\|, \|\widetilde{b}\|, \|c\|, \|\widetilde{c}\|, J \leq A, \|\delta(w,t,z)\|^r \leq J(z). \tag{23}$$

Setting $b'_t = b_t - \int \delta(t,z) 1_{\{\|\delta(t,z)\| \leq 1\}} \lambda(dz)$ and $Y'_t = \int_0^t b'_s ds + \int_0^t \sigma_s dW_s$, we have

$$Y_t = Y_0 + Y'_t + \sum_{s \leq t} \Delta Y_s.$$

The local estimator of the spot variance of the unobservable process Y' is given by,

$$\widehat{C}_i'^n = \frac{1}{k_n \Delta_n} \sum_{u=0}^{k_n - 1} (\Delta_{i+u}^n Y')(\Delta_{i+u}^n Y)'^\top = (\widehat{C}_i'^{n,gh})_{1 \leq g,h \leq d}. \tag{24}$$

Note that no jump truncation in needed in the definition of $\widehat{C}_i'^n$ since the process Y' is continuous. Therefore, it is more convenient to work with $\widehat{C}_i'^n$ rather than $\widehat{C}_i^n$ (defined in (13)). Let $[\widehat{H(C), G(C)}]_T^{LIN'}$ and $[\widehat{H(C), G(C)}]_T^{AN'}$ be the infeasible estimators obtained by replacing $\widehat{C}_i^n$ by $\widehat{C}_i'^n$ in the definition of $[\widehat{H(C), G(C)}]_T^{LIN}$ and $[\widehat{H(C), G(C)}]_T^{AN}$.

Step1: Dealing with price jumps

We prove that, as long as $(8p-1)/4(4p-r) \leq \varpi < \frac{1}{2}$, we have

$$\Delta_n^{-1/4}\Big([\widehat{H(C), G(C)}]_T^{LIN} - [\widehat{H(C), G(C)}]_T^{LIN'}\Big) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \Delta_n^{-1/4}\Big([\widehat{H(C), G(C)}]_T^{AN} - [\widehat{H(C), G(C)}]_T^{AN'}\Big) \xrightarrow{\mathbb{P}} 0. \tag{25}$$

To show this result, let us define the functions

$$R(x,y) = \sum_{g,h,a,b=1}^{d} \big(\partial_{gh} H \partial_{ab} G\big)(x)\big(y^{gh} - x^{gh}\big)\big(y^{ab} - x^{ab}\big), \; S(x,y) = \big(H(y) - H(x)\big)\big(G(y) - G(x)\big)$$
$$U(x) = \sum_{g,h,a,b=1}^{d} \big(\partial_{gh} H \partial_{ab} G\big)(x)\big(x^{ga}x^{hb} + x^{gb}x^{ha}\big),$$

for any $\mathbb{R}^d \times \mathbb{R}^d$ matrices x and y. The following decompositions hold,

$$[\widehat{H(C), G(C)}]_T^{AN} - [\widehat{H(C), G(C)}]_T^{AN'} = \frac{3}{2k_n} \sum_{i=1}^{[T/\Delta_n] - 2k_n + 1} \Big[\big(S(\widehat{C}_i^n, \widehat{C}_{i+k_n}^n) - S(\widehat{C}_i'^n, \widehat{C}_{i+k_n}'^n)\big) - \frac{2}{k_n}\big(U(\widehat{C}_i^n) - U(\widehat{C}_i'^n)\big)\Big],$$

$$[H(\widehat{C), G}(C)]_T^{LIN} - [H(\widehat{C), G}(C)]_T^{LIN'} = \frac{3}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\Big[\big(R(\widehat{C}_i^n, \widehat{C}_{i+k_n}^n) - R(\widehat{C}_i'^n, \widehat{C}_{i+k_n}'^n)\big) - \frac{2}{k_n}\big(U(\widehat{C}_i^n) - U(\widehat{C}_i'^n)\big)\Big].$$

By (3.11) in Jacod and Rosenbaum (2012), there exists a sequence of real numbers a_n converging to zero such that

$$\mathbb{E}(\|\widehat{C}_i^n - \widehat{C}_i'^n\|^q) \le K_q a_n \Delta_n^{(2q-r)\varpi+1-q}, \text{ for any } q > 0. \tag{26}$$

Since H and $G \in \mathcal{G}(p)$, it is easy to see that the functions R and S are continuously differentiable and satisfy

$$\|\partial J(x,y)\| \le K(1 + \|x\| + \|y\|)^{2p-1} \text{ for } 1 \le g,h,a,b \le d \text{ and } J \in \{S, R\}, \tag{27}$$
$$\|\partial U(x)\| \le K(1 + \|x\|)^{2p-1}, \tag{28}$$

where ∂J (respectively, ∂U) is a vector that collects the first order partial derivatives of the function J (respectively, U) with respect to all the elements of (x, y) (resp x). By Taylor expansion, Jensen inequality, (27) and (28), it can be shown that, for $J \in \{S, R\}$,

$$\begin{aligned}|J(\widehat{C}_i^n, \widehat{C}_{i+k_n}^n) - J(\widehat{C}_i'^n, \widehat{C}_{i+k_n}'^n)| &\le K(1 + \|\widehat{C}_i'^n\|^{2p-1} + \|\widehat{C}_{i+k_n}'^n\|^{2p-1})(\|\widehat{C}_i^n - \widehat{C}_i'^n\| + \|\widehat{C}_{i+k_n}^n - \widehat{C}_{i+k_n}'^n\|)\\ &\quad + K\|\widehat{C}_i^n - \widehat{C}_i'^n\|^{2p} + K\|\widehat{C}_{i+k_n}^n - \widehat{C}_{i+k_n}'^n\|^{2p} \text{ and}\\ |U(\widehat{C}_i^n) - U(\widehat{C}_i'^n)| &\le K(1 + \|\widehat{C}_i'^n\|^{2p-1})(\|\widehat{C}_i^n - \widehat{C}_i'^n\|) + K\|\widehat{C}_i^n - \widehat{C}_i'^n\|^{2p}.\end{aligned}$$

By (3.20) in Jacod and Rosenbaum (2012), we have $\mathbb{E}(\|\widehat{C}_i'^n\|^v) \le K_v$, for any $v \ge 0$. Hence by Hölder inequality, for $\epsilon > 0$ fixed,

$$\begin{aligned}\mathbb{E}(\|\widehat{C}_i'^n\|^{2p-2}\|\widehat{C}_i^n - \widehat{C}_i'^n\|) &\le \Big(\mathbb{E}(\|\widehat{C}_i^n - \widehat{C}_i'^n\|^{(1+\epsilon)})\Big)^{1/1+\epsilon}\Big(\mathbb{E}(\|\widehat{C}_i'^n\|^{(2p-2)(1+\epsilon)/\epsilon})\Big)^{\epsilon/1+\epsilon}\\ &\le K_p\Big(\mathbb{E}(\|\widehat{C}_i^n - \widehat{C}_i'^n\|^{(1+\epsilon)})\Big)^{1/1+\epsilon}\\ &\le K_p a_n \Delta_n^{(2-\frac{1}{1+\epsilon})\varpi + \frac{1}{1+\epsilon} - 1}\end{aligned}$$

Using the above result and (26), it easy to see that for (25) to hold, the following conditions are sufficient:

$$(2 - \frac{r}{1+\epsilon})\varpi + \frac{1}{1+\epsilon} - 1 - \frac{3}{4} \ge 0, \quad (4p - r)\varpi + 1 - 2p - \frac{3}{4} \ge 0, \quad \text{and} \quad (2 - r)\varpi + -\frac{3}{4} \ge 0.$$

Using the fact that $0 < \varpi < \frac{1}{2}$, and taking ϵ sufficiently close to zero, we can see that (25) holds if $(8p - 1)/4(4p - r) \le \varpi < \frac{1}{2}$, which completes the proof.

Step 2 : First approximation for the estimators

Taking advantage of Step 1, it is enough to derive the asymptotic distributions of $[H(\widehat{C), G}(C)]_T^{LIN\prime}$ and $[H(\widehat{C), G}(C)]_T^{AN\prime}$. We show that the two estimators $[H(\widehat{C), G}(C)]_T^{LIN'}$ and $[H(\widehat{C), G}(C)]_T^{AN'}$ can be approximated by a certain quantity with an error of approximation of order smaller than $\Delta_n^{-1/4}$. To see this, we set

$$\begin{aligned}[H(\widehat{C), G}(C)]_T^{A} = \frac{3}{2k_n}\sum_{g,h,a,b=1}^{d}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\Bigg(&\Big(\partial_{gh}H\partial_{ab}G\Big)(C_i^n)\Big[(\widehat{C}_{i+k_n}'^{n,gh} - \widehat{C}_i'^{n,gh})(\widehat{C}_{i+k_n}'^{n,ab} - \widehat{C}_i'^{n,ab})\\ &- \frac{2}{k_n}(\widehat{C}_i'^{n,ga}\widehat{C}_i'^{n,hb} + \widehat{C}_i'^{n,gb}\widehat{C}_i'^{n,ha})\Big]\Bigg),\end{aligned}$$

with $C_i^n = C_{(i-1)\Delta_n}$ and the superscript A being a short for the word "approximate". For notational simplicity, we do not index the above quantity by a prime although it depends on $\widehat{C}_i'^n$ instead $\widehat{C}_i^n$. We aim to prove that

$$\Delta_n^{-1/4}\Big([H(\widehat{C),G}(C)]_T^{LIN'} - [H(\widehat{C),G}(C)]_T^{A}\Big) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \Delta_n^{-1/4}\Big([H(\widehat{C),G}(C)]_T^{AN'} - [H(\widehat{C),G}(C)]_T^{A}\Big) \xrightarrow{\mathbb{P}} 0. \tag{29}$$

To prove (29), we introduce some new notation. Following Jacod and Rosenbaum (2012), we define

$$\alpha_i^n = (\Delta_i^n Y')(\Delta_i^n Y')^\top - C_i^n \Delta_n, \quad \beta_i^n = \widehat{C}_i'^n - C_i^n, \quad \text{and} \quad \gamma_i^n = \widehat{C}_{i+k_n}'^n - \widehat{C}_i'^n, \tag{30}$$

which satisfy

$$\beta_i^n = \frac{1}{k_n\Delta_n}\sum_{j=0}^{k_n-1}(\alpha_{i+j}^n + (C_{i+j}^n - C_i^n)\Delta_n) \quad \text{and} \quad \gamma_i^n = \beta_{i+k_n} - \beta_i^n + \Delta_n(C_{i+k_n}^n - C_i^n). \tag{31}$$

We have

$$[H(\widehat{C),G}(C)]_T^{LIN'} - [H(\widehat{C),G}(C)]_T^{A} = \frac{3}{2k_n}\sum_{g,h,a,b=1}^{d}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\psi_i^n(g,h,a,b),$$

$$[H(\widehat{C),G}(C)]_T^{AN'} - [H(\widehat{C),G}(C)]_T^{A} = \frac{3}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\Big(\chi_i^n - \sum_{g,h,a,b=1}^{d}(\partial_{gh}H\partial_{ab}G)(C_i^n)\gamma_i^{n,gh}\gamma_i^{n,ab}\Big),$$

with

$$\psi_i^n(g,h,a,b) = \Big(\big(\partial_{gh}H\partial_{ab}G\big)(\widehat{C}_i'^n) - \big(\partial_{gh}H\partial_{ab}G\big)(C_i^n)\Big)\gamma_i^{n,gh}\gamma_i^{n,ab},$$

$$\chi_i^n = \Big(H(\widehat{C}_{i+k_n}'^n) - H(\widehat{C}_i'^n)\Big)\Big(G(\widehat{C}_{i+k_n}'^n) - G(\widehat{C}_i'^n)\Big).$$

By Taylor expansion, we have

$$\big(\partial_{gh}S\partial_{ab}G\big)(\widehat{C}_i'^n) - \big(\partial_{gh}S\partial_{ab}G\big)(C_i^n) = \sum_{x,y=1}^{d}\Big(\partial^2_{xy,gh}S\partial_{ab}G + \partial^2_{xy,ab}G\partial_{gh}S\Big)(C_i^n)\beta_i^{n,xy}$$
$$+\frac{1}{2}\sum_{j,k,x,y=1}^{d}\Big(\partial^3_{jk,xy,gh}S\partial_{ab}G + \partial^2_{xy,gh}S\partial^2_{jk,ab}G + \partial^3_{jk,xy,ab}G\partial_{gh}S + \partial^2_{xy,ab}G\partial^2_{jk,gh}S\Big)(\widetilde{c}_i^n)\beta_i^{n,xy}\beta_i^{n,jk}$$

and

$$S(\widehat{C}_{i+k_n}'^n) - S(\widehat{C}_i'^n) = \sum_{gh}\partial_{gh}S(C_i^n)\gamma_i^{n,gh} + \sum_{j,k,g,h}\partial^2_{jk,gh}S(C_i^n)\gamma_i^{n,gh}\beta_i^{n,jk} + \frac{1}{2}\sum_{x,y,g,h}\partial^2_{xy,gh}S(C_i^n)\gamma_i^{n,gh}\gamma_i^{n,xy}$$
$$+\frac{1}{2}\sum_{x,y,j,k,g,h}\partial^3_{xy,jk,gh}S(CC_i^{n,S})\gamma_i^{n,gh}\beta_i^{n,xy}\beta_i^{n,jk} + \frac{1}{6}\sum_{j,k,x,y,g,h}\partial^3_{jk,xy,gh}S(C_i^{n,S})\gamma_i^{n,jk}\gamma_i^{n,gh}\gamma_i^{n,xy},$$

for $S \in \{H,G\}$, $\widetilde{c}_i^n = \lambda C_i^n + (1-\lambda)\widehat{C}_i'^n$, $C_i^{n,S} = \lambda_S\widehat{C}_i'^n + (1-\lambda_S)\widehat{C}_{i+k_n}'^n$, $CC_i^{n,S} = \mu_S C_i^n + (1-\mu_S)\widehat{C}_i'^n$ for $\lambda, \lambda_H, \mu_H, \lambda_G, \mu_G \in [0,1]$. Although $\widetilde{c}_i^n$ and λ depend on g, h, a, and b, we do not emphasize this in our notation to simplify the exposition.

We remind the reader some well-known results. For any continuous Itô process Z_t, we have

$$\mathbb{E}\Big(\sup_{w\in[0,s]}\Big\|Z_{t+w}-Z_t\Big\|^q\Big|\mathcal{F}_t\Big)\leq K_q s^{q/2}, \text{ and } \Big\|\mathbb{E}\Big(Z_{t+s}-Z_t\Big)\Big|\mathcal{F}_t\Big\|\leq Ks. \tag{32}$$

Set $\mathcal{F}_i^n=\mathcal{F}_{(i-1)\Delta_n}$. By (4.10) in Jacod and Rosenbaum (2013) we have,

$$\mathbb{E}\Big(\Big\|\alpha_i^n\Big\|^q\Big|\mathcal{F}_i^n\Big)\leq K_q\Delta_n^q \text{ for all } q\geq 0 \text{ and } \mathbb{E}\Big(\Big\|\sum_{j=0}^{k_n-1}\alpha_{i+j}^n\Big\|^q\Big|\mathcal{F}_i^n\Big)\leq K_q\Delta_n^q k_n^{q/2} \text{ whenever } q\geq 2. \tag{33}$$

Combining (41), (39), (40) with $Z=c$ and the Hölder inequality yields for $q\geq 2$,

$$\mathbb{E}\Big(\Big\|\beta_i^n\Big\|^q\Big|\mathcal{F}_i^n\Big)\leq K_q\Delta^{q/4}, \text{ and } \mathbb{E}\Big(\Big\|\gamma_i^n\Big\|^q\Big|\mathcal{F}_i^n\Big)\leq K_q\Delta^{q/4}. \tag{34}$$

The bound in the first equation of (42) is tighter than that in (4.11) of Jacod and Rosenbaum (2012) due to the absence of volatility jumps. This tighter bound will be useful later for deriving the asymptotic distribution for the approximate estimator (**Step** 3). By the boundedness of C_t and the polynomial growth assumption, we have

$$\Big|(\partial^3_{jk,xy,ab}G\partial_{gh}H+\partial^2_{xy,gh}H\partial^2_{jk,ab}G)(\widetilde{c}_i^n)\beta_i^{n,xy}\beta_i^{n,jk}\gamma_i^{n,gh}\gamma_i^{n,ab}\Big|\leq K(1+\|\widetilde{c}_i^n\|)^{2(p-2)}\|\beta_i^n\|^2\|\gamma_i^n\|^2.$$

Recalling $\widetilde{c}_i^n=\lambda C_i^n+(1-\lambda)\widehat{C}_i'^n$ and using the convexity of the function $x^{2(p-2)}$, we can refine the last inequality as follows:

$$\Big|(\partial^3_{jk,xy,ab}G\partial_{gh}H+\partial^2_{xy,gh}H\partial^2_{jk,ab}G)(\widetilde{c}_i^n)\beta_i^{n,xy}\beta_i^{n,jk}\gamma_i^{n,gh}\gamma_i^{n,ab}\Big|\leq K\big(1+\|\beta_i^n\|^{2(p-2)}\big)\|\beta_i^n\|^2\|\gamma_i^n\|^2. \tag{35}$$

By Taylor expansion, the polynomial growth assumption and using similar idea as for (35), we have

$$\chi_i^n-\sum_{g,h,a,b}(\partial_{gh}H\partial_{ab}G)(C_i^n)\gamma_i^{n,gh}\gamma_i^{n,ab}=\sum_{g,h,a,b,j,k}(\partial_{gh}H\partial^2_{jk,xy}G+\partial_{gh}G\partial^2_{jk,xy}H)(C_i^n)(\gamma_i^{n,gh}+\frac{1}{2}\beta_i^{n,gh})\gamma_i^{n,ab}\gamma_i^{n,jk}+\varphi_i^n$$

$$\sum_{g,h,a,b}\big(\partial_{gh}H\partial_{ab}G\big)(\widehat{C}_i'^n)-\big(\partial_{gh}H\partial_{ab}G\big)(C_i^n)=\sum_{g,h,a,b,x,y}(\partial_{gh}H\partial^2_{ab,xy}G+\partial_{ab}G\partial^2_{gh,xy}G)(C_i^n)(\beta_i^{n,xy})\gamma_i^{n,gh}\gamma_i^{n,ab}+\delta_i^n$$

with $\mathbb{E}(|\varphi_i^n|\big|\mathcal{F}_i^n)\leq K\Delta_n$ and $\mathbb{E}(|\delta_i^n|\big|\mathcal{F}_i^n)\leq K\Delta_n$ which follow by the Cauchy-Schwartz inequality together with (42). Given that $k_n=\theta(\Delta_n)^{-1/2}$, a direct implication of the previous inequalities is

$$\frac{3\Delta_n^{-1/4}}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\varphi_i^n\xrightarrow{\mathbb{P}}0 \quad\text{and}\quad \frac{3\Delta_n^{-1/4}}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\delta_i^n\xrightarrow{\mathbb{P}}0.$$

Therefore, in order to prove the two claims in (29), it suffices to show

$$\frac{3\Delta_n^{-1/4}}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\sum_{g,h,a,b,j,k}(\partial_{gh}H\partial^2_{jk,ab}G+\partial_{gh}H\partial^2_{jk,ab}G)(C_i^n)\gamma_i^{n,gh}\gamma_i^{n,ab}\gamma_i^{n,jk}\xrightarrow{\mathbb{P}}0, \tag{36}$$

$$\frac{3\Delta_n^{-1/4}}{2k_n}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\sum_{g,h,a,b,j,k}(\partial_{gh}H\partial^2_{jk,ab}G+\partial_{gh}H\partial^2_{jk,ab}G)(C_i^n)\beta_i^{n,gh}\gamma_i^{n,ab}\gamma_i^{n,jk}\xrightarrow{\mathbb{P}}0. \tag{37}$$

For any càdlàg bounded process Z, we set

$$\eta_{t,s}(Z) = \sqrt{\mathbb{E}\Big(\sup_{0<u\le s} \|Z_{t+u} - Z_t\|^2 | \mathcal{F}_i^n\Big)},$$
$$\eta_{i,j}^n(Z) = \sqrt{\mathbb{E}\Big(\sup_{0\le u\le j\Delta_n} \|Z_{(i-1)\Delta_n+u} - Z_{(i-1)\Delta_n}\|^2 | \mathcal{F}_i^n\Big)}.$$

In order to prove (36) and (37), we introduce the following lemmas.

Lemma 1. *For any càdlàg bounded process Z, for all $t, s > 0$, $j, k \ge 0$, set $\eta_{t,s} = \eta_{t,s}(Z)$. Then,*

$$\Delta_n \mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} \eta_{i,k_n}\Big) \longrightarrow 0, \qquad \Delta_n \mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} \eta_{i,2k_n}\Big) \longrightarrow 0,$$
$$\mathbb{E}\Big(\eta_{i+j,k} | \mathcal{F}_i^n\Big) \le \eta_{i,j+k} \quad \text{and} \quad \Delta_n \mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} \eta_{i,4k_n}\Big) \longrightarrow 0.$$

The first three claims of Lemma 6 are proved in Jacod and Rosenbaum (2012). The last result can be proved similarly to the first two.

Lemma 2. *Let Z be a continuous Itô process with drift b_t^Z and spot variance process C_t^Z, and set $\eta_{t,s} = \eta_{t,s}(b^Z, c^Z)$. Then, the following bounds hold:*

$$
\begin{aligned}
&|\mathbb{E}(Z_t|\mathcal{F}_0) - t b_0^Z| \le K t \eta_{0,t} \\
&|\mathbb{E}(Z_t^j Z_t^k - t C_0^{Z,jk} | \mathcal{F}_0)| \le K t^{3/2} (\sqrt{\Delta_n} + \eta_{0,t}) \\
&|\mathbb{E}\big((Z_t^j Z_t^k - t C_0^{Z,jk})(C_t^{Z,lm} - C_0^{Z,lm}) | \mathcal{F}_0\big)| \le K t^2 \\
&|\mathbb{E}(Z_t^j Z_t^k Z_t^l Z_t^m | \mathcal{F}_0) - \Delta_n^2 (C_0^{Z,jk} C_0^{Z,lm} + C_0^{Z,jl} C_0^{Z,km} + C_0^{Z,jm} C_0^{Z,kl})| \le K t^{5/2} \\
&|\mathbb{E}(Z_t^j Z_t^k Z_t^l | \mathcal{F}_0)| \le K t^2 \\
&|\mathbb{E}(\prod_{l=1}^{6} Z_t^{j_l} | \mathcal{F}_0) - \frac{\Delta_n^3}{6} \sum_{l<l'} \sum_{k<k'} \sum_{m<m'} C_0^{Z,j_l j_{l'}} C_0^{Z,j_k j_{k'}} C_0^{Z,j_m j_{m'}}| \le K t^{7/2}
\end{aligned}
$$

The first four claims of Lemma 7 are parts of Lemma 4.1 in Jacod and Rosenbaum (2012). The two remaining statements can be shown similarly.

Lemma 3. *Let ζ_i^n be a r-dimensional $\mathcal{F}_i^n$ measurable process satisfying $\|\mathbb{E}(\zeta_i^n | \mathcal{F}_{i-1}^n)\| \le L'$ and $\mathbb{E}\Big(\|\zeta_i^n\|^q \Big| \mathcal{F}_{i-1}^n\Big) \le L_q$. Also, let φ_i^n be a real-valued $\mathcal{F}_i^n$-measurable process with $\mathbb{E}\Big(\|\varphi_{i+j-1}^n\|^q \Big| \mathcal{F}_{i-1}^n\Big) \le L^q$ for $q \ge 2$ and $1 \le j \le 2k_n - 1$. Then, we have*

$$\mathbb{E}\left(\Big\| \sum_{j=1}^{2k_n-1} \varphi_{i+j-1}^n \zeta_{i+j}^n \Big\|^q \middle| \mathcal{F}_{i-1}^n\right) \le K_q L^q (L_q k_n^{q/2} + L'^q k_n^q).$$

Proof of Lemma 5

Set

$$\xi_i^n = \varphi_{i-1}^n \zeta_i^n, \qquad \xi_i^{'n} = \mathbb{E}(\xi_i | \mathcal{F}_{i-1}^n) = \mathbb{E}(\varphi_{i-1}^n \zeta_i^n | \mathcal{F}_{i-1}^n) = \varphi_{i-1}^n \mathbb{E}(\zeta_i^n | \mathcal{F}_{i-1}^n), \text{ and } \xi_i^{''n} = \xi_i^n - \xi_i^{'n}.$$

Given that $\|\mathbb{E}(\zeta_i^n|\mathcal{F}_{i-1}^n)\| \leq L'$, we have $\|\xi_i^{'n}\| \leq L'|\varphi_{i-1}^n|$. By the convexity of the function x^q, which holds for $q \geq 2$, we have

$$\| \sum_{j=1}^{2k_n-1} \xi_{i+j}^n \|^q \leq K\Big(\| \sum_{j=1}^{2k_n-1} \xi_{i+j}^{'n} \|^q + \| \sum_{j=1}^{2k_n-1} \xi_{i+j}^{''n} \|^q\Big).$$

Therefore, on the one hand we have

$$\| \sum_{j=1}^{2k_n-1} \xi_{i+j}^{'n} \|^q \leq K k_n^{q-1} \sum_{j=1}^{2k_n-1} \|\xi_{i+j}^{'n}\|^q \leq K k_n^{q-1} L'^q \sum_{j=1}^{2k_n-1} |\varphi_{i+j-1}^n|^q,$$

which by $\mathbb{E}\Big(\|\varphi_{i+j-1}^n\|^q \Big| \mathcal{F}_{i-1}^n\Big) \leq L^q$, satisfies

$$\mathbb{E}(\| \sum_{j=1}^{2k_n-1} \xi_{i+j}^{'n} \|^q | \mathcal{F}_{i-1}^n) \leq K L'^q k_n^{q-1} \sum_{j=1}^{2k_n-1} \mathbb{E}(|\varphi_{i+j-1}^n|^q | \mathcal{F}_{i-1}^n) \leq K L'^q k_n^q L^q.$$

On the other hand, we have $\mathbb{E}(\|\xi_{i+j}^{''n}\|^q|\mathcal{F}_{i-1}^n) \leq \mathbb{E}(\|\xi_{i+j}^n\|^q|\mathcal{F}_{i-1}^n) \leq L_q L^q$ and $\mathbb{E}(\xi_{i+j}^{''n}|\mathcal{F}_{i-1}^n) = 0$, where the first inequality is a consequence of $\mathbb{E}(\|\xi_{i+j}^{'n}\|^q|\mathcal{F}_{i-1}^n) \leq \mathbb{E}(\|\xi_{i+j}^n\|^q|\mathcal{F}_{i-1}^n) \leq L_q L^q$, which follows by the Jensen inequality and the law of iterated expectation. Hence, by Lemma B.2 of Aït-Sahalia and Jacod (2014) we have

$$\mathbb{E}(\| \sum_{j=1}^{2k_n-1} \xi_{i+j}^{''n} \|^q | \mathcal{F}_{i-1}^n) \leq K_q L^q L_q k_n^{q/2}.$$

To see the latter, we first prove that the required condition $\mathbb{E}(\|\xi_i^n\|^q|\mathcal{F}_{i-1}^n) \leq L_q L^q)$ in the Lemma B.2 of Aït-Sahalia and Jacod (2014) can be replaced by $\mathbb{E}(\|\xi_{i+j}^n\|^q|\mathcal{F}_{i-1}^n) \leq L_q L^q)$ for $1 \leq j \leq 2k_n - 1$ without altering the result.

Lemma 4. *We have:*

$$\begin{aligned}
&\Big|\mathbb{E}(\gamma_i^{n,jk}\gamma_i^{n,lm}\gamma_{i+2k_n}^{n,gh}\gamma_{i+2k_n}^{n,ab}|\mathcal{F}_i^n) - \frac{4}{k_n^2}(C_i^{n,ga}C_i^{n,hb} + C_i^{n,gb}C_i^{n,ha})(C_i^{n,jl}C_i^{n,km} + C_i^{n,jm}C_i^{n,kl}) \\
&- \frac{4\Delta_n}{3}(C_i^{n,jl}C_i^{n,km} + C_i^{n,jm}C_i^{n,kl})\overline{C}_i^{n,gh,ab} - \frac{4\Delta_n}{3}(C_i^{n,ga}C_i^{n,hb} - C_i^{n,gb}C_i^{n,ha})\overline{C}_i^{n,jk,lm} \\
&- \frac{4(k_n\Delta_n)^2}{9}\overline{C}_i^{n,gh,ab}\overline{C}_i^{n,jk,lm}\Big| \leq K\Delta_n(\Delta_n^{1/8} + \eta_{i,4k_n}^n).
\end{aligned}$$

Throughout, we use the expression "successive conditioning" to refer to the following equalities,

$$\begin{aligned}
&x_1y_1 - x_0y_0 = x_0(y_1 - y_0) + y_0(x_1 - x_0) + (x_1 - x_0)(y_1 - y_0), \\
&x_1y_1z_1 - x_0y_0z_0 = x_0y_0(z_1 - z_0) + x_0z_0(y_1 - y_0) + y_0z_0(x_1 - x_0) + x_0(y_0 - y_1)(z_0 - z_1) \\
&\qquad + y_0(x_0 - x_1)(z_0 - z_1) + z_0(x_0 - x_1)(y_0 - y_1) + (x_1 - x_0)(y_1 - y_0)(z_1 - z_0),
\end{aligned}$$

which hold for any real numbers x_0, y_0, z_0, x_1, y_1, and z_1.

Proof of Lemma 4

To prove Lemma 4, we first note that $\gamma_i^{n,jk}\gamma_i^{n,lm}$ is $\mathcal{F}_{i+2k_n}^n$-measurable. Then, by the law of iterated expectations, we have

$$\mathbb{E}\Big(\gamma_i^{n,jk}\gamma_i^{n,lm}\gamma_{i+2k_n}^{n,gh}\gamma_{i+2k_n}^{n,ab}|\mathcal{F}_i^n\Big) = \mathbb{E}\Big(\gamma_i^{n,jk}\gamma_i^{n,lm}\mathbb{E}(\gamma_{i+2k_n}^{n,gh}\gamma_{i+2k_n}^{n,ab}|\mathcal{F}_{i+2k_n}^n)|\mathcal{F}_i^n\Big).$$

By equation (3.27) in Jacod and Rosenbaum (2012), we have

$$|\mathbb{E}(\gamma_{i+2k_n}^{n,gh}\gamma_{i+2k_n}^{n,ab}|\mathcal{F}_{i+2k_n}^n) - \frac{2}{k_n}(C_{i+2k_n}^{n,ga}C_{i+2k_n}^{n,hb} + C_{i+2k_n}^{n,gb}C_{i+2k_n}^{n,ha}) - \frac{2k_n\Delta_n}{3}\overline{C}_{i+2k_n}^{n,gh,ab}| \leq K\sqrt{\Delta_n}(\Delta_n^{1/8} + \eta_{i+2k_n,2k_n}^n),$$
$$|\mathbb{E}(\gamma_i^{n,jk}\gamma_i^{n,lm}|\mathcal{F}_i^n) - \frac{2}{k_n}(C_i^{n,jl}C_i^{n,km} + C_i^{n,jm}C_i^{n,kl}) - \frac{2k_n\Delta_n}{3}\overline{C}_i^{n,jk,lm}| \leq K\sqrt{\Delta_n}(\Delta_n^{1/8} + \eta_{i,2k_n}^n).$$

Also,

$$\begin{aligned}
&|\mathbb{E}\Big(\gamma_i^{n,jk}\gamma_i^{n,lm}\Big[\mathbb{E}(\gamma_{i+2k_n}^{n,gh}\gamma_{i+2k_n}^{n,ab}\Big|\mathcal{F}_{i+2k_n}^n) - \frac{2}{k_n}(C_{i+2k_n}^{n,ga}C_{i+2k_n}^{n,hb} + C_{i+2k_n}^{n,gb}C_{i+2k_n}^{n,ha}) - \frac{2k_n\Delta_n}{3}\overline{C}_{i+2k_n}^{n,gh,ab}\Big]\Big|\mathcal{F}_i^n\Big)| \\
&\leq \sqrt{\Delta_n}\mathbb{E}(|\gamma_i^{n,jk}||\gamma_i^{n,lm}|(\Delta_n^{1/8} + \eta_{i+2k_n,2k_n}^n)|\Big|\mathcal{F}_i^n) \leq K\sqrt{\Delta_n}\Delta_n^{1/8}\mathbb{E}(|\gamma_i^{n,jk}||\gamma_i^{n,lm}|\Big|\mathcal{F}_i^n) \\
&+ K\sqrt{\Delta_n}\mathbb{E}(|\gamma_i^{n,jk}||\gamma_i^{n,lm}|\eta_{i+2k_n,2k_n}^n|\Big|\mathcal{F}_i^n) \leq K\Delta_n(\Delta_n^{1/8} + \eta_{i,4k_n}^n),
\end{aligned}$$

where the last inequality follows from Lemma 6. Using (40) successively with $Z = c$ and $Z = \overline{C}$ (recall that the latter holds under Assumption 2), together with the successive conditioning, we have

$$\begin{aligned}
&|\mathbb{E}\Big(\gamma_i^{n,jk}\gamma_i^{n,lm}\Big[\frac{2}{k_n}(C_{i+2k_n}^{n,ga}C_{i+2k_n}^{n,hb} + C_{i+2k_n}^{n,gb}C_{i+2k_n}^{n,ha}) + \frac{2k_n\Delta_n}{3}\overline{C}_{i+2k_n}^{n,gh,ab} - \frac{2}{k_n}(C_i^{n,ga}C_i^{n,hb} + C_i^{n,gb}C_i^{n,ha}) \\
&- \frac{2k_n\Delta_n}{3}\overline{C}_i^{n,gh,ab}\Big]\Big|\mathcal{F}_i^n\Big)| \leq K\Delta_n\Delta_n^{1/4}, \\
&|\mathbb{E}\Big(\gamma_i^{n,jk}\gamma_i^{n,lm}\Big[\frac{2}{k_n}(C_i^{n,ga}C_i^{n,hb} + C_i^{n,gb}C_i^{n,ha}) + \frac{2k_n\Delta_n}{3}\overline{C}_i^{n,gh,ab}\Big] - \Big[\frac{2}{k_n}(C_i^{n,jl}C_i^{n,km} + C_i^{n,jm}C_i^{n,kl}) + \frac{2k_n\Delta_n}{3}\overline{C}_i^{n,jk,lm}\Big] \\
&\times \Big[\frac{2}{k_n}(C_i^{n,ga}C_i^{n,hb} + C_i^{n,gb}C_i^{n,ha}) + \frac{2k_n\Delta_n}{3}\overline{C}_i^{n,gh,ab}\Big]\Big|\mathcal{F}_i^n\Big)| \leq K\Delta_n(\Delta_n^{1/8} + \eta_{i,2k_n}^n).
\end{aligned}$$

The last inequality yields the result.

Lemma 5. *Let ζ_i^n be a r-dimensional $\mathcal{F}_i^n$-measurable process satisfying $\|\mathbb{E}(\zeta_i^n|\mathcal{F}_{i-1}^n)\| \leq L'$ and $\mathbb{E}\Big(\|\zeta_i^n\|^q\Big|\mathcal{F}_{i-1}^n\Big) \leq L_q$. Also, let φ_i^n be a real-valued $\mathcal{F}_i^n$-measurable process with $\mathbb{E}\Big(\|\varphi_{i+j-1}^n\|^q\Big|\mathcal{F}_{i-1}^n\Big) \leq L^q$ for $q \geq 2$ and $1 \leq j \leq 2k_n - 1$. Then,*

$$\mathbb{E}\Bigg(\Big\|\sum_{j=1}^{2k_n-1}\varphi_{i+j-1}^n\zeta_{i+j}^n\Big\|^q\Bigg|\mathcal{F}_{i-1}^n\Bigg) \leq K_qL^q(L_qk_n^{q/2} + L'^qk_n^q).$$

We introduce some new notation. Following Jacod and Rosenbaum (2012), we define

$$\alpha_i^n = (\Delta_i^nY')(\Delta_i^nY')^\top - C_i^n\Delta_n, \quad \beta_i^n = \widehat{C}_i'^n - C_i^n, \quad \text{and} \quad \gamma_i^n = \widehat{C}_{i+k_n}'^n - \widehat{C}_i'^n, \tag{38}$$

which satisfy

$$\beta_i^n = \frac{1}{k_n\Delta_n}\sum_{j=0}^{k_n-1}(\alpha_{i+j}^n + (C_{i+j}^n - C_i^n)\Delta_n) \quad \text{and} \quad \gamma_i^n = \beta_{i+k_n} - \beta_i^n + \Delta_n(C_{i+k_n}^n - C_i^n). \tag{39}$$

We remind some well-known results. For any continuous Itô process Z_t, we have

$$\mathbb{E}\Big(\sup_{w\in[0,s]}\Big\|Z_{t+w}-Z_t\Big\|^q\Big|\mathcal{F}_t\Big)\leq K_q s^{q/2},\text{ and }\Big\|\mathbb{E}\Big(Z_{t+s}-Z_t\Big)\Big|\mathcal{F}_t\Big\|\leq Ks. \tag{40}$$

Set $\mathcal{F}_i^n=\mathcal{F}_{(i-1)\Delta_n}$. By (4.10) in Jacod and Rosenbaum (2013), we have

$$\mathbb{E}\Big(\Big\|\alpha_i^n\Big\|^q\Big|\mathcal{F}_i^n\Big)\leq K_q\Delta_n^q\text{ for all }q\geq 0\text{ and }\mathbb{E}\Big(\Big|\sum_{j=0}^{k_n-1}\alpha_{i+j}^n\Big\|^q\Big|\mathcal{F}_i^n\Big)\leq K_q\Delta_n^q k_n^{q/2}\text{ whenever }q\geq 2. \tag{41}$$

Combining (41), (39), (40) with $Z=c$ and the Hölder inequality yields, for $q\geq 2$,

$$\mathbb{E}\Big(\Big\|\beta_i^n\Big\|^q\Big|\mathcal{F}_i^n\Big)\leq K_q\Delta^{q/4},\quad\text{and}\quad\mathbb{E}\Big(\Big\|\gamma_i^n\Big\|^q\Big|\mathcal{F}_i^n\Big)\leq K_q\Delta^{q/4}. \tag{42}$$

For any càdlàg bounded process Z, we set

$$\eta_{t,s}(Z)=\sqrt{\mathbb{E}\Big(\sup_{0<u\leq s}\|Z_{t+u}-Z_t\|^2|\mathcal{F}_i^n\Big)},$$

$$\eta_{i,j}^n(Z)=\sqrt{\mathbb{E}\Big(\sup_{0\leq u\leq j\Delta_n}\|Z_{(i-1)\Delta_n+u}-Z_{(i-1)\Delta_n}\|^2|\mathcal{F}_i^n\Big)}.$$

Lemma 6. *For any càdlàg bounded process Z, for all $t,s>0$, $j,k\geq 0$, and set $\eta_{t,s}=\eta_{t,s}(Z)$. Then,*

$$\Delta_n\mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]}\eta_{i,k_n}\Big)\longrightarrow 0,\qquad\Delta_n\mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]}\eta_{i,2k_n}\Big)\longrightarrow 0,$$

$$\mathbb{E}\Big(\eta_{i+j,k}|\mathcal{F}_i^n\Big)\leq\eta_{i,j+k}\quad\textit{and}\quad\Delta_n\mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]}\eta_{i,4k_n}\Big)\longrightarrow 0.$$

The first three claims of Lemma 6 are proved in Jacod and Rosenbaum (2012). The last result can be proved similarly to the first two.

Lemma 7. *Let Z be a continuous Itô process with drift term b_t^Z and spot variance process C_t^Z, and set $\eta_{t,s}=\eta_{t,s}(b^Z,c^Z)$. Then, the following bounds hold:*

$$|\mathbb{E}(Z_t|\mathcal{F}_0)-tb_0^Z|\leq Kt\eta_{0,t}$$
$$|\mathbb{E}(Z_t^jZ_t^k-tC_0^{Z,jk}|\mathcal{F}_0)|\leq Kt^{3/2}(\sqrt{\Delta_n}+\eta_{0,t})$$
$$|\mathbb{E}((Z_t^jZ_t^k-tC_0^{Z,jk})(C_t^{Z,lm}-C_0^{Z,lm})|\mathcal{F}_0)|\leq Kt^2$$
$$|\mathbb{E}(Z_t^jZ_t^kZ_t^lZ_t^m|\mathcal{F}_0)-\Delta_n^2(C_0^{Z,jk}C_0^{Z,lm}+C_0^{Z,jl}C_0^{Z,km}+C_0^{Z,jm}C_0^{Z,kl})|\leq Kt^{5/2}$$
$$|\mathbb{E}(Z_t^jZ_t^kZ_t^l|\mathcal{F}_0)|\leq Kt^2$$
$$|\mathbb{E}(\prod_{l=1}^6 Z_t^{j_l}|\mathcal{F}_0)-\frac{\Delta_n^3}{6}\sum_{l<l'}\sum_{k<k'}\sum_{m<m'}C_0^{Z,j_lj_{l'}}C_0^{Z,j_kj_{k'}}C_0^{Z,j_mj_{m'}}|\leq Kt^{7/2}$$

The first four claims of Lemma 7 are parts of Lemma 4.1 in Jacod and Rosenbaum (2012). The two remaining statements can be shown similarly.

Lemma 8. *The following results hold:*

$$|\mathbb{E}(\beta_i^{n,jk}\beta_i^{n,lm}\beta_i^{n,gh}|\mathcal{F}_i^n)|\leq K\Delta_n^{3/4}(\Delta_n^{1/4}+\eta_{i,k_n}^n), \tag{43}$$

$$|\mathbb{E}(\beta_i^{n,jk}\beta_i^{n,lm}(c_{i+k_n}^{n,gh}-c_i^{n,gh})|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4}+\eta_{i,k_n}^n), \tag{44}$$
$$|\mathbb{E}(\beta_i^{n,jk}(c_{i+k_n}^{n,lm}-c_i^{n,lm})(c_{i+k_n}^{n,gh}-c_i^{n,gh})|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4}+\eta_{i,k_n}^n), \tag{45}$$
$$|\mathbb{E}(\beta_i^{n,jk}\gamma_i^{n,lm}\gamma_i^{n,gh}|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4}+\eta_{i,2k_n}^n), \tag{46}$$
$$|\mathbb{E}(\gamma_i^{n,jk}\gamma_i^{n,lm}\gamma_i^{n,gh}|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4}+\eta_{i,2k_n}^n). \tag{47}$$

Proof of (43) in Lemma 8

We start by obtaining some useful bounds for some quantities of interest. First, using the second statement in Lemma 7 applied to $Z = Y'$, we have

$$|\mathbb{E}(\alpha_i^{n,jk}|\mathcal{F}_i^n)| \leq K\Delta_n^{3/2}(\sqrt{\Delta_n}+\eta_{i,1}^n). \tag{48}$$

Second, by repeated application of the Cauchy-Schwartz inequality and making use of the third and last statements in Lemma 7 as well as (40) with $Z = c$, it can be shown that

$$\left|\mathbb{E}(\alpha_i^{n,jk}\alpha_i^{n,lm}|\mathcal{F}_i^n) - \Delta_n^2\left(C_i^{n,jl}C_i^{n,km}+C_i^{n,jm}C_i^{n,kl}\right)\right| \leq K\Delta_n^{5/2}. \tag{49}$$

Next, by successive conditioning and using the bound in (40) for $Z = c$ as well as (48) and (49) , we have for $0 \leq u \leq k_n - 1$,

$$\left|\mathbb{E}(\alpha_{i+u}^{n,jk}|\mathcal{F}_i^n)\right| \leq K\Delta_n^{3/2}(\sqrt{\Delta_n}+\eta_{i,u}^n), \tag{50}$$

$$\left|\mathbb{E}(\alpha_{i+u}^{n,jk}\alpha_{i+u}^{n,lm}|\mathcal{F}_i^n) - \Delta_n^2\left(C_i^{n,jl}C_i^{n,km}+C_i^{n,jm}C_i^{n,kl}\right)\right| \leq K\Delta_n^{5/2}, \tag{51}$$

To show (43), we first observe that $\beta_i^{n,jk}\beta_i^{n,lm}\beta_i^{n,gh}$ can be decomposed as

$$\begin{aligned}
\beta_i^{n,jk}\beta_i^{n,lm}\beta_i^{n,gh} &= \frac{1}{k_n^3\Delta_n^3}\sum_{u=0}^{k_n-1}\zeta_{i,u}^{n,jk}\zeta_{i,u}^{n,lm}\zeta_{i,u}^{n,gh} + \frac{1}{k_n^3\Delta_n^3}\sum_{u=0}^{k_n-2}\sum_{v=u+1}^{k_n-1}\Big[\zeta_{i,u}^{n,jk}\zeta_{i,v}^{n,lm}\zeta_{i,v}^{n,gh}+\zeta_{i,u}^{n,gh}\zeta_{i,v}^{n,jk}\zeta_{i,v}^{n,lm} \\
&+ \zeta_{i,u}^{n,lm}\zeta_{i,v}^{n,gh}\zeta_{i,v}^{n,jk}\Big] + \frac{1}{k_n^3\Delta_n^3}\sum_{u=0}^{k_n-2}\sum_{v=u+1}^{k_n-1}\big[\zeta_{i,u}^{n,jk}\zeta_{i,u}^{n,lm}\zeta_{i,v}^{n,gh}+\zeta_{i,u}^{n,gh}\zeta_{i,u}^{n,jk}\zeta_{i,v}^{n,lm}+\zeta_{i,u}^{n,lm}\zeta_{i,u}^{n,gh}\zeta_{i,v}^{n,jk}\big] \\
&+ \frac{1}{k_n^3\Delta_n^3}\sum_{u=0}^{k_n-3}\sum_{v=u+1}^{k_n-2}\sum_{w=v+1}^{k_n-1}\Big[\zeta_{i,u}^{n,jk}\zeta_{i,v}^{n,lm}\zeta_{i,w}^{n,gh}+\zeta_{i,u}^{n,jk}\zeta_{i,v}^{n,gh}\zeta_{i,w}^{n,lm}+\zeta_{i,u}^{n,lm}\zeta_{i,v}^{n,jk}\zeta_{i,w}^{n,gh}+\zeta_{i,u}^{n,lm}\zeta_{i,v}^{n,gh}\zeta_{i,w}^{n,jk} \\
&+ \zeta_{i,u}^{n,gh}\zeta_{i,v}^{n,lm}\zeta_{i,w}^{n,jk}+\zeta_{i,u}^{n,gh}\zeta_{i,v}^{n,jk}\zeta_{i,w}^{n,lm}\Big],
\end{aligned}$$

with $\zeta_{i,u}^n = \alpha_{i+u}^n + (C_{i+u}^n - C_i^n)\Delta_n$, which satisfies $\mathbb{E}(\|\zeta_{i,u}^n\|^q|\mathcal{F}_i^n) \leq K\Delta_n^q$ for $q \geq 2$.
Set

$$\xi_i^n(1) = \frac{1}{k_n^3\Delta_n^3}\sum_{u=0}^{k_n-1}\zeta_{i,u}^{n,jk}\zeta_{i,u}^{n,lm}\zeta_{i,u}^{n,gh}, \quad \xi_i^n(2) = \frac{1}{k_n^3\Delta_n^3}\sum_{u=0}^{k_n-2}\sum_{v=u+1}^{k_n-1}\zeta_{i,u}^{n,jk}\zeta_{i,v}^{n,lm}\zeta_{i,v}^{n,gh}$$
$$\xi_i^n(3) = \frac{1}{k_n^3\Delta_n^3}\sum_{u=0}^{k_n-2}\sum_{v=u+1}^{k_n-1}\zeta_{i,u}^{n,jk}\zeta_{i,u}^{n,lm}\zeta_{i,v}^{n,gh} \quad \text{and} \quad \xi_i^n(4) = \frac{1}{k_n^3\Delta_n^3}\sum_{u=0}^{k_n-3}\sum_{v=u+1}^{k_n-2}\sum_{w=v+1}^{k_n-1}\zeta_{i,u}^{n,jk}\zeta_{i,v}^{n,lm}\zeta_{i,w}^{n,gh}.$$

The following bounds can be established,

$$|\mathbb{E}(\xi_i^n(1)|\mathcal{F}_i^n)| \leq K\Delta_n, \quad |\mathbb{E}(\xi_i^n(2)|\mathcal{F}_i^n)| \leq K\Delta_n, \quad |\mathbb{E}(\xi_i^n(3)|\mathcal{F}_i^n)| \leq K\Delta_n \quad \text{and}$$

$$|\mathbb{E}(\xi_i^n(4)|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,k_n}).$$

Proof of $|\mathbb{E}(\xi_i^n(1)|\mathcal{F}_i^n)| \leq K\Delta_n$

The result readily follows from an application of the Cauchy Schwartz inequality together with the bound $\mathbb{E}(\|\zeta_{i+u}^n\|^q|\mathcal{F}_i^n) \leq K_q\Delta_n^q$ for $q \geq 2$.

Proof of $|\mathbb{E}(\xi_i^n(2)|\mathcal{F}_i^n)| \leq K\Delta_n$

Using the law of iterated expectation, we have, for $u < v$,

$$\mathbb{E}(\zeta_{i+u}^{n,jk}\zeta_{i+v}^{n,lm}\zeta_{i+v}^{n,gh}|\mathcal{F}_i^n) = \mathbb{E}(\zeta_{i+u}^{n,jk}\mathbb{E}(\zeta_{i+v}^{n,lm}\zeta_{i+v}^{n,gh}|\mathcal{F}_{i+u+1}^n)|\mathcal{F}_i^n). \tag{52}$$

By successive conditioning, (49), and the Cauchy-Schwartz inequality, we also have

$$|\mathbb{E}(\zeta_{i,v}^{n,lm}\zeta_{i,v}^{n,gh}|\mathcal{F}_{i+u+1}^n) - \Delta_n^2(C_{i+u+1}^{n,lg}C_{i+u+1}^{n,mh} + C_{i+u+1}^{n,lh}C_{i+u+1}^{n,mg}) - \Delta_n^2(C_{i+u+1}^{n,gh} - C_i^{n,gh})(C_{i+u+1}^{n,lm} - C_i^{n,lm})| \leq K\Delta_n^{5/2}.$$

Given that $\mathbb{E}(|\zeta_{i+u}^{n,jk}|^q|\mathcal{F}_i^n) \leq \Delta_n^q$, the approximation error involved in replacing $\mathbb{E}(\zeta_{i+v}^{n,lm}\zeta_{i+v}^{n,gh}|\mathcal{F}_{i+u+1}^n)$ by $\Delta_n^2(C_{i+u+1}^{n,lg}C_{i+u+1}^{n,mh} + C_{i+u+1}^{n,lh}C_{i+u+1}^{n,mg}) + \Delta_n^2(C_{i+u+1}^{n,gh} - C_i^{n,gh})(C_{i+u+1}^{n,lm} - C_i^{n,lm})$ in (52) is smaller than $\Delta_n^{7/2}$. From (3.9) in Jacod and Rosenbaum (2012) we have

$$|\mathbb{E}(\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,lm} - C_{i+u}^{n,lm})|\mathcal{F}_i^n)| \leq K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_{i,k_n}^n). \tag{53}$$

Since $(C_{i+u}^n - C_i^n)$ is $\mathcal{F}_{i+u}^n$-measurable, we use the successive conditioning, the Cauchy-Schwartz inequality, (48), (49), and the fifth statement in Lemma 7 applied to $Z = c$ to obtain

$$\begin{aligned}
&|\mathbb{E}(\alpha_{i+u}^{n,gh}(C_{i+u}^{n,lm} - C_i^{n,lm})(C_{i+u}^{n,jk} - C_i^{n,jk})|\mathcal{F}_i^n)| \leq K\Delta_n^{5/2}\\
&|\mathbb{E}(\alpha_{i+u}^{n,jk}\alpha_{i+u}^{n,lm}(C_{i+u}^{n,gh} - C_i^{n,gh})|\mathcal{F}_i^n)| \leq K\Delta_n^{5/2}\\
&|\mathbb{E}\big((C_{i+u}^{n,lm} - C_i^{n,lm})(C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+u}^{n,gh} - C_i^{n,gh})\big)|\mathcal{F}_i^n)| \leq K\Delta_n,
\end{aligned} \tag{54}$$

which can be proved using . The following inequalities can be established easily using (48), the successive conditioning together with (40) for $Z = c$,

$$\Big|\mathbb{E}(\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,lg}C_{i+u+1}^{n,mh} + C_{i+u+1}^{n,lh}C_{i+u+1}^{n,mg})|\mathcal{F}_i^n)\Big| \leq K\Delta_n^{3/2}$$

$$\Big|\mathbb{E}\Big((C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+u+1}^{n,lg}C_{i+u+1}^{n,mh} + C_{i+u+1}^{n,lh}C_{i+u+1}^{n,mg})|\mathcal{F}_i^n\Big)\Big| \leq K\Delta_n^{1/2}$$

$$\Big|\mathbb{E}(\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,gh} - C_i^{n,gh})(C_{i+u+1}^{n,lm} - C_i^{n,lm})|\mathcal{F}_i^n)\Big| \leq K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_{i,k_n}^n).$$

The last three inequalities together yield $|\mathbb{E}(\xi_i^n(2)|\mathcal{F}_i^n)| \leq K\Delta_n$.

Proof of $|\mathbb{E}(\xi_i^n(3)|\mathcal{F}_i^n)| \leq K\Delta_n$

First, note that, for $u < v$, we have

$$\mathbb{E}(\zeta_{i+u}^{n,jk}\zeta_{i+u}^{n,lm}\zeta_{i+v}^{n,gh}|\mathcal{F}_i^n) = \mathbb{E}(\zeta_{i+u}^{n,jk}\zeta_{i+u}^{n,lm}\mathbb{E}(\zeta_{i+v}^{n,gh}|\mathcal{F}_{i+u+1}^n)|\mathcal{F}_i^n). \tag{55}$$

By successive conditioning and (48) , we have

$$|\mathbb{E}(\alpha_{i+w}^{n,gh}|\mathcal{F}_{i+v+1}^n)| \leq K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_{i+v+1,w-v}). \tag{56}$$

Using the first statement of Lemma applied to $Z = c$, it can be shown that

$$|\mathbb{E}\big((C^{n,gh}_{i+w} - C^{n,gh}_{i+v+1}))|\mathcal{F}^n_i\big) - \Delta_n(w-v-1)\widetilde{b}^{n,gh}_{i+v+1}| \le K(w-v-1)\Delta_n\eta_{i+v+1,w-v} \le K\Delta_n^{1/2}\eta_{i+v+1,w-v}.$$

The last two inequalities together imply

$$\Big|\mathbb{E}\Big(\zeta^{n,gh}_{i+w}|\mathcal{F}^n_{i+v+1}\Big) - (C^{n,gh}_{i+v+1} - C^{n,gh}_i)\Delta_n - \Delta_n^2(w-v-1)\widetilde{b}^{n,gh}_{i+v+1}\Big| \le K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_{i+v+1,w-v}). \quad (57)$$

Since $\mathbb{E}(|\zeta^{n,jk}_{i,u}|^q|\mathcal{F}^n_i) \le \Delta_n^q$, the error induced by replacing $\mathbb{E}(\zeta^{n,gh}_{i+v}|\mathcal{F}^n_{i+u+1})$ by $(C^{n,gh}_{i+v+1} - C^{n,gh}_i)\Delta_n + \Delta_n^2(w-v-1)\widetilde{b}^{n,gh}_{i+v+1}$ in (55) is smaller that $\Delta_n^{7/2}$.
Using Cauchy Schwartz inequality, successive conditioning, (54), (40) for $Z = c$ and the boundedness of $\widetilde{b}_t$ and C_t we obtain

$$\begin{aligned}
&\Big|\mathbb{E}\Big(\alpha^{n,jk}_{i+u}\alpha^{n,lm}_{i+u}(C^{n,jk}_{i+u+1} - C^{n,gh}_i)|\mathcal{F}^n_{i+u}\Big)\Big| \le K\Delta_n^{5/2}\\
&\Big|\mathbb{E}\Big(\alpha^{n,jk}_{i+u}\alpha^{n,lm}_{i+u}\widetilde{b}^{n,gh}_{i+u+1}|\mathcal{F}^n_{i+u}\Big)\Big| \le K\Delta_n^2\\
&\Big|\mathbb{E}\Big(\alpha^{n,jk}_{i+u}(C^{n,lm}_{i+u} - C^{n,lm}_i)(C^{n,gh}_{i+u+1} - C^{n,gh}_i)|\mathcal{F}^n_i\Big)\Big| \le K\Delta_n^{1/4}\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta^n_{i,k_n})\\
&\Big|\mathbb{E}\Big(\alpha^{n,jk}_{i+u}(C^{n,lm}_{i+u} - C^{n,lm}_i)\widetilde{b}^{n,gh}_{i+u+1}|\mathcal{F}^n_i\Big)\Big| \le \Delta_n^{5/4}\\
&\Big|\mathbb{E}\Big((C^{n,jk}_{i+u} - C^{n,gh}_i)(C^{n,lm}_{i+u} - C^{n,lm}_i)\widetilde{b}^{n,gh}_{i+u+1}|\mathcal{F}^n_i\Big)\Big| \le K\Delta_n^{1/2}\\
&\Big|\mathbb{E}\Big((C^{n,jk}_{i+u} - C^{n,jk}_i)(C^{n,lm}_{i+u} - C^{n,lm}_i)(C^{n,gh}_{i+u+1} - C^{n,gh}_i)|\mathcal{F}^n_i\Big)\Big| \le K\Delta_n.
\end{aligned}$$

The above inequalities together yield $|\mathbb{E}(\xi^n_i(3)|\mathcal{F}^n_i)| \le K\Delta_n$.

Proof of $|\mathbb{E}(\xi^n_i(4)\big|\mathcal{F}^n_i)| \le K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta^n_{i,k_n})$

We first observe that $\xi^n_i(4)$ can be rewritten as

$$\xi^n_i(4) = \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}\zeta^{n,jk}_{i+u}\zeta^{n,lm}_{i+v}\zeta^{n,gh}_{i+w},$$

where

$$\begin{aligned}
\zeta^{n,jk}_{i+u}\zeta^{n,lm}_{i+v}\zeta^{n,gh}_{i+w} = \Big[&\alpha^{n,jk}_{i+u}\alpha^{n,lm}_{i+v}\alpha^{n,gh}_{i+w} + \alpha^{n,jk}_{i+u}\Delta_n\alpha^{n,lm}_{i+v}(C^{n,gh}_{i+w} - C^{n,gh}_i) + \alpha^{n,jk}_{i+u}\Delta_n(C^{n,lm}_{i+v} - C^{n,lm}_i)\alpha^{n,gh}_{i+w}\\
&+ \Delta_n^2\alpha^{n,jk}_{i+u}(C^{n,lm}_{i+v} - C^{n,lm}_i)(C^{n,gh}_{i+w} - C^{n,gh}_i) + \Delta_n(C^{n,jk}_{i+u} - C^{n,jk}_i)\alpha^{n,lm}_{i+v}\alpha^{n,gh}_{i+w} + \Delta_n^2(C^{n,jk}_{i+u} - C^{n,jk}_i)\alpha^{n,lm}_{i+v}(C^{n,gh}_{i+w} - C^{n,gh}_i)\\
&+ \Delta_n^2(C^{n,jk}_{i+u} - C^{n,jk}_i)(C^{n,lm}_{i+v} - C^{n,lm}_i)\alpha^{n,gh}_{i+w} + \Delta_n^3(C^{n,jk}_{i+u} - C^{n,jk}_i)(C^{n,lm}_{i+v} - C^{n,lm}_i)(C^{n,gh}_{i+w} - C^{n,gh}_i)\Big].
\end{aligned}$$

Based on the above decomposition, we set

$$\xi^n_i(4) = \sum_{j=1}^{8}\chi(j),$$

with $\chi(j)$ defined below. We aim to show that $|\mathbb{E}(\chi(j)\big|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4}+\eta_{i,k_n}^n)$, $j=1,\ldots,8$. First, set

$$\chi(1) = \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\alpha_{i+v}^{n,lm}\alpha_{i+w}^{n,gh}.$$

Upon changing the order of the summation, we have

$$\chi(1) = \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\alpha_{i+v}^{n,lm}\alpha_{i+w}^{n,gh}.$$

Define also

$$\chi'(1) = \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\alpha_{i+v}^{n,lm}\mathbb{E}(\alpha_{i+w}^{n,gh}|\mathcal{F}_{i+v+1}^n).$$

Note that $\mathbb{E}(\chi(1)|\mathcal{F}_i^n)=\mathbb{E}(\chi'(1)|\mathcal{F}_i^n)$.
It is easy to see that by Lemma 5, we have for $q\geq 2$,

$$\mathbb{E}\Big(\Big\|\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big\|^q\Big|\mathcal{F}_i^n\Big) \leq K_q\Delta_n^{3q/4}.$$

The Cauchy-Schwartz inequality yields,

$$\mathbb{E}\bigg(\Big|\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\alpha_{i+v}^{n,lm}\mathbb{E}(\alpha_{i+w}^{n,gh}|\mathcal{F}_{i+v+1}^n)\Big|\Big|\mathcal{F}_i^n\bigg) \leq Kk_n^2\Big[\mathbb{E}\Big(\Big|\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big|^4\Big|\mathcal{F}_i^n\Big)\Big]^{1/4}\Big[\mathbb{E}\Big(\Big|\alpha_{i+v}^{n,lm}\Big|^4\Big|\mathcal{F}_i^n\Big)\Big]^{1/4}$$
$$\times\Big[\mathbb{E}\Big(\Big|\mathbb{E}(\alpha_{i+w}^{n,gh}|\mathcal{F}_{i+v+1}^n)\Big|^2\Big|\mathcal{F}_i^n\Big)\Big]^{1/2} \leq K\Delta_n k_n^2\Delta_n^{3/4}\Delta_n^{3/2}(\sqrt{\Delta_n}+\eta_{i,k_n}^n),$$

where the last iteration is obtained using (56) as well as the inequality $(a+b)^{1/2}\leq a^{1/2}+b^{1/2}$, which holds for positive real numbers a and b, and the third statement in Lemma 6.
It follows from this result that

$$|\mathbb{E}\Big(\chi(1)\big|\mathcal{F}_i^n\Big)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n}+\eta_{i,k_n}^n).$$

Next, we introduce

$$\chi(2) = \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\Delta_n(C_{i+u}^{n,jk}-C_i^{n,jk})\Big)\alpha_{i+v}^{n,lm}\alpha_{i+w}^{n,gh},$$
$$\chi(3) = \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+v}^{n,jk}\Big)\Delta_n(C_{i+u}^{n,lm}-C_i^{n,lm})\alpha_{i+w}^{n,gh},$$
$$\chi(4) = \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\Delta_n(C_{i+u}^{n,jk}-C_i^{n,jk})\Big)\Delta_n(C_{i+u}^{n,lm}-C_i^{n,lm})\alpha_{i+w}^{n,gh}.$$

Given that for $q\geq 2$, we have

$$\mathbb{E}\Big(\Big\|\sum_{u=0}^{v-1}\Delta_n(C_{i+u}^{n,jk}-C_i^{n,jk})\Big\|^q\Big|\mathcal{F}_i^n\Big) \leq K_q\Delta_n^{3q/4} \text{ and } \mathbb{E}(\|C_{i+u}^{n,jk}-C_i^{n,jk}\|^q\big|\mathcal{F}_i^n) \leq K_q\Delta_n^{q/4},$$

one can follow essentially the same steps as for $\chi(1)$ to show that

$$|\mathbb{E}(\chi(2)|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n}+\eta_{i,k_n}^n) \quad \text{and} \quad |\mathbb{E}(\chi(j)|\mathcal{F}_i^n)| \leq K\Delta_n(\sqrt{\Delta_n}+\eta_{i,k_n}^n) \quad \text{for} \quad j=3,4.$$

Define

$$\begin{aligned}
\chi(5) &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\alpha_{i+v}^{n,lm}\Delta_n(C_{i+w}^{n,gh}-C_i^{n,gh})\\
\chi'(5) &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\alpha_{i+v}^{n,lm}\Delta_n\mathbb{E}\big((C_{i+w}^{n,gh}-C_i^{n,gh})|\mathcal{F}_{i+v+1}^n\big)\\
\chi(6) &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\Delta_n(C_{i+u}^{n,jk}-C_i^{n,jk})\Big)\alpha_{i+v}^{n,lm}\Delta_n(C_{i+w}^{n,gh}-C_i^{n,gh})\\
\chi(7) &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\Delta_n(C_{i+v}^{n,lm}-C_i^{n,lm})\Delta_n(C_{i+w}^{n,gh}-C_i^{n,gh}),
\end{aligned}$$

where we have $\mathbb{E}(\chi(5)|\mathcal{F}_i^n)=\mathbb{E}(\chi'(5)|\mathcal{F}_i^n)$. Recalling (57), we further decompose $\chi'(5)$ as,

$$\chi'(5)=\sum_{j=1}^{5}\chi(5)[j],$$

with

$$\begin{aligned}
\chi\prime(5)[1] &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\alpha_{i+v}^{n,lm}\Big(\mathbb{E}\Big(C_{i+w}^{n,gh}-C_i^{n,gh}|\mathcal{F}_{i+v+1}^n\Big)-(C_{i+v+1}^{n,gh}-C_i^{n,gh})\Delta_n-\widetilde{b}_{i+v+1}^{n,gh}\Delta_n^2(w-v-1)\Big)\\
\chi\prime(5)[2] &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Delta_n(C_{i+v}^{n,gh}-C_i^{n,gh})\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\alpha_{i+v}^{n,lm}\\
\chi\prime(5)[3] &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\Delta_n(C_{i+v+1}^{n,gh}-C_{i+v}^{n,gh})\alpha_{i+v}^{n,lm}\\
\chi\prime(5)[4] &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\Delta_n^2(w-v-1)(\widetilde{b}_{i+v+1}^{n,gh}-\widetilde{b}_{i+v}^{n,gh})\alpha_{i+v}^{n,lm}\\
\chi\prime(5)[5] &= \frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\Delta_n^2(w-v-1)\widetilde{b}_{i+v}^{n,gh}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\alpha_{i+v}^{n,lm}.
\end{aligned}$$

Using (57), (56), (53) and following the same strategy proof as for $\chi(1)$, it can be shown that

$$|\mathbb{E}\Big(\chi\prime(5)[j]|\mathcal{F}_i^n\Big)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n}+\eta_{i,k_n}^n), \quad \text{for} \quad j=1,\ldots,5,$$

which in turn implies

$$|\mathbb{E}\Big(\chi(5)|\mathcal{F}_i^n\Big)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n}+\eta_{i,k_n}^n), \quad \text{for} \quad j=1,\ldots,5.$$

The term $\chi(6)$ can be handled similarly to $\chi(5)$, hence we conclude that

$$|\mathbb{E}\Big(\chi(6)|\mathcal{F}_i^n\Big)| \leq K\Delta_n^{3/4}(\sqrt{\Delta_n}+\eta_{i,k_n}^n).$$

Next, we set

$$\chi(7)=\frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\bigg(\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\Delta_n(C_{i+v}^{n,lm}-C_i^{n,lm})\Delta_n(C_{i+w}^{n,gh}-C_i^{n,gh})\bigg).$$

Define

$$\chi(7)[1]=\frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\bigg(\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\Delta_n(C_{i+v}^{n,lm}-C_i^{n,lm})\Delta_n(C_{i+v+1}^{n,gh}-C_{i+v}^{n,gh})\bigg)$$
$$\chi(7)[2]=\frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\bigg(\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\Delta_n(C_{i+v}^{n,lm}-C_i^{n,lm})\Delta_n(C_{i+v}^{n,gh}-C_i^{n,gh})\bigg)$$
$$\chi(7)[3]=\frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\bigg(\sum_{v=0}^{w-1}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\Delta_n(C_{i+v}^{n,lm}-C_i^{n,lm})\Delta_n^2(w-v-1)(\widetilde{b}_{i+v+1}^{n,gh}-\widetilde{b}_{i+v}^{n,gh})\bigg)$$
$$\chi(7)[4]=\frac{1}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\bigg(\sum_{v=0}^{w-1}\Delta_n^2(w-v-1)\widetilde{b}_{i+v}^{n,gh}\Big(\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}\Big)\Delta_n(C_{i+v}^{n,lm}-C_i^{n,lm})\bigg),$$

so that

$$\chi(7)=\sum_{j=1}^{4}\chi(7)[j].$$

Similar to calculations used for $\chi(1)$, it can be shown that

$$|\mathbb{E}(\chi(7)[j]\big|\mathcal{F}_i^n)|\le K\Delta_n^{1/4}(\Delta_n^{1/4}+\eta_{i,k_n}),\quad \text{for}\quad j=1,\dots,3.$$

To handle the remaining term $\chi(7)[4]$, we set

$$\chi(7)[4][1]=\frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,lm}-C_{i+u}^{n,lm})(C_{i+u+1}^{n,gh}-C_{i+u}^{n,gh})$$

$$\chi(7)[4][2]=\frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}(C_{i+u}^{n,gh}-C_i^{n,gh})\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,lm}-C_{i+u}^{n,lm})$$

$$\chi'(7)[4][2]=\frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}(C_{i+u}^{n,gh}-C_i^{n,gh})\mathbb{E}(\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,lm}-C_{i+u}^{n,lm})|\mathcal{F}_{i+u}^n)$$

$$\chi(7)[4][3]=\frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}(C_{i+u}^{n,lm}-C_i^{n,lm})\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,gh}-C_{i+u}^{n,gh})$$

$$\chi(7)[4][4]=\frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}(C_{i+u}^{n,lm}-C_i^{n,lm})(C_{i+u}^{n,gh}-C_i^{n,gh})\alpha_{i+u}^{n,jk}$$

$$\chi(7)[4][5]=\frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}(C_{i+u}^{n,lm}-C_i^{n,lm})\alpha_{i+u}^{n,jk}(C_{i+v}^{n,gh}-C_{i+u+1}^{n,gh})$$

$$\chi'(7)[2][5]=\frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}(C_{i+u}^{n,lm}-C_i^{n,lm})\alpha_{i+u}^{n,jk}\mathbb{E}((C_{i+v}^{n,gh}-C_{i+u+1}^{n,gh}|\mathcal{F}_{i+u}^n)$$

$$\chi(7)[4][6]=\frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,lm}-C_{i+u}^{n,lm})(C_{i+v}^{n,gh}-C_{i+u+1}^{n,gh})$$

$$\chi(7)[4][7] = \frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}(C_{i+u}^{n,gh} - C_i^{n,gh})\alpha_{i+u}^{n,jk}(C_{i+v}^{n,lm} - C_{i+u+1}^{n,lm})$$
$$\chi(7)[4][8] = \frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}(C_{i+u+1}^{n,gh} - C_{i+u}^{n,gh})(C_{i+v}^{n,lm} - C_{i+u+1}^{n,lm})$$
$$\chi(7)[4][9] = \frac{\Delta_n^2}{(k_n\Delta_n)^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}\alpha_{i+u}^{n,jk}(C_{i+v}^{n,lm} - C_{i+u+1}^{n,lm})(C_{i+v}^{n,gh} - C_{i+u+1}^{n,gh}),$$

which satisfy,

$$\chi(7)[4] = \sum_{j=1}^{9}\chi(7)[4][j].$$

By using arguments similar to those used for $\chi(1)$, it can be shown that

$$|\mathbb{E}(\chi(7)[4][j]\big|\mathcal{F}_i^n)| \leq K\Delta_n^{1/4}(\Delta_n^{1/4} + \eta_{i,k_n}), \quad \text{for} \quad j = 1, \ldots, 8,$$

which yields

$$|\mathbb{E}(\chi(7)\big|\mathcal{F}_i^n)| \leq K\Delta_n^{1/4}(\Delta_n^{1/4} + \eta_{i,k_n}).$$

Next, define

$$\chi(8) = \frac{1}{k_n^3}\sum_{w=2}^{k_n-1}\sum_{v=0}^{w-1}\sum_{u=0}^{v-1}(C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+v}^{n,lm} - C_i^{n,lm})(C_{i+w}^{n,gh} - C_i^{n,gh}).$$

This term can be further decomposed into 6 components. Successive conditioning and existing bounds give

$$|\mathbb{E}\Big((C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+v}^{n,lm} - C_{i+u}^{n,lm})(C_{i+w}^{n,gh} - C_{i+v}^{n,gh})\big|\mathcal{F}_i^n\Big)| \leq K\Delta_n$$
$$|\mathbb{E}\Big((C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+v}^{n,lm} - C_{i+u}^{n,lm})(C_{i+v}^{n,gh} - C_{i+u}^{n,gh})\big|\mathcal{F}_i^n\Big)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,k_n})$$
$$|\mathbb{E}\Big((C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+v}^{n,lm} - C_{i+u}^{n,lm})(C_{i+u}^{n,gh} - C_i^{n,gh})\big|\mathcal{F}_i^n\Big)| \leq K\Delta_n$$
$$|\mathbb{E}\Big((C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+u}^{n,lm} - C_i^{n,lm})(C_{i+w}^{n,gh} - C_{i+v}^{n,gh})\big|\mathcal{F}_i^n\Big)| \leq K\Delta_n$$
$$|\mathbb{E}\Big((C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+u}^{n,lm} - C_i^{n,lm})(C_{i+v}^{n,gh} - C_{i+u}^{n,gh})\big|\mathcal{F}_i^n\Big)| \leq K\Delta_n$$
$$|\mathbb{E}\Big((C_{i+u}^{n,jk} - C_i^{n,jk})(C_{i+u}^{n,lm} - C_i^{n,lm})(C_{i+u}^{n,gh} - C_i^{n,gh})\big|\mathcal{F}_i^n\Big)| \leq K\Delta_n$$

These bounds can be used to deduce

$$|\mathbb{E}(\chi(8)\big|\mathcal{F}_i^n)| \leq K\Delta_n.$$

This completes the proof.

Proof of (44) and (45) in Lemma 8

Observe that

$$\beta_i^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})(C_{i+k_n}^{n,gh}-C_i^{n,gh})=\frac{1}{k_n\Delta_n}\sum_{u=0}^{k_n-1}\varsigma_{i,u}^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})(C_{i+k_n}^{n,gh}-C_i^{n,gh}),$$

$$\beta_i^{n,jk}\beta_i^{n,lm}(C_{i+k_n}^{n,gh}-C_i^{n,gh})=\frac{1}{k_n^2\Delta_n^2}\sum_{u=0}^{k_n-1}\varsigma_{i,u}^{n,jk}\varsigma_{i,u}^{n,lm}(C_{i+k_n}^{n,gh}-C_i^{n,gh})+\frac{1}{k_n^2\Delta_n^2}\sum_{u=0}^{k_n-2}\sum_{v=0}^{k_n-1}\varsigma_{i,u}^{n,jk}\varsigma_{i,v}^{n,lm}(C_{i+k_n}^{n,gh}-C_i^{n,gh})$$
$$+\frac{1}{k_n^2\Delta_n^2}\sum_{u=0}^{k_n-2}\sum_{v=0}^{k_n-1}\varsigma_{i,u}^{n,lm}\varsigma_{i,v}^{n,jk}(C_{i+k_n}^{n,gh}-C_i^{n,gh}).$$

Hence, (44) and (45) can be proved using the same strategy as for (43).

Proof of (46) and (47) in Lemma 8

Note that we have

$$\begin{aligned}&\gamma_i^{n,jk}\gamma_i^{n,lm}\beta_i^{n,gh}=\beta_i^{n,gh}\beta_{i+k_n}^{n,jk}\beta_{i+k_n}^{n,lm}+\beta_i^{n,gh}\beta_i^{n,jk}\beta_i^{n,lm}-\beta_i^{n,gh}\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}-\beta_i^{n,gh}\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}\\&+\beta_i^{n,gh}\beta_{i+k_n}^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})-\beta_i^{n,gh}\beta_i^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})+\beta_i^{n,gh}\beta_{i+k_n}^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})-\beta_i^{n,gh}\beta_i^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})\\&+\beta_i^{n,gh}(C_{i+k_n}^{n,jk}-C_i^{n,jk})(C_{i+k_n}^{n,lm}-C_i^{n,lm}),\end{aligned}$$

and

$$\begin{aligned}&\gamma_i^{n,gh}\gamma_i^{n,jk}\gamma_i^{n,lm}=\beta_{i+k_n}^{n,gh}\beta_{i+k_n}^{n,jk}\beta_{i+k_n}^{n,lm}+\beta_{i+k_n}^{n,gh}\beta_i^{n,jk}\beta_i^{n,lm}-\beta_{i+k_n}^{n,gh}\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}-\beta_{i+k_n}^{n,gh}\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}\\&+\beta_{i+k_n}^{n,gh}\beta_{i+k_n}^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})-\beta_{i+k_n}^{n,gh}\beta_i^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})+\beta_{i+k_n}^{n,gh}\beta_{i+k_n}^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})-\beta_{i+k_n}^{n,gh}\beta_i^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})\\&+\beta_{i+k_n}^{n,gh}(C_{i+k_n}^{n,jk}-C_i^{n,jk})(C_{i+k_n}^{n,lm}-C_i^{n,lm})-\beta_i^{n,gh}\beta_{i+k_n}^{n,jk}\beta_{i+k_n}^{n,lm}-\beta_i^{n,gh}\beta_i^{n,jk}\beta_i^{n,lm}+\beta_i^{n,gh}\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}+\beta_i^{n,gh}\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}\\&-\beta_i^{n,gh}\beta_{i+k_n}^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})+\beta_i^{n,gh}\beta_i^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})-\beta_i^{n,gh}\beta_{i+k_n}^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})+\beta_i^{n,gh}\beta_i^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})\\&-\beta_i^{n,gh}(C_{i+k_n}^{n,jk}-C_i^{n,jk})(C_{i+k_n}^{n,lm}-C_i^{n,lm})+\beta_{i+k_n}^{n,jk}\beta_{i+k_n}^{n,lm}(C_{i+k_n}^{n,gh}-C_i^{n,gh})+\beta_i^{n,jk}\beta_i^{n,lm}(C_{i+k_n}^{n,gh}-C_i^{n,gh})\\&-\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}(C_{i+k_n}^{n,gh}-C_i^{n,gh})-\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}(C_{i+k_n}^{n,gh}-C_i^{n,gh})+\beta_{i+k_n}^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})(C_{i+k_n}^{n,gh}-C_i^{n,gh})\\&-\beta_i^{n,jk}(C_{i+k_n}^{n,lm}-C_i^{n,lm})(C_{i+k_n}^{n,gh}-C_i^{n,gh})+\beta_{i+k_n}^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})(C_{i+k_n}^{n,gh}-C_i^{n,gh})\\&-\beta_i^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})(C_{i+k_n}^{n,gh}-C_i^{n,gh})+(C_{i+k_n}^{n,jk}-C_i^{n,jk})(C_{i+k_n}^{n,lm}-C_i^{n,lm})(C_{i+k_n}^{n,gh}-C_i^{n,gh}).\end{aligned}$$

From (39), notice that β_i^n is $\mathcal{F}_{i+k_n}^n$-measurable and satisfies $\|\mathbb{E}(\beta_i^n|\mathcal{F}_i^n)\|\le K\Delta_n^{1/2}$.
Using the law of iterated expectations and existing bounds, it can be shown that

$$\begin{aligned}&|\mathbb{E}(\beta_i^{n,lm}\beta_{i+k_n}^{n,jk}|\mathcal{F}_i^n)|\le K\Delta_n^{3/4}.\\&|\mathbb{E}(\beta_i^{n,lm}\beta_i^{n,gh}\beta_{i+k_n}^{n,jk}|\mathcal{F}_i^n)|\le K\Delta_n\\&|\mathbb{E}(\beta_i^{n,lm}(C_{i+k_n}^{n,gh}-C_i^{n,gh})\beta_{i+k_n}^{n,jk}|\mathcal{F}_i^n)|\le K\Delta_n\\&|\mathbb{E}(\beta_{i+k_n}^{n,lm}(C_{i+k_n}^{n,jk}-C_i^{n,jk})|\mathcal{F}_i^n)|\le K\Delta_n^{3/4}\\&|\mathbb{E}((C_{i+k_n}^{n,jk}-C_i^{n,jk})(C_{i+k_n}^{n,lm}-C_i^{n,lm})(C_{i+k_n}^{n,gh}-C_i^{n,gh})|\mathcal{F}_i^n)|\le K\Delta_n.\end{aligned}\tag{58}$$

By Lemma 3.3 in Jacod and Rosenbaum (2012), we have

$$|\mathbb{E}(\beta_{i+k_n}^{n,gh}\beta_{i+k_n}^{n,ab}|\mathcal{F}_{i+k_n}^n)-\frac{1}{k_n}(C_{i+k_n}^{n,ga}C_{i+k_n}^{n,hb}+C_{i+k_n}^{n,gb}C_{i+k_n}^{n,ha})-\frac{k_n\Delta_n}{3}\overline{C}_{i+k_n}^{n,gh,ab}|\le K\sqrt{\Delta_n}(\Delta_n^{1/8}+\eta_{i+k_n,k_n}^n).$$

Hence, for $\varphi_i^{n,gh} \in \{\beta_i^{n,gh}, C_{i+k_n}^{n,gh} - C_i^{n,gh}\}$, which satisfies $\mathbb{E}(|\varphi_i^{n,gh}|^q \big| \mathcal{F}_i^n) \leq K\Delta_n^{q/4}$ and $\mathbb{E}(\varphi_i^{n,gh} | \mathcal{F}_i^n) \leq K\Delta_n^{1/2}$, it can be proved that

$$|\mathbb{E}(\varphi_i^{n,gh}\beta_{i+k_n}^{n,jk}\beta_{i+k_n}^{n,lm}|\mathcal{F}_i^n) - \mathbb{E}\Big(\varphi_i^{n,gh}\Big[\frac{1}{k_n}(C_{i+k_n}^{n,jl}C_{i+k_n}^{n,km} + C_{i+k_n}^{n,jm}C_{i+k_n}^{n,kl}) - \frac{k_n\Delta_n}{3}\overline{C}_{i+k_n}^{n,jk,lm}\Big]|\mathcal{F}_i^n\Big)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,2k_n}^n).$$

Next, successive conditioning and existing bounds give

$$\begin{aligned}
&|\mathbb{E}(\varphi_i^{n,gh}\overline{C}_{i+k_n}^{n,jk,lm})| \leq K\Delta_n^{1/4}(\Delta_n^{1/4} + \eta_{i,k_n}^n) \\
&|\mathbb{E}(\varphi_i^{n,gh}C_{i+k_n}^{n,jl}C_{i+k_n}^{n,km})| \leq K\Delta_n^{1/2},
\end{aligned}$$

which implies

$$|\mathbb{E}(\varphi_i^{n,gh}\beta_{i+k_n}^{n,jk}\beta_{i+k_n}^{n,lm}|\mathcal{F}_i^n)| \leq K\Delta_n^{3/4}(\Delta_n^{1/4} + \eta_{i,2k_n}^n). \tag{59}$$

It is easy to see that (43), (58) and (59) and the inequality $\eta_{i,k_n}^n \leq \eta_{i,2k_n}^n$ together yield (46) and (47).

Step 3: Asymptotic Distribution of the approximate estimator

First, we decompose the approximate estimator as

$$[H(\widehat{C), G}(C)]_T^{(A)} = [H(\widehat{C), G}(C)]_T^{(A1)} - [H(\widehat{C), G}(C)]_T^{(A2)},$$

with

$$[H(\widehat{C), G}(C)]_T^{(A1)} = \frac{3}{2k_n}\sum_{g,h,a,b=1}^{d}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}(\partial_{gh}H\partial_{ab}G)(C_{i-1}^n)(\widehat{C}_{i+k_n}^{\prime n,gh} - \widehat{C}_i^{\prime n,gh})(\widehat{C}_{i+k_n}^{\prime n,ab} - \widehat{C}_i^{\prime n,ab}),$$

and

$$[H(\widehat{C), G}(C)]_T^{(A2)} = \frac{3}{k_n^2}\sum_{g,h,a,b=1}^{d}\sum_{i=1}^{[T/\Delta_n]-2k_n+1}(\partial_{gh}H\partial_{ab}G)(\widehat{C}_i^{\prime n})(\widehat{C}_i^{\prime n,ga}\widehat{C}_i^{\prime n,hb} + \widehat{C}_i^{\prime n,gb}\widehat{C}_i^{\prime n,ha}).$$

In this section, we use the notation $C_{i-1}^n = C_{(i-1)\Delta_n}$ and $\mathcal{F}_i = \mathcal{F}_{(i-1)\Delta_n}$ to simplify the exposition. Given the polynomial growth assumption satisfied by H and G and the fact that $k_n = \theta(\Delta_n)^{-1/2}$, by Theorem 2.2 in Jacod and Rosenbaum (2012) we have

$$\frac{1}{\sqrt{\Delta_n}}\left([H(\widehat{C), G}(C)]_T^{(A2)} - \frac{3}{\theta^2}\sum_{g,h,a,b=1}^{d}\int_0^T(\partial_{gh}H\partial_{ab}G)(C_t)(c_t^{ga}c_t^{hb} + c_i^{gb}c_t^{ha})dt\right) = O_p(1),$$

which yields

$$\frac{1}{\Delta_n^{1/4}}\left([H(\widehat{C), G}(C)]_T^{(A2)} - \frac{3}{\theta^2}\sum_{g,h,a,b=1}^{d}\int_0^T(\partial_{gh}H\partial_{ab}G)(C_t)(c_t^{ga}c_t^{hb} + c_i^{gb}c_t^{ha})dt\right) \xrightarrow{\mathbb{P}} 0.$$

To study the asymptotic behavior of $[H(\widehat{C), G}(C)]_T^{(A1)}$, we follow Aït-Sahalia and Jacod (2014) and define the following multidimensional quantities

$$\zeta(1)_i^n = \frac{1}{\Delta_n}\Delta_i^nY'(\Delta_i^nY')^\top - C_{i-1}^n, \qquad \zeta(2)_i^n = \Delta_i^nc,$$

$$\zeta'(u)_i^n = \mathbb{E}(\zeta(u)_i^n | \mathcal{F}_{i-1}^n), \qquad \zeta''(u)_i^n = \zeta(u)_i^n - \zeta'(u)_i^n,$$

with

$$\zeta^r(u)_i^n = \left(\zeta^r(u)_i^{n,gh}\right)_{1\le g,h\le d}.$$

We also define, for $m \in \{0,\dots,2k_n-1\}$ and $j,l\in\mathbb{Z}$,

$$\varepsilon(1)_m^n = \begin{cases} -1 & if\ 0\le m<k_n \\ +1 & if\ k_n\le m<2k_n, \end{cases}$$

$$\varepsilon(2)_m^n = \sum_{q=m+1}^{2k_n-1} \varepsilon(1)_q^n = (m+1)\wedge(2k_n-m-1),$$

$$z_{u,v}^n = \begin{cases} 1/\Delta_n & if\ u=v=1 \\ 1 & \text{otherwise}, \end{cases}$$

$$\gamma(u,v;m)_{j,l}^n = \frac{3}{2k_n^3} \sum_{q=0\vee(j-m)}^{(l-m-1)\vee(2k_n-m-1)} \varepsilon(u)_q^n \varepsilon(u)_{q+m}^n, \qquad \Gamma(u,v)_m^n = \gamma(u,v;m)_{0,2k_n}^n,$$
$$M(u,v;u',v')_n = z_{u,v}^n z_{u',v'}^n \sum_{m=1}^{2k_n-1} \Gamma(u,v)_m^n \Gamma(u',v')_m^n.$$

The following decompositions hold,

$$\widehat{C}_i'^n = C_{i-1}^n + \frac{1}{k_n}\sum_{j=0}^{k_n-1}\sum_{u=1}^{2} \overline{\varepsilon}(u)_j^n \zeta(u)_{i+j}^n, \qquad \widehat{C}_{i+k_n}'^n - \widehat{C}_i'^n = \frac{1}{k_n}\sum_{j=0}^{2k_n-1}\sum_{u=1}^{2} \varepsilon(u)_j^n \zeta(u)_{i+j}^n,$$
$$\gamma_i^{n,gh}\gamma_i^{n,ab} = \frac{1}{k_n^2}\sum_{u=1}^{2}\sum_{v=1}^{2}\left(\sum_{j=0}^{2k_n-1} \varepsilon(u)_j^n\varepsilon(v)_j^n \zeta(u)_{i+j}^{n,gh}\zeta(v)_{i+j}^{n,ab} + \sum_{j=0}^{2k_n-2}\sum_{q=j+1}^{2k_n-1} \varepsilon(u)_j^n\varepsilon(v)_q^n \zeta(u)_{i+j}^{n,gh}\zeta(v)_{i+q}^{n,ab} \right.$$
$$\left. + \sum_{j=1}^{2k_n-1}\sum_{q=0}^{j-1} \varepsilon(u)_j^n\varepsilon(v)_q^n \zeta(u)_{i+j}^{n,gh}\zeta(v)_{i+q}^{n,ab}\right).$$

A change of the order of the summation in the last term gives

$$\gamma_i^{n,gh}\gamma_i^{n,ab} = \frac{1}{k_n^2}\sum_{u=1}^{2}\sum_{v=1}^{2}\left(\sum_{j=0}^{2k_n-1} \varepsilon(u)_j^n\varepsilon(v)_j^n \zeta(u)_{i+j}^{n,gh}\zeta(v)_{i+j}^{n,ab} + \sum_{j=0}^{2k_n-2}\sum_{q=j+1}^{2k_n-1} \varepsilon(u)_j^n\varepsilon(v)_q^n \zeta(u)_{i+j}^{n,gh}\zeta(v)_{i+q}^{n,ab} \right.$$
$$\left. + \sum_{j=0}^{2k_n-2}\sum_{q=j+1}^{2k_n-1} \varepsilon(v)_j^n\varepsilon(u)_q^n \zeta(v)_{i+j}^{n,ab}\zeta(u)_{i+q}^{n,gh}\right).$$

Therefore, we can further rewrite $\widehat{[H(C),G(C)]}_T^{(A1)}$ as

$$\widehat{[H(C),G(C)]}_T^{(A1)} = \widehat{[H(C),G(C)]}_T^{(A11)} + \widehat{[H(C),G(C)]}_T^{(A12)} + \widehat{[H(C),G(C)]}_T^{(A13)}, \text{with}$$

$$[\widehat{H(C),G(C)}]_T^{(A1w)} = \sum_{g,h,a,b=1}^{d} \sum_{u,v=1}^{2} \widehat{A1w}(H,gh,u;G,ab,v)_T^n, \quad w=1,2,3,$$

and,

$$\widehat{A11}(H,gh,u;G,ab,v)_T^n = \frac{3}{2k_n^3} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \sum_{j=0}^{2k_n-1} (\partial_{gh}H\partial_{ab}G)(C_{i-1}^n)\varepsilon(u)_j^n \varepsilon(v)_j^n \zeta(u)_{i+j}^{n,gh}\zeta(v)_{i+j}^{n,ab},$$

$$\widehat{A12}(H,gh,u;G,ab,v)_T^n = \frac{3}{2k_n^3} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \sum_{j=0}^{2k_n-2} \sum_{q=j+1}^{2k_n-1} (\partial_{gh}H\partial_{ab}G)(C_{i-1}^n)\varepsilon(u)_j^n \varepsilon(v)_q^n \zeta(u)_{i+j}^{n,gh}\zeta(v)_{i+q}^{n,ab},$$

$$\widehat{A13}(H,gh,u;G,ab,v)_T^n = \frac{3}{2k_n^3} \sum_{i=1}^{[T/\Delta_n]-2k_n+1} \sum_{j=0}^{2k_n-2} \sum_{q=j+1}^{2k_n-1} (\partial_{gh}H\partial_{ab}G)(C_{i-1}^n)\varepsilon(v)_j^n \varepsilon(u)_q^n \zeta(v)_{i+j}^{n,ab}\zeta(u)_{i+q}^{n,gh},$$

where we clearly have $\widehat{A13}(H,gh,u;G,ab,v)_T^n = \widehat{A12}(G,ab,v;H,gh,u)_T^n$. By a change of the order of the summation,

$$\widehat{A11}(H,gh,u;G,ab,v)_T^n = \frac{3}{2k_n^3} \sum_{i=1}^{[T/\Delta_n]} \sum_{j=0\vee(i+2k_n-1-[T/\Delta_n])}^{(2k_n-1)\wedge(i-1)} (\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n \varepsilon(v)_j^n \zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab},$$

$$\widehat{A12}(H,gh,u;G,ab,v)_T^n = \frac{3}{2k_n^3} \sum_{i=2}^{[T/\Delta_n]} \sum_{m=1}^{(i-1)\wedge(2k_n-1)} \sum_{j=0\vee(i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1)\wedge(i-m-1)} (\partial_{gh}H\partial_{ab}G)(C_{i-1-j-m}^n)\times$$
$$\varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \zeta_{gh}(u)_{i-m}^n \zeta_{ab}(v)_i^n.$$

Set

$$\widetilde{A11}(H,gh,u;G,ab,v)_T^n = \frac{3}{2k_n^3} \sum_{i=2k_n}^{[T/\Delta_n]} \sum_{j=0}^{2k_n-1} (\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n \varepsilon(v)_j^n \zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab},$$

$$\widetilde{A12}(H,gh,u;G,ab,v)_T^n = \frac{3}{2k_n^3} \sum_{i=2k_n}^{[T/\Delta_n]} \sum_{m=1}^{(i-1)\wedge(2k_n-1)} \sum_{j=0}^{(2k_n-m-1)} (\partial_{gh}H\partial_{ab}G)(C_{i-j-1-m}^n)\varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \zeta_{gh}(u)_{i-m}^n \zeta_{ab}(v)_i^n,$$

and

$$\begin{aligned}\overline{A11}(H,gh,u;G,ab,v)_T^n &= \frac{3}{2k_n^3} \sum_{i=2k_n}^{[T/\Delta_n]} \Big(\sum_{j=0}^{2k_n-1} \varepsilon(u)_j^n \varepsilon(v)_j^n \Big) (\partial_{gh}H\partial_{ab}G)(C_{i-2k_n}^n) \zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab} \\ &= \Gamma(u,v)_0^n \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh}H\partial_{ab}G)(C_{i-2k_n}^n) \zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab},\end{aligned}$$

$$\begin{aligned}\overline{A12}(H,gh,u;G,ab,v)_T^n &= \frac{3}{2k_n^3} \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh}H\partial_{ab}G)(C_{i-2k_n}^n) \sum_{m=1}^{(i-1)\wedge(2k_n-1)} \sum_{j=0}^{(2k_n-m-1)} \varepsilon(u)_j^n \varepsilon(v)_{j+m}^n \zeta_{gh}(u)_{i-m}^n \zeta_{ab}(v)_i^n \\ &= \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh}H\partial_{ab}G)(C_{i-2k_n}^n) \rho_{gh}(u,v)_i^n \zeta_{ab}(v)_i^n,\end{aligned}$$

with

$$\rho_{gh}(u,v)_i^n = \sum_{m=1}^{2k_n-1} \Gamma(u,v)_m^n \zeta_{gh}(u)_{i-m}^n.$$

The following results hold:

$$\frac{1}{\Delta_n^{1/4}}\left(\widehat{A1w}(H,gh,u;G,ab,v)_T^n - \widetilde{A1w}(H,gh,u;G,ab,v)_T^n\right) \xrightarrow{\mathbb{P}} 0 \quad \text{for all} \quad (H,gh,u,G,ab,v) \text{ and } w=1,2. \tag{60}$$

$$\frac{1}{\Delta_n^{1/4}}\left(\widetilde{A1w}(H,gh,u;G,ab,v)_T^n - \overline{A1w}(H,gh,u;G,ab,v)_T^n\right) \xrightarrow{\mathbb{P}} 0 \quad \text{for all} \quad (H,gh,u,G,ab,v) \text{ and } w=1,2. \tag{61}$$

Proof of (60) for $w=1$

The proof is similar to Step 5 on page 548 of Aït-Sahalia and Jacod (2014). Our proof deviates from the latter reference by the fact that, in all the sums, the terms $\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}$ are scaled by random variables rather that constant real numbers. First, observe that we can write

$$\begin{aligned}
&\widehat{A11} - \widetilde{A11} = \widetilde{\widehat{A11}}(1) + \widetilde{\widehat{A11}}(2) + \widetilde{\widehat{A11}}(3) \quad \text{with} \\
&\widetilde{\widehat{A11}}(1) = \sum_{i=1}^{(2k_n-1)\wedge[T/\Delta_n]} \left(\frac{3}{2k_n^3}\sum_{j=0\vee(i+2k_n-1-[T/\Delta_n])}^{(2k_n-1)\wedge(i-1)} (\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n\right)\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}, \\
&\widetilde{\widehat{A11}}(2) = \sum_{i=[T/\Delta_n]-2k_n+2}^{[T/\Delta_n]} \frac{3}{2k_n^3}\left(\sum_{j=0\vee(i+2k_n-1-[T/\Delta_n])}^{(2k_n-1)\wedge(i-1)} (\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n\right. \\
&\qquad \left. - \sum_{j=0}^{(2k_n-1)} (\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n\right)\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}, \\
&\widetilde{\widehat{A11}}(3) = \sum_{i=2k_n}^{[T/\Delta_n]-2k_n+1} \frac{3}{2k_n^3}\left(\sum_{j=0\vee(i+2k_n-1-[T/\Delta_n])}^{(2k_n-1)\wedge(i-1)} (\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n\right. \\
&\qquad \left. - \sum_{j=0}^{(2k_n-1)} (\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n\right)\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}.
\end{aligned}$$

It is easy to see that $\widetilde{\widehat{A12}}(3)=0$. Using (40) with $Z=c$ and (41), it can be shown that

$$\mathbb{E}(\|\zeta(1)_i^n\|^q|\mathcal{F}_{i-1}^n) \leq K_q, \quad \mathbb{E}(\|\zeta(2)_i^n\|^q|\mathcal{F}_{i-1}^n) \leq K_q\Delta_n^{q/2}. \tag{62}$$

The polynomial growth assumption on H and G and the boundedness of C_t imply that $|(\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)| \leq K$. Hence, the random quantities $\left(\frac{3}{2k_n^3}\sum_{j=0\vee(i+2k_n-1-[T/\Delta_n])}^{(2k_n-1)\wedge(i-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n\right)$ and $\frac{3}{2k_n^3}\sum_{j=0}^{(2k_n-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n$ are $\mathcal{F}_{i-1}^n-$ measurable and are bounded by $\widetilde{\gamma}_{u,v}^n$ defined as

$$\widetilde{\gamma}_{u,v}^n = \begin{cases} K & \text{if } (u,v)=(2,2) \\ K/k_n & \text{if } (u,v)=(1,2),(2,1) \\ K/k_n^2 & \text{if } (u,v)=(1,1). \end{cases}$$

Similarly, the quantity,

$$\frac{3}{2k_n^3}\Bigg(\sum_{j=0\vee(i+2k_n-1-[T/\Delta_n])}^{(2k_n-1)\wedge(i-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n-\sum_{j=0}^{(2k_n-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-j-1}^n)\varepsilon(u)_j^n\varepsilon(v)_j^n\Bigg),$$

is $\mathcal{F}_{i-1}^n-$ measurable and bounded by $2\widetilde{\gamma}_{u,v}^n$. Note also that, by (62) and the Cauchy Schwartz inequality, we have,

$$\mathbb{E}(|\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}||\mathcal{F}_{i-1}^n)\leq\mathbb{E}(\|\zeta(u)_i^n\|^2|\mathcal{F}_{i-1}^n)^{1/2}\mathbb{E}(\|\zeta(v)_i^n\|^2|\mathcal{F}_{i-1}^n)^{1/2}\leq\begin{cases}K\Delta_n & \text{if }(u,v)=(2,2)\\ K\Delta_n^{1/2} & \text{if }(u,v)=(1,2),(2,1)\\ K & \text{if }(u,v)=(1,1).\end{cases}$$

The above bounds, together with the fact that $k_n=\theta\Delta_n^{-1/2}$, give $\mathbb{E}(|\widetilde{\widetilde{A11}}(1)|)\leq K\Delta_n^{1/2}$ and $\mathbb{E}(|\widetilde{\widetilde{A11}}(2)|)\leq K\Delta_n^{1/2}$ for all (u,v). These two results together imply $\widetilde{\widetilde{A11}}(1)=o(\Delta_n^{-1/4})$ and $\widetilde{\widetilde{A11}}(2)=o(\Delta_n^{-1/4})$, which yields the result.

Proof of (60) for $w=2$

We proceed similarly to Step 6 on page 548 of Aït-Sahalia and Jacod (2014). First, observe that we have

$$\widehat{A12}-\widetilde{A12}=\widetilde{\widetilde{A12}}(1)+\widetilde{\widetilde{A12}}(2)\quad\text{with}$$

$$\widetilde{\widetilde{A12}}(1)=\sum_{i=2}^{(2k_n-1)\wedge[T/\Delta_n]}\Bigg(\sum_{m=1}^{(i-1)}\frac{3}{2k_n^3}\Big(\sum_{j=0\vee(i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1)\wedge(i-m-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-1-j-m}^n)\varepsilon(u)_j^n\varepsilon(v)_{j+m}^n\Big)\zeta_{gh}(u)_{i-m}^n\Bigg)\zeta_{ab}(v)_i^n,$$

$$\widetilde{\widetilde{A12}}(2)=\sum_{i=[T/\Delta_n]-2k_n+2}^{[T/\Delta_n]}\Bigg(\sum_{m=1}^{(i-1)\wedge(2k_n-1)}\Big(\frac{3}{2k_n^3}\sum_{j=0\vee(i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1)\wedge(i-m-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-1-j-m}^n)\varepsilon(u)_j^n\varepsilon(v)_{j+m}^n\Big)$$
$$-\sum_{j=0}^{(2k_n-m-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-1-j-m}^n)\varepsilon(u)_j^n\varepsilon(v)_{j+m}^n\Big)\zeta_{gh}(u)_{i-m}^n\Bigg)\zeta_{ab}(v)_i^n.$$

It is easy to see that the quantity

$$\kappa_i^{m,n}=\frac{3}{2k_n^3}\Big(\sum_{j=0\vee(i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1)\wedge(i-m-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-1-j-m}^n)\varepsilon(u)_j^n\varepsilon(v)_{j+m}^n\Big)$$

is $\mathcal{F}_{i-m-1}^n$ measurable and bounded by $\widetilde{\gamma}_{u,v}^n$. Let

$$\kappa_i^n=\sum_{m=1}^{(i-1)}\frac{3}{2k_n^3}\Big(\sum_{j=0\vee(i+2k_n-1-m-[T/\Delta_n])}^{(2k_n-m-1)\wedge(i-m-1)}(\partial_{gh}H\partial_{ab}G)(C_{i-1-j-m}^n)\varepsilon(u)_j^n\varepsilon(v)_{j+m}^n\Big)\zeta_{gh}(u)_{i-m}^n.$$

It follows that κ_i^n is $\mathcal{F}_{i-1}^n$-measurable. We have

$$\mathbb{E}(|\kappa_i^{m,n}|^z|\mathcal{F}_0)\leq(\widetilde{\gamma}_{u,v}^n)^z$$

$$|\mathbb{E}(\zeta(u)^n_{i-m}|\mathcal{F}_{i-m-1})| \leq \begin{cases} K\sqrt{\Delta_n} & \text{if } u=1 \\ K\Delta_n & \text{if } u=2 \end{cases}, \qquad \mathbb{E}(\|\zeta(u)^n_{i-m}\|^z|\mathcal{F}_{i-m-1}) \leq \begin{cases} K_z & \text{if } u=1 \\ K_z\Delta_n^{z/2} & \text{if } u=2 \end{cases}$$

Using Lemma 5, we deduce that for $z \geq 2$,

$$\mathbb{E}(|\kappa_i^n|^z) \leq \begin{cases} K_z(\widetilde{\gamma}^n_{u,v})^z k_n^{z/2} & \text{if } u=1 \\ K_z(\widetilde{\gamma}^n_{u,v})^z/k_n^{z/2} & \text{if } u=2 \end{cases} \leq \begin{cases} K_z/k_n^{-3z/2} & \text{if } v=1 \\ K_z k_n^{-z/2} & \text{if } v=2 \end{cases}$$

Using the above result, and similarly to step 6 on page 548 of Aït-Sahalia and Jacod (2014), we obtain that $\frac{1}{\Delta_n^{1/4}}\widetilde{A12}(1) \overset{\mathbb{P}}{\Rightarrow} 0$. A similar argument yields $\frac{1}{\Delta_n^{1/4}}\widetilde{A12}(2) \overset{\mathbb{P}}{\Rightarrow} 0$, which completes the proof of (60) for $w=2$.

Proof of (61) for $w = 1$

Define

$$\Theta(u,v)_0^{(C),i,n} = \frac{3}{2k_n^3}\sum_{j=0}^{2k_n-1}\Big((\partial_{gh}H\partial_{ab}G)(C^n_{i-j-1}) - (\partial_{gh}H\partial_{ab}G)(C^n_{i-2k_n})\Big)\varepsilon(u)^n_j\varepsilon(v)^n_j.$$

By Taylor expansion, the polynomial growth assumption on H and G and using (40) with $Z=c$, we have

$$\Big|\mathbb{E}\Big((\partial_{gh}H\partial_{ab}G)(C^n_{i-j-1}) - (\partial_{gh}H\partial_{ab}G)(C^n_{i-2k_n})|\mathcal{F}^n_{i-2k_n}\Big)\Big| \leq K(k_n\Delta_n) \leq K\sqrt{\Delta_n} \quad \text{for} \quad j=0,\dots,2k_n-1$$
$$\mathbb{E}(|(\partial_{gh}H\partial_{ab}G)(C^n_{i-j-1}) - (\partial_{gh}H\partial_{ab}G)(C^n_{i-2k_n})|^q|\mathcal{F}^n_{i-2k_n})| \leq K(k_n\Delta_n)^{q/2} \leq K\Delta_n^{q/4} \quad \text{for} \quad q \geq 2$$

Next, observe that $\Theta(u,v)_0^{(C),i,n}$ is $\mathcal{F}^n_{i-1}$ -measurable and satisfies $|\Theta(u,v)_0^{(C),i,n}| \leq \widetilde{\gamma}^n_{u,v}$, $|\mathbb{E}\Big(\Theta(u,v)_0^{(C),i,n}|\mathcal{F}^n_{i-2k_n}\Big)| \leq K\Delta_n^{1/2}\widetilde{\gamma}^n_{u,v}$ and $\mathbb{E}\Big(|\Theta(u,v)_0^{(C),i,n}|^q|\mathcal{F}^n_{i-2k_n}\Big) \leq K_q\Delta_n^{q/4}(\widetilde{\gamma}^n_{u,v})^q$ where the latter follows from the Hölder inequality. We aim to prove that

$$\widehat{E} = \frac{1}{\Delta_n^{1/4}}\left[\sum_{i=2k_n}^{[T/\Delta_n]}\Theta(u,v)_0^{(C),i,n}\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}\right]$$

converges to zero in probability for any H, G, g, h, a, and b with $u,v=1,2$.
To show this result, we first introduce the following quantities:

$$\widehat{E}(1) = \frac{1}{\Delta_n^{1/4}}\left[\sum_{i=2k_n}^{[T/\Delta_n]}\Theta(u,v)_0^{(C),i,n}\mathbb{E}(\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}|\mathcal{F}^n_{i-1})\right]$$
$$\widehat{E}(2) = \frac{1}{\Delta_n^{1/4}}\left[\sum_{i=2k_n}^{[T/\Delta_n]}\Theta(u,v)_0^{(C),i,n}\big(\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab} - \mathbb{E}(\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}|\mathcal{F}^n_{i-1})\big)\right],$$

with $\widehat{E} = \widehat{E}(1) + \widehat{E}(2)$. By Cauchy-Schwartz inequality, we have

$$\mathbb{E}(|\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}|^q) \leq (\widehat{\gamma}^n_{u,v})^{q/2}, \text{where } \ \widehat{\gamma}^n_{u,v} = \begin{cases} K & \text{if } (u,v)=(1,1) \\ K\Delta_n & \text{if } (u,v)=(1,2),(2,1) \\ K\Delta_n^2 & \text{if } (u,v)=(2,2) \end{cases}$$

Since $\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}$ is $\mathcal{F}_i^n$-measurable, the martingale property of $\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab} - \mathbb{E}(\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}|\mathcal{F}_{i-1}^n)$ implies, for all (u,v),

$$\mathbb{E}(|\widehat{E}(2)|^2) \le K\Delta_n^{-3/2}(\Delta_n^{1/4}\widetilde{\gamma}_{u,v}^n)^2\widehat{\gamma}_{u,v}^n \le K\Delta_n.$$

The latter inequality implies $\widehat{E}(2) \overset{\mathbb{P}}{\Rightarrow} 0$ for all (u,v). It remains to show that $\widehat{E}(1) \overset{\mathbb{P}}{\Rightarrow} 0$. We remind some bounds under Assumption 2, see (B.83) in Aït-Sahalia and Jacod (2014),

$$|\mathbb{E}(\zeta(1)_i^{n,gh}\zeta(2)_i^{n,ab}|\mathcal{F}_{i-1}^n)| \le K\Delta_n, \tag{63}$$
$$|\mathbb{E}(\zeta(1)_i^{n,gh}\zeta(1)_i^{n,ab}|\mathcal{F}_{i-1}^n) - (C_{i-1}^{n,ga}C_{i-1}^{n,hb} + C_{i-1}^{n,gb}C_{i-1}^{n,ha})| \le K\Delta_n^{1/2}, \tag{64}$$
$$|\mathbb{E}(\zeta(2)_i^{n,gh}\zeta(2)_i^{n,ab}|\mathcal{F}_{i-1}^n - \overline{C}_{i-1}^{n,gh,ab}\Delta_n)| \le K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_i^n). \tag{65}$$

Case $(u,v) \in \{(1,2),(2,1)\}$. By (63) we have

$$\mathbb{E}(|\widehat{E}(1)|) \le K\frac{T}{\Delta_n}\frac{1}{\Delta_n^{1/4}}(\Delta_n^{1/4}\widetilde{\gamma}_{u,v}^n\Delta_n) \le K\Delta_n^{1/2} \quad \text{so} \quad \widehat{E}(1) \overset{\mathbb{P}}{\Rightarrow} 0.$$

Case $(u,v) \in \{(1,1),(2,2)\}$. Set

$$\widehat{E}'(1) = \frac{1}{\Delta_n^{1/4}}\left[\sum_{i=2k_n}^{[T/\Delta_n]}\Theta(u,v)_0^{(C),i,n}V_{i-2k_n}^n\right]$$
$$\widehat{E}''(1) = \frac{1}{\Delta_n^{1/4}}\left[\sum_{i=2k_n}^{[T/\Delta_n]}\Theta(u,v)_0^{(C),i,n}(V_{i-1}^n - V_{i-2k_n}^n)\right]$$
$$\widehat{E}'''(1) = \frac{1}{\Delta_n^{1/4}}\left[\sum_{i=2k_n}^{[T/\Delta_n]}\Theta(u,v)_0^{(C),i,n}\Big(\mathbb{E}(\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}|\mathcal{F}_{i-1}^n) - V_{i-1}^n\Big)\right]$$

where

$$V_{i-1}^n = \begin{cases} C_{i-1}^{n,ga}C_{i-1}^{n,hb} + C_{i-1}^{n,gb}C_{i-1}^{n,ha} & \text{if } (u,v) = (2,2) \\ \overline{C}_{i-1}^{n,gh,ab}\Delta_n & \text{if } (u,v) = (1,1) \\ 0 & \text{otherwise} \end{cases}$$

Note that we have $\widehat{E}(1) = \widehat{E}'(1) + \widehat{E}''(1) + \widehat{E}'''(1)$. Using (64) and (65), it can be shown that

$$\mathbb{E}(|\widehat{E}'''(1)|) \le \begin{cases} K\frac{1}{\Delta_n^{5/4}}(\Delta_n^{1/4}\widetilde{\gamma}_{u,v}^n)\Delta_n^{1/2} & \text{if } (u,v) = (1,1) \\ K\frac{1}{\Delta_n^{5/4}}(\Delta_n^{1/4}\widetilde{\gamma}_{u,v}^n)\Delta_n^{3/2} & \text{if } (u,v) = (2,2) \end{cases} \le K\Delta_n^{1/2} \quad \text{in all cases.}$$

Next, we prove $\widehat{E}'(1) \overset{\mathbb{P}}{\Rightarrow} 0$. To this end, write

$$\widehat{E}'(1) = \frac{1}{\Delta_n^{1/4}}\left[\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\Theta(u,v)_0^{(C),i-1+2k_n,n}V_{(i-1)\Delta_n}\right].$$

The fact that the summand in the last sum is $\mathcal{F}_{i+2k_n-2}^n$-measurable and lemma B.8 in Aït-Sahalia and Jacod (2014) imply that it is sufficient to show

$$\frac{1}{\Delta_n^{1/4}}\left[\sum_{i=1}^{[T/\Delta_n]-2k_n+1}|\mathbb{E}(\Theta(u,v)_0^{(C),i-1+2k_n,n}V_{(i-1)\Delta_n}|\mathcal{F}_{i-1}^n)|\right] \overset{\mathbb{P}}{\Rightarrow} 0 \quad \text{and}$$

$$\frac{2k_n-2}{\Delta_n^{1/2}}\left[\sum_{i=1}^{[T/\Delta_n]-2k_n+1}\mathbb{E}\Big(|\Theta(u,v)_0^{(C),i-1+2k_n,n}V_{(i-1)\Delta_n})|^2\Big)\right]\Rightarrow 0.$$

The first result readily follows from the inequality

$$|\mathbb{E}(\Theta(u,v)_0^{(C),i-1+2k_n,n}V_{(i-1)\Delta_n}|\mathcal{F}_{i-1}^n)|\leq\begin{cases}K\Delta_n^{1/2}\widetilde{\gamma}_{u,v}^n & \text{if } (u,v)=(1,1)\\ K\Delta_n^{1/2}\widetilde{\gamma}_{u,v}^n\Delta_n & \text{if } (u,v)=(2,2)\end{cases}\leq K\Delta_n^{3/2}\quad\text{in all cases}$$

while the second is a direct consequence of

$$\mathbb{E}(|\Theta(u,v)_0^{(C),i-1+2k_n,n}V_{(i-1)\Delta_n}|^2)\leq\begin{cases}K\Delta_n^{1/2}(\widetilde{\gamma}_{u,v}^n)^2 & \text{if } (u,v)=(1,1)\\ K\Delta_n^{1/2}(\widetilde{\gamma}_{u,v}^n)^2\Delta_n^2 & \text{if } (u,v)=(2,2)\end{cases}\leq K\Delta_n^{5/2}\quad\text{in all cases.}$$

Finally, to prove that $\widehat{E}''(1)\stackrel{\mathbb{P}}{\Longrightarrow}0$, we use the fact that

$$\begin{aligned}\mathbb{E}(|\Theta(u,v)_0^{(C),i,n}(V_{(i-1)\Delta_n}-V_{(i-2k_n)\Delta_n})|)&\leq\mathbb{E}(|\Theta(u,v)_0^{(C),i,n}|^2)^{1/2}\mathbb{E}(|V_{(i-1)\Delta_n}-V_{(i-2k_n)\Delta_n}|^2)^{1/2}\\&\leq\begin{cases}K\Delta_n^{1/2}\widetilde{\gamma}_{u,v}^n & \text{if } (u,v)=(1,1)\\ K\Delta_n^{1/4}\widetilde{\gamma}_{u,v}^n\Delta_n\Delta_n^{1/4} & \text{if } (u,v)=(2,2)\end{cases},\end{aligned}$$

which follows by the Cauchy-Schwartz inequality and earlier bounds. In particular, successive conditioning together with Assumption 2 imply that for $(u,v)=(1,1)$ and $(2,2)$, $\mathbb{E}(|V_{(i-1)\Delta_n}-V_{(i-2k_n)\Delta_n}|^2)\leq\Delta_n^{1/2}$.

Proof of (61) for $w=2$

Our aim here is to show that

$$\widehat{E}(2)=\frac{1}{\Delta_n^{1/4}}\sum_{i=2k_n}^{[T/\Delta_n]}\left(\sum_{m=1}^{2k_n-1}\left(\frac{3}{2k_n^3}\sum_{j=0}^{2k_n-m-1}\left[(\partial_{gh}H\partial_{ab}G)(c_{i-j-m-1}^n)-(\partial_{gh}H\partial_{ab}G)(c_{i-2k_n}^n)\right]\varepsilon(u)_j^n\varepsilon(v)_{j+m}^n\right)\times\right.$$
$$\left.\zeta(u)_{i-m}^{n,gh}\right)\zeta(v)_i^{n,ab}\stackrel{\mathbb{P}}{\Longrightarrow}0.$$

For this purpose, we introduce some new notation. For any $0\leq m\leq 2k_n-1$, set

$$\Theta(u,v)_m^{(C),i,n}=\frac{3}{2k_n^3}\sum_{j=0}^{2k_n-m-1}\left[(\partial_{gh}H\partial_{ab}G)(c_{i-j-m-1}^n)-(\partial_{gh}H\partial_{ab}G)(c_{i-2k_n}^n)\right]\varepsilon(u)_j^n\varepsilon(v)_{j+m}^n$$
$$\rho(u,v)^{(C),i,n,gh}=\sum_{m=1}^{2k_n-1}\Theta(u,v)_m^{(C),i,n}\zeta(u)_{i-m}^{n,gh}.$$

It is easy to see that $\Theta(u,v)_m^{(C),i,n}$ is $\mathcal{F}_{i-m-1}^n$ measurable and satisfies, by Hölder inequality,

$$|\Theta(u,v)_m^{(C),i,n}|\leq\widetilde{\gamma}_{u,v}^n\quad\text{and}\quad\mathbb{E}\Big(|\Theta(u,v)_m^{(C),i,n}|^q\big|\mathcal{F}_{i-2k_n}^n\Big)\leq K_q\Delta_n^{q/4}(\widetilde{\gamma}_{u,v}^n)^q.$$

Lemma 5 implies that for $q \geq 2$,

$$\mathbb{E}(|\rho(u,v)^{(C),i,n,gh}|^q) \leq \begin{cases} K_q(\Delta_n^{1/4}\widetilde{\gamma}_{u,v}^n)^q k_n^{q/2} & \text{if } u=1 \\ K_q(\Delta_n^{1/4}\widetilde{\gamma}_{u,v}^n)^q / k_n^{q/2} & \text{if } u=2 \end{cases} \leq \begin{cases} K_q/k_n^{2q} & \text{if } v=1 \\ K_q k_n^q & \text{if } v=2 \end{cases}. \tag{66}$$

Set

$$\widehat{E}'(2) = \frac{1}{\Delta_n^{1/4}} \sum_{i=2k_n}^{[T/\Delta_n]} \rho(u,v)^{(C),i,n,gh}\mathbb{E}(\zeta(v)_i^{n,ab}|\mathcal{F}_{i-1}^n),$$

$$\widehat{E}''(2) = \frac{1}{\Delta_n^{1/4}} \sum_{i=2k_n}^{[T/\Delta_n]} \rho(u,v)^{(C),i,n,gh}(\zeta(v)_i^{n,ab} - \mathbb{E}(\zeta(v)_i^{n,ab}|\mathcal{F}_{i-1}^n)).$$

The martingale increments property implies $\mathbb{E}(|\widehat{E}''(2)|^2) \leq K\Delta_n^{1/2}$ in all the cases, which in turn implies $\widehat{E}''(2) \stackrel{\mathbb{P}}{\Longrightarrow} 0$. Next, using the bounds on $\rho(u,v)^{(C),i,n,gh}$ and similarly to step 7 on page 549 of Aït-Sahalia and Jacod (2014), we obtain that $\widehat{E}'(2) \stackrel{\mathbb{P}}{\Longrightarrow} 0$.

Return to the proof of Theorem 1

So far, we have proved that

$$\frac{1}{\Delta_n^{1/4}}\Big([\widehat{H(C), G(C)}]_T^{(A1)} - \sum_{g,h,a,b=1}^{d}\sum_{u,v=1}^{2} \overline{A11}(H,gh,u;G,ab,v)_T^n + \overline{A12}(H,gh,u;G,ab,v)_T^n$$
$$+ \overline{A12}(G,ab,v;H,gh,u)_T^n\Big) \stackrel{\mathbb{P}}{\longrightarrow} 0.$$

We next show that,

$$\frac{1}{\Delta_n^{1/4}} \sum_{i=2k_n}^{[T/\Delta_n]} (\partial_{gh}H\partial_{ab}G)(C_{i-2k_n}^n)\rho_{gh}(u,v)_i^n\zeta'_{ab}(v)_i^n \stackrel{\mathbb{P}}{\Longrightarrow} 0, \quad \forall \quad (u,v) \tag{67}$$

$$\frac{1}{\Delta_n^{1/4}}\Big(\overline{A11}(H,gh,u;G,ab,v) - \int_0^T (\partial_{gh}H\partial_{ab}G)(C_t)\overline{C}_t^{gh,ab}dt\Big) \stackrel{\mathbb{P}}{\Longrightarrow} 0 \text{ when } (u,v)=(2,2) \tag{68}$$

$$\frac{1}{\Delta_n^{1/4}}\Big(\overline{A11}(H,gh,u;G,ab,v) - \frac{3}{\theta^2}\int_0^T (\partial_{gh}H\partial_{ab}G)(C_t)(C_t^{ga}C_t^{hb} + C_t^{gb}C_t^{ha})dt\Big) \stackrel{\mathbb{P}}{\Longrightarrow} 0 \text{ when } (u,v)=(1,1) \tag{69}$$

$$\frac{1}{\Delta_n^{1/4}}\overline{A11}(H,gh,u;G,ab,v) \stackrel{\mathbb{P}}{\Longrightarrow} 0 \text{ when } (u,v)=(1,2),(2,1) \tag{70}$$

which will in turn imply

$$\frac{1}{\Delta_n^{1/4}}\Big([\widehat{H(C), G(C)}]_T^{(A)} - [H(C),G(C)]_T - \frac{3}{2k_n^3}\sum_{g,h,a,b}^{d}\sum_{u,v=1}^{2}\sum_{i=2k_n}^{[T/\Delta_n]}\Big[(\partial_{gh}H\partial_{ab}G)(C_{i-2k_n}^n)\rho_{gh}(u,v)_i^n\zeta''_{ab}(v)_i^n \tag{71}$$

$$+ (\partial_{ab}H\partial_{gh}G)(C_{i-2k_n}^n)\rho_{ab}(v,u)_i^n\zeta''_{gh}(v)_i^n\Big]\Big) \stackrel{\mathbb{P}}{\Longrightarrow} 0. \tag{72}$$

(67) can be proved easily following steps similar to step 7 on page 549 of Aït-Sahalia and Jacod (2014) and using the bounds of $\rho(u,v)_i^{n,gh}$ in (66) . To show (68),(69) and (70), we set

$$\overline{\overline{A11}}(H,gh,u;G,ab,v)=\Gamma(u,v)_0^n\sum_{i=2k_n}^{[T/\Delta_n]}(\partial_{gh}H\partial_{ab}G)(C_{i-1})\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}.$$

Then it holds that,

$$\frac{1}{\Delta_n^{1/4}}\Big(\overline{\overline{A11}}(H,gh,u;G,ab,v)-\overline{A11}(H,gh,u;G,ab,v)\Big)\overset{\mathbb{P}}{\Rightarrow}0.$$

This result can be proved following similar steps as for (60) in case $w=1$ by replacing $\Theta(u,v)_0^{(C),i,n}$ by $\Gamma(u,v)_0^n((\partial_{gh}H\partial_{ab}G)(C_{i-1})-(\partial_{gh}H\partial_{ab}G)(C_{i-2k_n}))$, which has the same bounds as the former. Next, decompose $\overline{\overline{A11}}$ as follows,

$$\begin{aligned}\overline{\overline{A11}}(H,gh,u;G,ab,v)=\Gamma(u,v)_0^n\Bigg[&\sum_{i=2k_n}^{[T/\Delta_n]}(\partial_{gh}H\partial_{ab}G)(C_{i-1})V_{i-1}^n\\&+\sum_{i=2k_n}^{[T/\Delta_n]}(\partial_{gh}H\partial_{ab}G)(C_{i-1})\Big(\mathbb{E}(\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}|\mathcal{F}_{i-1}^n)-V_{i-1}^n\Big)\\&+\sum_{i=2k_n}^{[T/\Delta_n]}(\partial_{gh}H\partial_{ab}G)(C_{i-1})\Big(\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}-\mathbb{E}(\zeta(u)_i^{n,gh}\zeta(v)_i^{n,ab}|\mathcal{F}_{i-1}^n)\Big)\Bigg].\end{aligned}$$

We follow the proof of (61) for $w=1$, and we replace $\Theta(u,v)_0^{(C),i,n}$ by $\Gamma(u,v)_0^n(\partial_{gh}H\partial_{ab}G)(C_{i-1})$, which satisfies only the condition $|\Gamma(u,v)_0^n(\partial_{gh}H\partial_{ab}G)(C_{i-1})|\leq\tilde{\gamma}_{u,v}^n$. This calculation shows that that the last two terms in the above decomposition of vanish at a rate slower that $\Delta_n^{1/4}$. Therefore,

$$\frac{1}{\Delta_n^{1/4}}\left(\overline{\overline{A11}}(H,gh,u;G,ab,v)-\Gamma(u,v)_0^n\Big(\sum_{i=2k_n}^{[T/\Delta_n]}(\partial_{gh}H\partial_{ab}G)(C_{i-1})V_{i-1}^n\Big)\right)\Rightarrow0.$$

As a consequence, for $(u,v)=(1,2)$ and $(2,1)$,

$$\frac{1}{\Delta_n^{1/4}}\overline{\overline{A11}}(H,gh,u;G,ab,v)\Rightarrow0.$$

The results follow from the following observation,

$$\frac{1}{\Delta_n^{1/4}}\left(\Gamma(u,v)_0^n\Big(\sum_{g,h,a,b=1}^{d}\sum_{i=2k_n}^{[T/\Delta_n]}(\partial_{gh}H\partial_{ab}G)(C_{i-1})V_{i-1}^n(u,v)\Big)-\frac{3}{\theta^2}\int_0^T(\partial_{gh}H\partial_{ab}G)(C_t)(C_t^{ga}C_t^{hb}+C_t^{gb}C_t^{ha})dt\right)\Rightarrow0,$$

$$\text{for } (u,v)=(2,2)$$

$$\frac{1}{\Delta_n^{1/4}}\left(\sum_{g,h,a,b=1}^{d}\Gamma(u,v)_0^n\Big(\sum_{i=2k_n}^{[T/\Delta_n]}(\partial_{gh}H\partial_{ab}G)(C_{i-1})V_{i-1}^n(u,v)\Big)-[H(C),G(C)]_T\right)\Rightarrow0,\quad\text{for }(u,v)=(1,1).$$

Set

$$\xi(H,gh,u;G,ab,v)_i^n=\frac{1}{\Delta_n^{1/4}}(\partial_{gh}H\partial_{ab}G)(C_{i-2k_n}^n)\rho_{gh}(u,v)_i^n\zeta_{ab}''(v)_i^n,$$

$$Z(H, gh, u; G, ab, v)_t^n = \Delta_n^{1/4} \sum_{i=2k_n}^{[t/\Delta_n]} \xi(H, gh, u; G, ab, v)_i^n.$$

Notice that (71) implies

$$\frac{1}{\Delta_n^{1/4}} \Big(\widehat{[H(C), G(C)]}_T^{(A)} - [H(C), G(C)]_T \Big) \stackrel{\mathcal{L}}{=} \sum_{g,h,a,b=1}^{d} \sum_{u,v=1}^{2} \frac{1}{\Delta_n^{1/4}} \Big(Z(H, gh, u; G, ab, v)_T^n + Z(H, ab, v; G, gh, u)_T^n \Big). \tag{73}$$

Next, observe that to derive the asymptotic distribution of $\Big(\widehat{[H_1(C), G_1(C)]}_T^{(A)}, \ldots, \widehat{[H_\kappa(C), G_\kappa(C)]}_T^{(A)} \Big)$, it suffices to study the joint asymptotic behavior of the family of processes $\frac{1}{\Delta_n^{1/4}} Z(H, gh, u; G, ab, v)_T^n$.
It is easy to see that $\xi(H, gh, u; G, ab, v)_i^n$ are martingale increments, relative to the discrete filtration $(\mathcal{F}_i^n)$. Therefore, by Theorem 2.2.15 of Jacod and Protter (2012), to obtain the joint asymptotic distribution of $\frac{1}{\Delta_n^{1/4}} Z(H, gh, u; G, ab, v)_T^n$, it is enough to prove the following three properties, for all $t > 0$, all $(H, gh, u; G, ab, v), (H', g'h', u'; G', a'b', v')$ and all martingales N which are either bounded and orthogonal to W, or equal to one component W^j,

$$A\Big((H, gh, u; G, ab, v), (H', g'h', u'; G', a'b', v')\Big)_t^n := \sum_{i=2k_n}^{[t/\Delta_n]} \mathbb{E}(\xi(H, gh, u; G, ab, v)_i^n \xi(H', g'h', u'; G', a'b', v')_i^n | \mathcal{F}_{i-1}^n)$$
$$\stackrel{\mathbb{P}}{\Longrightarrow} A\Big((H, gh, u; G, ab, v), (H', g'h', u'; G', a'b', v')\Big)_t$$
$$\sum_{i=2k_n}^{[t/\Delta_n]} \mathbb{E}(|\xi(H, gh, u; G, ab, v)_i^n|^4 | \mathcal{F}_{i-1}^n) \stackrel{\mathbb{P}}{\Longrightarrow} 0$$
$$B(N; H, gh, u; G, ab, v)_t^n := \sum_{i=2k_n}^{[t/\Delta_n]} \mathbb{E}(\xi(H, gh, u; G, ab, v)_i^n \Delta_i^n N | \mathcal{F}_{i-1}^n) \stackrel{\mathbb{P}}{\Longrightarrow} 0.$$

Using the polynomial growth assumption on H_r and G_r, the second and the third results can be proved by a natural extension to the multivariate case of (B.105) and (B.106) in Aït-Sahalia and Jacod (2014). Define

$$V_{ab}^{a'b'}(v, v')_t = \begin{cases} (C_t^{aa'} C_t^{bb'} + C_t^{ab'} C_t^{ba'}) & \text{if} \quad (v, v') = (1, 1) \\ \overline{C}_t^{ab,a'b'} & \text{if} \quad (v, v') = (2, 2) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\overline{V}_{gh}^{g'h'}(u, u')_t = \begin{cases} (C_t^{gg'} C_t^{hh'} + C_t^{gh'} C_t^{hg'}) & \text{if} \quad (u, u') = (1, 1) \\ \overline{C}_t^{gh,g'h'} & \text{if} \quad (u, u') = (2, 2) \\ 0 & \text{otherwise.} \end{cases}$$

Once again using the polynomial growth assumption on H_r and G_r and following steps similar to the proof of (B.104) in Aït-Sahalia and Jacod (2014), one can show that

$$A\Big((H, gh, u; G, ab, v), (H', g'h', u'; G', a'b', v')\Big)_t = M(u, v; u', v') \int_0^t (\partial_{gh} H \partial_{ab} G \partial_{g'h'} H \partial_{a'b'} G)(C_s) V_{ab}^{a'b'}(v, v')_s \overline{V}_{gh}^{g'h'}(u, u')_s ds,$$

with

$$M(u,v;u',v') = \begin{cases} 3/\theta^3 & \text{if} \quad (u,v;u',v') = (1,1;1,1) \\ 3/4\theta & \text{if} \quad (u,v;u',v') = (1,2;1,2),(2,1;2,1) \\ 151\theta/280 & \text{if} \quad (u,v;u',v') = (2,2;2,2) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have
$A\Big((H,gh,u;G,ab,v),(H',g'h',u';G',a'b',v')\Big)_T =$

$$\begin{cases} \frac{3}{\beta^3}\int_0^T (\partial_{gh}H\partial_{ab}G\partial_{g'h'}H'\partial_{a'b'}G')(C_t)(C_t^{gg'}C_t^{hh'}+C_t^{gh'}C_t^{hg'})(C_t^{aa'}C_t^{bb'}+C_t^{ab'}C_t^{ba'})dt & \text{if} \quad (u,v;u',v')=(1,1;1,1) \\ \frac{3}{4\beta}\int_0^T (\partial_{gh}H\partial_{ab}G\partial_{g'h'}H'\partial_{a'b'}G')(C_t)(C_t^{gg'}C_t^{hh'}+C_t^{gh'}C_t^{hg'})\overline{C}_t^{ab,a'b'}dt & \text{if} \quad (u,v;u',v')=(1,2;1,2) \\ \frac{3}{4\beta}\int_0^T (\partial_{gh}H\partial_{ab}G\partial_{g'h'}H'\partial_{a'b'}G')(C_t)(C_t^{aa'}C_t^{bb'}+C_t^{ab'}C_s^{ba'})\overline{t}_s^{gh,g'h'}dt & \text{if} \quad (u,v;u',v')=(2,1;2,1) \\ \frac{151\beta}{280}\int_0^T (\partial_{gh}H\partial_{ab}G\partial_{g'h'}H'\partial_{a'b'}G')(C_t)\overline{C}_s^{ab,a'b'}\overline{C}_t^{gh,g'h'}dt & \text{if} \quad (u,v;u',v')=(2,2;2,2) \\ 0 & \text{otherwise.} \end{cases}$$

Using (73), we deduce that the asymptotic covariance between $[H_r(\widehat{C), G_r}(C)]_T^{(A)}$ and $[H_s(\widehat{C), G_s}(C)]_T^{(A)}$ is given by

$$\sum_{g,h,a,b=1}^{d}\sum_{g',h',a',b'=1}^{d}\sum_{u,v,u',v'=1}^{2}\Bigg(A\Big((H_r,gh,u;G_r,ab,v),(H_s,g'h',u';G_s,a'b',v')\Big)_T$$
$$+A\Big((H_r,gh,u;G_r,ab,v),(H_s,a'b',v';G_s,g'h',u')\Big)_T + A\Big((H_r,ab,v;G_r,gh,u),(H_s,g'h',u';G_s,a'b',v')\Big)_T$$
$$+A\Big((H_r,ab,v;H_r,gh,u),(H_s,a'b',v';G_s,g'h',u')\Big)_T\Bigg).$$

After some simple calculations, the above expression can be rewritten as

$$\sum_{g,h,a,b=1}^{d}\sum_{j,k,l,m=1}^{d}\Bigg(\frac{6}{\theta^3}\int_0^T \big(\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s(C_t)\big)\Big[(C_t^{gj}C_t^{hk}+C_t^{gk}C_t^{hj})(C_t^{al}C_t^{bm}+C_t^{am}C_t^{bl})$$
$$+(C_t^{aj}C_t^{bk}+C_t^{ak}C_t^{bj})(C_t^{gl}C_t^{hm}+C_t^{gm}C_t^{hl})\Big]dt$$
$$+\frac{151\theta}{140}\int_0^t \big(\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s(C_t)\big)\Big[\overline{C}^{gh,jk}\overline{C}^{ab,lm}+\overline{C}^{ab,jk}\overline{C}^{gh,lm}\Big]dt$$
$$+\frac{3}{2\theta}\int_0^t \big(\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s(C_t)\big)\Big[(C_t^{gj}C_t^{hk}+C_t^{gk}C_t^{hj})\overline{C}_t^{ab,lm}+(C_t^{al}C_t^{bm}+C_t^{am}C_t^{bl})\overline{C}_t^{gh,jk}$$
$$+(C_t^{gl}C_s^{hm}+C_t^{gm}C_s^{hl})\overline{C}_t^{ab,jk}+(C_t^{aj}C_t^{bk}+C_t^{ak}C_t^{bj})\overline{C}_t^{gh,lm}\Big]dt\Bigg),$$

which completes the proof.

B.2 Proof of Theorem 2

Using the polynomial growth assumption on H_r, G_r, H_s and G_s and Theorem 2.2 in Jacod and Rosenbaum (2012), one can show that

$$\frac{6}{\theta^3}\widehat{\Omega}_T^{r,s,(1)} \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(1)}.$$

Next, by equation (3.27) in Jacod and Rosenbaum (2012), we have

$$\frac{3}{2\theta}[\widehat{\Omega}_T^{r,s,(3)} - \frac{6}{\theta}\widehat{\Omega}_T^{r,s,(1)}] \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(3)}.$$

Finally, to show that

$$\frac{151\theta}{140}\frac{9}{4\theta^2}[\widehat{\Omega}_T^{r,s,(2)} + \frac{4}{\theta^2}\widehat{\Omega}_T^{r,s,(1)} - \frac{4}{3}\widehat{\Omega}_T^{r,s,(3)}] \xrightarrow{\mathbb{P}} \Sigma_T^{r,s,(2)},$$

we first observe that as in Step 1, the approximation error induced by replacing $\widehat{C}_i^n$ by $\widehat{C}_i'^n$ is negligible. For $1 \le g,h,a,b,j,k,l,m \le d$ and $1 \le r,s \le d$, we define

$$\begin{aligned}
\widehat{W}_T^n &= \sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\partial_{gh}H_r\partial_{ab}G_r\partial_{gh}H_s\partial_{lm}G_s)(\widehat{C}_i^n)\gamma_i^{n,gh}\gamma_i^{n,jk}\gamma_{i+2k_n}^{n,ab}\gamma_{i+2k_n}^{n,lm} \\
\widehat{w}(1)_i^n &= (\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(C_i^n)\mathbb{E}(\gamma_i^{n,gh}\gamma_i^{n,jk}\gamma_{i+2k_n}^{n,ab}\gamma_{i+2k_n}^{n,lm}|\mathcal{F}_i^n) \\
\widehat{w}(2)_i^n &= (\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(C_i^n)(\gamma_i^{n,gh}\gamma_i^{n,jk}\gamma_{i+2k_n}^{n,ab}\gamma_{i+2k_n}^{n,lm} - \mathbb{E}(\gamma_i^{n,gh}\gamma_i^{n,jk}\gamma_{i+2k_n}^{n,ab}\gamma_{i+2k_n}^{n,lm}|\mathcal{F}_i^n)) \\
\widehat{w}(3)_i^n &= \Big((\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(\widehat{C}_i^n) - (\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(C_i^n)\Big)\gamma_i^{n,gh}\gamma_i^{n,jk}\gamma_{i+2k_n}^{n,ab}\gamma_{i+2k_n}^{n,lm} \\
\widehat{W}(u)_t^n &= \sum_{i=1}^{[T/\Delta_n]-4k_n+1} \widehat{w}_i(u), \ u = 1,2,3.
\end{aligned}$$

Note that we have $\widehat{W}_t^n = \widehat{W}(1)_t^n + \widehat{W}(2)_t^n + \widehat{W}(3)_t^n$. By Taylor expansion and using repeatedly the boundedness of C_t, we have

$$|\widehat{w}(3)_i^n| \le (1 + \|\beta_i^n\|^{4(p-1)})\|\beta_i^n\|\|\gamma_i^n\|^2\|\gamma_{i+2k_n}^n\|^2,$$

which implies $\mathbb{E}(|\widehat{w}(3)_i^n|) \le K\Delta_n^{5/4}$ and $\widehat{W}(3)_t^n \xrightarrow{\mathbb{P}} 0$. Using Cauchy-Schwartz inequality and the bound $\mathbb{E}(\|\gamma_i^n\|^q|\mathcal{F}_i^n) \le K\Delta_n^{q/4}$, we have $\mathbb{E}(|\widehat{w}(2)_i^n|^2) \le K\Delta_n^2$. Observing furthermore that $\widehat{w}(2)_i^n$ is $\mathcal{F}_{i+4k_n}-$measurable, we use Lemma B.8 in Aït-Sahalia and Jacod (2014) to show that $\widehat{W}(2)_t^n \xrightarrow{\mathbb{P}} 0$. Also, define

$$\begin{aligned}
w_i^n &= (\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(C_i^n)\Big[\frac{4}{k_n^2\Delta_n}(C_i^{n,ga}C_i^{n,hb} + C_i^{n,gb}C_i^{n,ha})(C_i^{n,jl}C_i^{n,km} + C_i^{n,jm}C_i^{n,kl}) \\
&+ \frac{4}{3}(C_i^{n,jl}C_i^{n,km} + C_i^{n,jm}C_i^{n,kl})\overline{C}_i^{n,gh,ab} + \frac{4}{3}(C_i^{n,ga}C_i^{n,hb} + C_i^{n,gb}C_i^{n,ha})\overline{C}_i^{n,jk,lm} + \frac{4(k_n^2\Delta_n)}{9}\overline{C}_i^{n,gh,ab}\overline{C}_i^{n,jk,lm}\Big], \\
W_T^n &= \Delta_n \sum_{i=1}^{[T/\Delta_n]-4k_n+1} w_i^n.
\end{aligned}$$

The cadlag property of c and $\overline{C}$, $k_n\sqrt{\Delta_n} \longrightarrow \theta$, and the Riemann integral argument imply $W_T^n \xrightarrow{\mathbb{P}} W_T$ where

$$\begin{aligned}
W_T &= \int_0^T (\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(C_t)\Big[\frac{4}{\theta^2}(C_t^{ga}C_t^{hb} + C_t^{gb}C_t^{ha})(C_t^{jl}C_t^{km} + C_t^{jm}C_t^{kl}) + \frac{4}{3}(C_t^{jl}C_t^{km} + C_t^{jm}C_t^{kl})\overline{C}_t^{gh,ab} \\
&+ \frac{4}{3}(C_t^{ga}C_i^{hb} + C_t^{gb}C_t^{ha})\overline{C}_t^{jk,lm} + \frac{4\theta^2}{9}\overline{C}_t^{gh,ab}\overline{C}_t^{jk,lm}\Big]dt.
\end{aligned}$$

In addition, by Lemma 4, we have

$$\mathbb{E}(|\widehat{W}(1)_T^n - W_T^n|) \le \Delta_n\mathbb{E}\Bigg(\sum_{i=1}^{[T/\Delta_n]-4k_n+1} (\Delta_n^{1/8} + \eta_{i,4k_n})\Bigg).$$

Hence, by the third result of Lemma 6 we have $\widehat{W}_T^n \xrightarrow{\mathbb{P}} W_t$, from which it can be deduced that

$$\frac{9}{4\theta^2}\Big[\widehat{W}(1)_T^n + \frac{4}{k_n^2}\sum_{i=1}^{[T/\Delta_n]-4k_n+1}(\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(\widehat{C}_i^n)[C_i^n(jk,lm)C_i^n(gh,ab)]$$
$$-\frac{2}{k_n}\sum_{i=1}^{[T/\Delta_n]-4k_n+1}(\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(\widehat{C}_i^n)C_i^n(gh,ab)\gamma_i^{n,jk}\gamma_i^{n,lm}$$
$$-\frac{2}{k_n}\sum_{i=1}^{[T/\Delta_n]-4k_n+1}(\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(\widehat{C}_i^n)C_i^n(jk,lm)\gamma_i^{n,gh}\gamma_i^{n,ab}\Big]$$
$$\xrightarrow{\mathbb{P}}\int_0^T(\partial_{gh}H_r\partial_{ab}G_r\partial_{jk}H_s\partial_{lm}G_s)(C_t)\overline{C}_t^{gh,ab}\overline{C}_t^{jk,lm}dt.$$

The result follows from the above convergence, a symmetry argument, and straightforward calculations.