

Inference from high-frequency data: A subsampling approach*

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Abstract

In this paper, we show how to estimate the asymptotic (conditional) covariance matrix, which appears in many central limit theorems in high-frequency estimation of asset return volatility. We provide a recipe for the estimation of this matrix by subsampling; an approach that computes rescaled copies of the original statistic based on local stretches of high-frequency data, and then it studies the sampling variation of these. We show that our estimator is consistent both in frictionless markets and models with additive microstructure noise. We derive a rate of convergence for it and are also able to determine an optimal rate for its tuning parameters (e.g., the number of subsamples). Subsampling does not require an extra set of estimators to do inference, which renders it trivial to implement. As a variance-covariance matrix estimator, it has the attractive feature that it is positive semi-definite by construction. Moreover, the subsampler is to some extent automatic, as it does not exploit explicit knowledge about the structure of the asymptotic covariance. It therefore tends to adapt to the problem at hand and be robust against misspecification of the noise process. As such, this paper facilitates assessment of the sampling errors inherent in high-frequency estimation of volatility. We highlight the finite sample properties of the subsampler in a Monte Carlo study, while some initial empirical work demonstrates its use to draw feasible inference about volatility in financial markets.

JEL Classification: C10; C80.

Keywords: bipower variation; high-frequency data; microstructure noise; positive semi-definite estimation; pre-averaging; stochastic volatility; subsampling.

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1. Introduction

Volatility is a key ingredient in the assessment and prediction of financial risk, be it in asset- and derivatives pricing (e.g., Black and Scholes, 1973; Sharpe, 1964), portfolio selection (e.g., Markowitz, 1952), or risk management and hedging (e.g., Jorion, 2006).

Around the turn of the millennium, the advent of financial high-frequency data led to a surge in the nonparametric measurement of volatility (see, e.g., Andersen, Bollerslev, and Diebold, 2010; Barndorff-Nielsen and Shephard, 2007). High-frequency data are recorded at the tick-by-tick level and store information about the time, price (i.e., a bid-ask quote or transaction price), and size of individual orders and executions. In theory, the harnessing of high-frequency information leads to a perfect, error-free measure of ex-post volatility via the realized variance; a sum of squared intraday log-returns (e.g., Andersen and Bollerslev, 1998; Barndorff-Nielsen and Shephard, 2002).

After the initial—pioneering—work, the literature turned towards addressing two inherent shortcomings of realized variance. Firstly, realized variance can only estimate quadratic variation, and it does not separate continuous, diffusive volatility from discontinuous jump risk. This motivated the development and application of estimators that can robustly measure very general functionals of volatility, also in the presence of jumps (e.g., Aït-Sahalia and Jacod, 2011; Andersen, Dobrev, and Schaumburg, 2012; Barndorff-Nielsen and Shephard, 2004; Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard, 2006; Christensen, Oomen, and Podolskij, 2010; Corsi, Pirino, and Renò, 2010; Mancini, 2009; Mykland, Shephard, and Sheppard, 2012). Secondly, when applied to tick-by-tick data realized variance is severely biased by common sources of noise, which form an integral part of any realistic model for securities' prices (e.g., Hansen and Lunde, 2006b; Zhou, 1996). This paved the way for the next cohort of estimators that were designed to be more resistant to noise, e.g., Aït-Sahalia, Mykland, and Zhang (2005); Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008); Jacod, Li, Mykland, Podolskij, and Vetter (2009); Podolskij and Vetter (2009a); Zhang (2006). As such, much progress has been made and today there is no shortage of estimators that can provide consistent estimates of volatility functionals in various contexts (either plain vanilla, or robustly to jumps or noise—or both).

Meanwhile, inference about volatility—e.g., construction of confidence intervals or hypothesis tests—has received less attention. This is surprising, because whether the sample is small or large, as long as it is finite,

there is necessarily some sampling error left in the estimate, and when confidence intervals are computed in practice, high-frequency estimators of volatility are often found to contain sizable errors (e.g., Barndorff-Nielsen, Hansen, Lunde, and Shephard, 2008). The distinction between a realized measure of volatility and its population target is critical, because failing to properly take sampling uncertainty into account can severely distort parameter estimation of stochastic volatility models and be detrimental to the construction and evaluation of forecasts of volatility (e.g., Andersen and Bollerslev, 1998; Andersen, Bollerslev, and Meddahi, 2005, 2011; Hansen and Lunde, 2006a, 2014; Patton, 2011).

There are several problems associated with drawing inference about volatility functionals in high-frequency data. The first and foremost is of course to figure out the relevant distribution theory. The next hurdle is then to find a good proxy for the asymptotic (conditional) variance of the estimator. This is a formidable challenge in practice, because the asymptotic variance often depends on parameters that are substantially more difficult to back out from the available sample of high-frequency data. The expression for the asymptotic variance typically also rests heavily on the properties of the data and it is bound to change depending on these. This is an unpleasant concern with real high-frequency data, which are contaminated by market microstructure frictions. While the noise is often assumed to be i.i.d. and independent of the efficient price, there is some empirical and theoretical support for a serially correlated, heteroscedastic, and, potentially, endogenous noise process at the tick level (e.g., Ait-Sahalia, Mykland, and Zhang, 2011; Diebold and Strasser, 2013; Hansen and Lunde, 2006b; Kalnina and Linton, 2008). An estimator of the asymptotic variance designed for i.i.d. and independent noise can not be expected to give valid inference, if the underlying conditions are violated. In practice, it is not trivial to verify the conditions imposed on the noise (e.g., Hautsch and Podolskij, 2013), which makes it more pressing to find estimators that are robust against modeling criteria. Finally, in multivariate analysis, inference would at some stage require an estimate of the asymptotic covariance matrix. Here, the proposed estimator should ideally be positive semi-definite, while, in contrast, some existing estimators of the asymptotic covariance matrix in the high-frequency setting are not assured to be that. As we show in this paper, this runs smack into problems in practically relevant and realistic settings (see Table 1 in Section 4).

In this paper, we propose to use subsampling for assessing the uncertainty embedded in high-frequency estimation of functionals of financial volatility. Subsampling is based on creating several—properly rescaled—

estimates of the parameter(s) of interest using local stretches of sample data and then studying the sampling variation of these. It was originally developed in the context of stationary time series in the long-span domain (e.g., Politis and Romano, 1994; Politis, Romano, and Wolf, 1999). The term appeared in the high-frequency literature in Zhang, Mykland, and Aït-Sahalia (2005), who proposed a two-scale realized variance based on price subsampling. This is different from traditional subsampling and actually does not work for asymptotic variance estimation, because it leads to an overlapping samples problem in the subsampled returns, causing the subsample estimates to be too strongly correlated in large samples. This was pointed out by Kalnina and Linton (2007) and Kalnina (2011), who propose an alternative inference strategy based on infill return subsampling, which leads to better asymptotic properties. Kalnina (2012) extends these ideas to inference about a multivariate parameter, while Ikeda (2015) and Varneskov (2014) consider subsample estimation of the asymptotic variance of the realized kernel.

As an inferential tool, subsampling has several attractive features from a practical point of view. First, subsampling is intuitive and relatively easy to compute, because it does not require an extra set of estimators; it uses copies of the original statistic. Second, in the multivariate context, it leads to variance-covariance matrix estimates that are positive semi-definite by construction. And, third, subsampling does not explicitly take the structure of the asymptotic variance into account. It is to a large degree automatic and has an innate ability to adapt to the problem at hand, which makes it highly robust against design criteria, as shown by Kalnina (2011). This type of analysis, where inference is effectively carried out by bypassing the asymptotic variance, is also the focal point of a recent paper by Mykland and Zhang (2014), who propose a so-called Observed Asymptotic Variance, which, as our approach, is based on the comparison of adjacent estimators.

This paper builds on these ideas. We contribute to the current state of the extant literature in several directions, which are currently not available. First, we propose to subsample bipower variation in frictionless and noisy markets. Second, we derive an asymptotic theory for subsample estimation of the asymptotic variance-covariance matrix of this popular statistic. We show that our estimator can be made consistent in both settings. In addition, we provide a decomposition of the leading error terms of the statistic, from which we are able to get some insights about how to configure the subsampler by optimally choosing its tuning parameters (e.g., the number of subsamples). This leads to a rate of convergence for the subsampler; a result that has, to the best of

our knowledge, not been derived in earlier work.

The rest of this paper goes as follows. Section 2 introduces the setting. In Section 3, we derive the theory first without and then with noise. In Section 4, we do numerical simulations in order to inspect the finite sample performance of our estimator. In Section 5, we confront our framework with some real high-frequency data, while the Appendix contains the proofs of our results.

2. Theoretical framework

We consider a scalar process $X = (X_t)_{t \geq 0}$, which represents the log-price of some financial security. It is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and adapted to $(\mathcal{F}_t)_{t \geq 0}$. We assume that X can be described by a continuous Itô semimartingale (or stochastic volatility model), as expressed by the equation:

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s, \quad (2.1)$$

where X_0 is the starting price, $a = (a_t)_{t \geq 0}$ is a predictable and locally bounded drift process, $\sigma = (\sigma_t)_{t \geq 0}$ is an adapted, càdlàg volatility process, while $W = (W_t)_{t \geq 0}$ is a standard Brownian motion.¹

In these models, natural measures of ex-post volatility can be written, for some suitable function f ,

$$IV(f)_t = \int_0^t f(\sigma_s) ds, \quad (2.2)$$

i.e. integrated functions of the diffusion coefficient.

To maintain a streamlined exposition, we only study the univariate setting in this paper. All of our theoretical results extend directly and without any changes (apart from additional notation) to multivariate X , see, e.g., Kalnina (2012) for related research in that direction.

We also point out that a and σ are left unspecified in this paper, and our results are completely nonparametric (within this class of models). While we do not impose assumptions a priori, we sometimes need to add additional regularity conditions on σ of the following type.

¹A basic result in financial economics states that if X is drawn from an arbitrage-free, frictionless market, it necessarily has a semimartingale structure (e.g., Back, 1991; Delbaen and Schachermayer, 1994). If further X has continuous sample paths, as assumed in this paper, then weak assumptions ensure that X can be represented as in Eq. (2.1). Later on, we provide further details about the robustness of our results, when X exhibits jumps. We explicitly cover the noisy setting with market frictions in Section 3.3.

Assumption (V): σ is of the form:

$$\sigma_t = \sigma_0 + \int_0^t \tilde{a}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \tilde{v}_s dB_s, \quad (2.3)$$

where σ_0 is its initial value, $\tilde{a} = (\tilde{a}_t)_{t \geq 0}$, $\tilde{\sigma} = (\tilde{\sigma}_t)_{t \geq 0}$ and $\tilde{v} = (\tilde{v}_t)_{t \geq 0}$ are adapted, càdlàg stochastic processes, while $B = (B_t)_{t \geq 0}$ is a standard Brownian motion that is independent of W .

We are in the high-frequency setting. We suppose that historical data of X is available in the time frame $[0, 1]$, i.e. we set $t = 1$. In this interval, we assume that X is recorded at equidistant time points $t_i = i/n$, for $i = 0, 1, \dots, n$, so that $n + 1$ is the total number of log-price observations in the sample. We define the n increments, or log-returns, of X as:

$$\Delta_i^n X = X_{i/n} - X_{(i-1)/n}, \quad \text{for } i = 1, \dots, n. \quad (2.4)$$

The asymptotic theory we derive below is then infill, i.e. we are at some point going to let $n \rightarrow \infty$.

2.1. Bipower variation

The econometric challenge is that the objects of interest appearing in Eq. (2.2) are latent, but they can be estimated from the available sample of high-frequency data in Eq. (2.4). A popular statistic, which is well-suited to do this, is the bipower variation of Barndorff-Nielsen and Shephard (2004). It is based on the summation of products of the absolute value of adjacent high-frequency returns.² Here, we adopt the more general definition of bipower variation from Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2006):³

$$V(f, g)^n = \frac{1}{n} \sum_{i=1}^{n-1} f(\sqrt{n} \Delta_i^n X) g(\sqrt{n} \Delta_{i+1}^n X), \quad (2.5)$$

where $f = (f_1, \dots, f_m)'$ and $g = (g_1, \dots, g_m)'$ are \mathbb{R}^m -valued functions, which are assumed to fulfill the following condition.

²We should note that bipower variation is nested within a broader class of multipower variations, which adds additional lags in Eq. (2.5), see, e.g. Barndorff-Nielsen, Shephard, and Winkel (2006). All of our theoretical results extend directly to this framework.

³As we frequently cite this paper, it will henceforth be referred to as BGJPS6.

Assumption (K): The function $h : \mathbb{R} \mapsto \mathbb{R}$ is even and continuously differentiable. Moreover, both $h(x)$ and its derivative $h'(x)$ are at most of polynomial growth.

Both Assumption (V) and (K) are standard conditions for the validity of central limit theorems for classical high-frequency statistics; see, e.g., BGJPS6. The following proposition, which is adapted from that paper, then describes the limiting properties of bipower variation.

Proposition 2.1 *Assume that X is a continuous Itô semimartingale as in Eq. (2.1), where the volatility process σ follows Assumption (V) and Assumption (K) is true for each component of $f = (f_1, \dots, f_m)'$ and $g = (g_1, \dots, g_m)'$. Then, as $n \rightarrow \infty$, it holds that*

$$\sqrt{n} \left(V(f, g)^n - V(f, g) \right) \xrightarrow{d_s} \text{MN}(0, \Sigma), \quad (2.6)$$

where

$$V(f, g) = \int_0^1 \rho_{\sigma_s}(f) \rho_{\sigma_s}(g) ds, \quad (2.7)$$

and “ $\xrightarrow{d_s}$ ” means convergence in law stably (as described below). Moreover, $\rho_x(f) = \mathbb{E}[f(xZ)]$ for $x \in \mathbb{R}$ and $Z \sim \text{N}(0, 1)$. Finally, Σ is the $m \times m$ asymptotic conditional covariance matrix, which has elements

$$\begin{aligned} \Sigma_{ij} = \int_0^1 & \left[\rho_{\sigma_s}(f_i f_j) \rho_{\sigma_s}(g_i g_j) + \rho_{\sigma_s}(f_i) \rho_{\sigma_s}(g_j) \rho_{\sigma_s}(f_j g_i) \right. \\ & \left. + \rho_{\sigma_s}(f_j) \rho_{\sigma_s}(g_i) \rho_{\sigma_s}(f_i g_j) - 3 \rho_{\sigma_s}(f_i) \rho_{\sigma_s}(f_j) \rho_{\sigma_s}(g_i) \rho_{\sigma_s}(g_j) \right] ds. \end{aligned} \quad (2.8)$$

Proof. See BGJPS6.

The proposition provides the necessary foundation for making inference about bipower variation. It shows that $V(f, g)^n$ is consistent for $V(f, g)$. Moreover, the asymptotic distribution of $V(f, g)^n$ is mixed normal, i.e. it has a random variance-covariance matrix Σ , which is independent from the randomness of the normal distribution. To transform this into a probabilistic statement based on the standard normal distribution, it would be tempting to look at Eq. (2.6) and deduce that $\Sigma^{-1/2} \sqrt{n} \left(V(f, g)^n - V(f, g) \right) \xrightarrow{d} \text{N}(0, I_m)$, where “ \xrightarrow{d} ” is convergence in law and I_m is the m -dimensional identity matrix. We cannot readily jump to this conclusion from the regular definition of convergence in law, however, unless Σ is constant. In our setting, where Σ is stochastic, we therefore exploit an alternative, stronger type of convergence, which recovers the above feature.

More formally, let Z_n be a sequence of random variables. Also, let Z be a random variable, which is defined on some appropriate extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original space $(\Omega, \mathcal{F}, \mathbb{P})$. Z_n is then said to converge stably to Z , and we shall write $Z_n \xrightarrow{d_s} Z$, if for any bounded, continuous function h and any bounded, \mathcal{F} -measurable random variable Y , the convergence $\mathbb{E}[Yh(Z_n)] \rightarrow \mathbb{E}'[Yh(Z)]$ holds, as $n \rightarrow \infty$. It follows that stable convergence implies convergence in law by taking $Y = 1$. We refer to Jacod and Shiryaev (2003) for further details of this concept.

Proposition 2.1 is based on Assumption **(K)**, which assumes f and g are differentiable. It can be extended to non-differentiable functions, given Assumption **(H')** and **(K')** from BGJPS6. Assumption **(H')** says σ should be bounded away from zero, while Assumption **(K')** puts suitable restrictions on the set, where f and g are not differentiable. We refer to BGJPS6 for a concise, mathematical statement of these conditions. Throughout the paper, we explain how this weaker setting affects our results.

A leading example of a statistic that falls in the latter category, and which is also used extensively in applied work, is the original bipower variation of Barndorff-Nielsen and Shephard (2004). It takes $f_i(x) = |x|^{q_i}$ and $g_i(x) = |x|^{r_i}$, for $1 \leq i \leq m$ and $q_i, r_i \geq 0$, which is not always differentiable at $x = 0$. Still, if Assumption **(H')** and **(K')** from BGJPS6 are fulfilled, Proposition 2.1 holds. Here,

$$V(f_i, g_i)^n = \frac{1}{n} \sum_{i=1}^{n-1} |\Delta_i^n X|^{q_i} |\Delta_{i+1}^n X|^{r_i}, \quad V(f_i, g_i) = \mu_{q_i} \mu_{r_i} \int_0^1 |\sigma_s|^{q_i+r_i} dS, \quad (2.9)$$

where $\mu_q = \mathbb{E}[|Z|^q]$ and $Z \sim \mathbf{N}(0, 1)$. Also, Σ has the form:

$$\Sigma_{ij} = (\mu_{q_i+q_j} \mu_{r_i+r_j} + \mu_{q_i} \mu_{r_j} \mu_{q_j+r_i} + \mu_{q_j} \mu_{r_i} \mu_{q_i+r_j} - 3\mu_{q_i} \mu_{q_j} \mu_{r_i} \mu_{r_j}) \int_0^1 |\sigma_s|^{q_i+q_j+r_i+r_j} dS, \quad (2.10)$$

and the link between Eq. (2.2) and Eq. (2.4) is made apparent.

3. Subsample estimation of Σ

In order to do inference about bipower variation based on Proposition 2.1, we also need to estimate Σ . Although there are some existing estimators out there, they are typically not general enough, or else they are inherently flawed, as we show below.

In this paper, we therefore propose to estimate Σ by subsampling of high-frequency data. This was suggested by Kalnina and Linton (2007) and Kalnina (2011) for the asymptotic variance of the two-scale realized

variance, following earlier work in the classical time series literature (e.g., Politis and Romano, 1994; Politis, Romano, and Wolf, 1999). Subsampling is an attractive procedure. First, all it does is to compute rescaled copies of the original statistic based on local stretches of high-frequency data, and then it studies the sampling variation of these. This makes it highly intuitive and trivial to implement. Second, it leads to estimators of Σ that are positive semi-definite by construction. This is critical in practice, as shown in Section 4. And third, it has an innate robustness against the statistical properties of microstructure noise, as discussed in Section 3.3

3.1. Subsampling for power variation

In order to develop a complete theory for positive semi-definite estimation of the asymptotic, conditional covariance matrix Σ of bipower variation using subsampling, we will start by analyzing the power variation case, i.e. $f = (f_1, \dots, f_m)'$. This should not only help to build intuition for the general case, but it turns out the results are interesting in their own right, because the rates of convergence differ according to whether we consider power variation, or multipower variation of order bigger than one.

Thus, we define the power variation estimator:

$$V(f)^n = \frac{1}{n} \sum_{i=1}^n f(\sqrt{n}\Delta_i^n X). \quad (3.1)$$

Then, we propose to set:

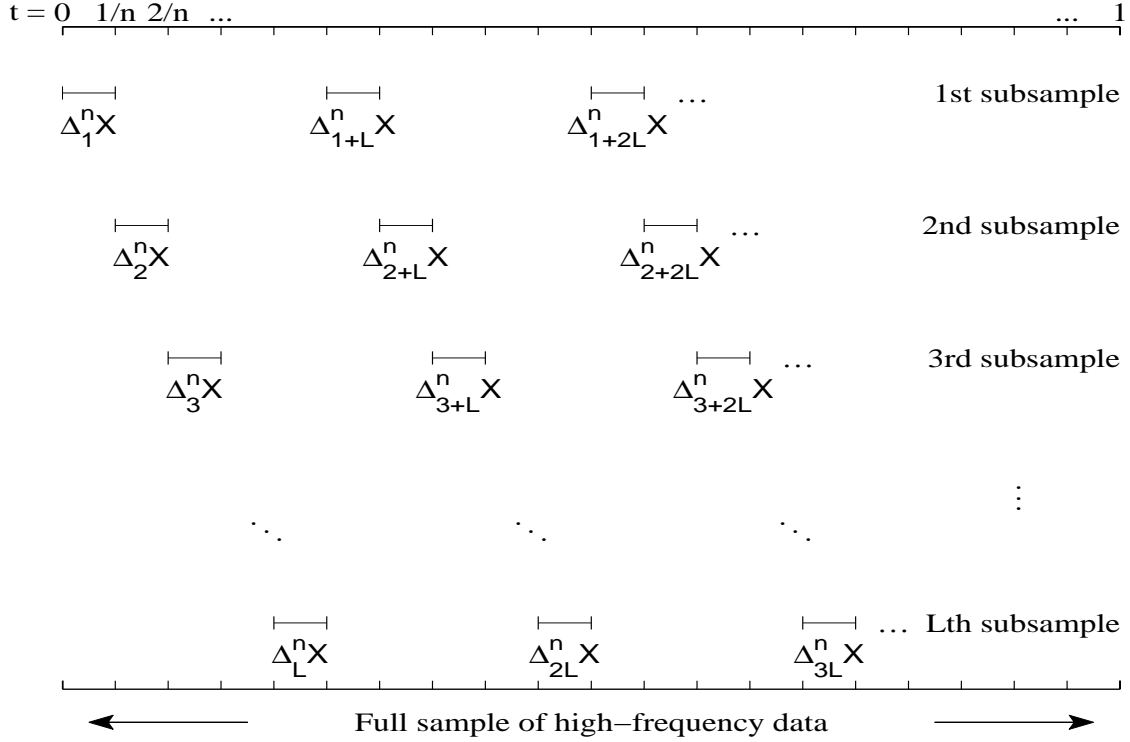
$$\hat{\Sigma}_n = \frac{1}{L} \sum_{l=1}^L \left(\sqrt{\frac{n}{L}} \left(V_l(f)^n - V(f)^n \right) \right) \left(\sqrt{\frac{n}{L}} \left(V_l(f)^n - V(f)^n \right) \right)', \quad (3.2)$$

where, assuming L divides n ,

$$V_l(f)^n = \frac{1}{n/L} \sum_{i=1}^{n/L} f(\sqrt{n}\Delta_{(i-1)L+l}^n X). \quad (3.3)$$

In Figure 1, we illustrate the construction of the subsamples for power variation. As the figure shows, we start by splitting the full sample of high-frequency data into L smaller subsamples. We assign log-returns successively to each subsample, starting from the first and going back to it after the L th subsample has been reached. We continue this process until the entire sample is exhausted. The l th subsample therefore consists of the increments $(\Delta_{(i-1)L+l}^n X)_{i=1, \dots, n/L}$.

Figure 1: Infill return subsampling for power variation.



Note. The figure shows how the full sample of available high-frequency data $(\Delta_i^n X)_{i=1}^n$ is split into smaller samples of size n/L in order to compute subsampled estimates of power variation.

We then compute the statistics $V_l(f)^n$, which is the power variation estimator based on the l th subsample. In order to estimate Σ , we look at the sampling variation of the $V_l(f)^n$'s, after they have been suitably demeaned and scaled.

As we prove in a moment, we can choose the number of subsamples L , such that $\hat{\Sigma}_n$ is a consistent estimator of Σ . Intuitively, under suitable conditions on L , $V_l(f)^n \xrightarrow{p} V(f)$, where $V(f) = \int_0^1 \rho_{\sigma_s}(f) ds$, and its asymptotic distribution (more or less) follows from Proposition 2.1, except that its rate of convergence is $(n/L)^{-1/2}$, i.e. $\sqrt{\frac{n}{L}}(V_l(f)^n - V(f)) \xrightarrow{d_s} \text{MN}(0, \Sigma)$. Moreover, as each subsample is based on non-overlapping increments, the $V_l(f)^n$'s are, asymptotically, conditionally independent. This suggests that by averaging the sum of outer products of $\sqrt{\frac{n}{L}}(V_l(f)^n - V(f))$, we should get a consistent estimator of Σ , which is then positive semi-definite by construction.

While $V_l(f)^n$ should in principle be centered around $V(f)$, the whole problem to begin with is of course

that $V(f)$ is latent. So it has to be replaced by a consistent estimator to get a feasible estimator of Σ that can be computed from data. In Eq. (3.2), we therefore propose to plug in the full sample power variation estimate $V(f)^n$. This does not affect the asymptotics, because $V(f)^n$ converges much faster than $V_t(f)^n$.

We pause here for a moment to reflect a bit on the setup. It turns out that the assumptions behind Proposition 2.1 are, by construction, also necessary to prove the consistency of $\hat{\Sigma}_n$. Moreover, as we show later, they are also sufficient, if $L \rightarrow \infty$ and $n/L \rightarrow \infty$. To be able to derive a convergence rate for $\hat{\Sigma}_n$ and provide an optimal choice of the parameter L , however, we need some additional structure, which is not standard in the high-frequency literature.

First, we are going to assume that all of the driving terms in both X and σ can be modeled as Brownian semimartingales, i.e. we shall require that:

Assumption (H): σ follows Assumption (V), and each of $a, \tilde{a}, \tilde{\sigma}$ and \tilde{v} is of the form in Eq. (2.3).

The second condition we impose is a highly technical requirement that concerns the Malliavin smoothness of the random variables appearing in Assumption (H). We summarize some selected elements and notations from Malliavin calculus, which are relevant to our paper, in Section A.6 at the end of the Appendix, while the list of necessary conditions for the main text are comprised as Assumption (M).

Assumption (M): We assume that for any $0 \leq t \leq r \leq s$: $\sigma_s, \tilde{\sigma}_s, \tilde{v}_s, D_t(\sigma_s), D_t(\tilde{\sigma}_s), D_t(\tilde{v}_s) \in \mathbb{D}_{1,2}$ and

$$\|D_t(\sigma_s)\|_{L^{32}} + \|D_t(\tilde{\sigma}_s)\|_{L^{32}} + \|D_t(\tilde{v}_s)\|_{L^{32}} \leq C, \tag{3.4}$$

$$\|D_t(D_r(\sigma_s))\|_{L^{16}} + \|D_t(D_r(\tilde{\sigma}_s))\|_{L^{16}} + \|D_t(D_r(\tilde{v}_s))\|_{L^{16}} \leq C.$$

Moreover, f'' exists and has polynomial growth.

To put this assumption into context, note that if the involved stochastic processes are solutions to stochastic differential equations, then condition (3.4) is fulfilled given a sufficient smoothness of the corresponding drift term and volatility function. We refer to Eq. (A.47) for the computation of the Malliavin derivative in this case.

The next result then leads to a rate of convergence for $\hat{\Sigma}_n$.

Theorem 3.1 *Assume that X is a continuous Itô semimartingale as in Eq. (2.1), where Assumption **(H)** is true, as is Assumption **(K)** for each component of $f = (f_1, \dots, f_m)'$. Moreover, Assumption **(M)** is fulfilled. As $n \rightarrow \infty$, $L \rightarrow \infty$, and $n/L \rightarrow \infty$, it holds that*

$$\hat{\Sigma}_n - \Sigma = \underbrace{O_p\left(\frac{1}{\sqrt{L}}\right)}_{\text{CLT}} + \underbrace{O_p\left(\frac{L}{n}\right)}_{\text{blocking}}. \quad (3.5)$$

Proof. See Appendix.

Theorem 3.1 presents the leading errors inherent in $\hat{\Sigma}_n$. The first term, $1/\sqrt{L}$, intuitively follows from a central limit theorem result, because $\hat{\Sigma}_n$ is an empirical mean of L asymptotically, conditionally independent statistics. However, it is not easy to apply this relationship for a formal derivation of the error rate. The second error is more subtle, but it can be explained by the block distance L/n between successive increments within a subsampling statistic, and from the smoothness and semimartingale form of the volatility process σ . As such, we should point out that the proof of Theorem 3.1 is much more complex compared to subsampling of i.i.d. observations.

We can find the fastest rate of convergence by balancing both errors. This requires:

$$L = O(n^{2/3}), \quad (3.6)$$

such that

$$\hat{\Sigma}_n - \Sigma = O_p(n^{-1/3}). \quad (3.7)$$

We believe that this rate is sharp and that it cannot be improved within the context of subsampling high-frequency data, but we shall not attempt to prove this.

We can combine the consistency of $\hat{\Sigma}_n$ from Theorem 3.1 with the convergence in distribution in Eq. (2.6). If we then appeal to the properties of stable convergence, we get the following feasible result:

$$\hat{\Sigma}_n^{-1/2} \sqrt{n} \left(V(f)^n - V(f) \right) \xrightarrow{d} N(0, I_m), \quad (3.8)$$

which can be used to construct confidence intervals for $V(f)$ or do hypothesis testing. If the convergence had not been stable in law, this result would not follow in general.

The stable central limit theorem of Proposition 2.1 is also valid for a non-differentiable function f given Assumption **(H')** and **(K')** from BGJPS6, but it appears out of reach to derive a convergence rate for $\hat{\Sigma}_n$ under these weaker conditions. We can nonetheless show that $\hat{\Sigma}_n$ still converges in probability to Σ , which—as discussed above—is relevant for applied work.

Theorem 3.2 *Assume that X is a continuous Itô semimartingale as in Eq. (2.1), where Assumption **(V)** is true, as are Assumption **(H')** and **(K')** from BGJPS6. As $n \rightarrow \infty$, $L \rightarrow \infty$, and $n/L \rightarrow \infty$, it holds that*

$$\hat{\Sigma}_n \xrightarrow{p} \Sigma. \quad (3.9)$$

Proof. See Appendix.

Remark Theorem 3.1 and 3.2 are both robust to the presence of jumps in X , if the associated central limit theorem in Proposition 2.1 is. So, for example, if $f_i(x) = |x|^{q_i}$, we need $q_i \in (0, 1)$ to ensure that $\hat{\Sigma}_n$ is jump-robust (e.g., Barndorff-Nielsen, Shephard, and Winkel, 2006). To strengthen this statement to any $q_i > 0$, we could base our statistic on truncated power variations (e.g., Jing, Liu, and Kong, 2014; Mancini, 2009), but we leave this option for future work. We refer to Jacod and Protter (2012) for a detailed analysis of this topic.

3.2. Subsampling for bipower variation

In the previous section, we presented a subsampling estimator for the asymptotic conditional covariance matrix of power variation. If we are interested in bipower (or multipower) variation, the theory derived there does not readily apply. This is because the summands in Eq. (2.5) are, asymptotically, 1-dependent, which the subsampling approach shown in Figure 1 does not adequately capture.

In order to consistently estimate Σ in the bipower case, we use an intuitive blocking approach, which is described next. We define the i th block of high-frequency data by taking:

$$B_i(p) = \left\{ j : (i-1)p \leq j \leq ip \right\}, \quad (3.10)$$

where $p \geq 2$ is an integer, and $i \geq 1$.

$B_i(p)$ is composed of the sample of adjacent log-price observations $X_{(i-1)p/n}, \dots, X_{ip/n}$. From this, we can compute p consecutive returns $\Delta_{(i-1)p+1}^n X, \dots, \Delta_{ip}^n X$. Therefore, $B_i(p)$ plays the role of the interval

$[(i-1)/n, i/n]$ for power variation, which was used to compute a single return $\Delta_i^n X$. The only change is that we need to make this interval longer, such that we can consistently estimate the covariance structure of $V(f, g)^n$. As the $B_i(p)$'s are based on non-overlapping increments, it still holds that bipower variations computed from different subsamples are, asymptotically, conditionally independent.

We reset $\hat{\Sigma}_n$ as follows:

$$\hat{\Sigma}_n = \frac{1}{L} \sum_{l=1}^L \left(\sqrt{\frac{n}{L}} \left(V_l(f, g)^n - V(f, g)^n \right) \right) \left(\sqrt{\frac{n}{L}} \left(V_l(f, g)^n - V(f, g)^n \right) \right)', \quad (3.11)$$

where, assuming Lp divides n ,

$$\begin{aligned} V_l(f, g)^n &= \frac{Lp}{n} \sum_{i=1}^{n/Lp} v_{(i-1)L+l}(f, g)^n, \\ v_i(f, g)^n &= \frac{1}{p-1} \sum_{j, j+1 \in B_i(p)} f(\sqrt{n}\Delta_j^n X) g(\sqrt{n}\Delta_{j+1}^n X). \end{aligned} \quad (3.12)$$

Note that n/Lp is the number of blocks assigned to each subsample, and that the subsample statistic $v_i(f, g)^n$ is computed only from data within the i th block $B_i(p)$. As in the above, we definitely require $n \rightarrow \infty$, $p \rightarrow \infty$, $L \rightarrow \infty$, and $n/pL \rightarrow \infty$ to prove the asymptotic theory for $\hat{\Sigma}_n$. It turns out, however, we need a slightly stronger condition for the last part to ensure consistency. This is because the rate $\sqrt{\frac{n}{L}}$ in the definition of Eq. (3.11) corresponds to the martingale part of $V_l(f, g)^n - V(f, g)^n$, while the statistic $V_l(f, g)^n - V(f, g)^n$ also has a bias term, which is of order Lp/n . Thus, to make the bias negligible with respect to the martingale part, we need $n/Lp^2 \rightarrow \infty$. Hence, our “minimal” assumptions are based on this condition.

Theorem 3.3 *Assume that X is a continuous Itô semimartingale as in Eq. (2.1), where Assumption **(H)** is true, as is Assumption **(K)** for each component of $f = (f_1, \dots, f_m)'$ and $g = (g_1, \dots, g_m)'$. Moreover, Assumption **(M)** is fulfilled. As $n \rightarrow \infty$, $p \rightarrow \infty$, $L \rightarrow \infty$, and $n/Lp^2 \rightarrow \infty$, it holds that*

$$\hat{\Sigma}_n - \Sigma = \underbrace{O_p\left(\frac{1}{\sqrt{L}}\right)}_{\text{CLT}} + \underbrace{O_p\left(\frac{Lp^2}{n}\right)}_{\text{blocking}} + \underbrace{O_p\left(\frac{1}{p}\right)}_{\text{HAC}}. \quad (3.13)$$

Proof. See Appendix.

The first two errors in Eq. (3.13) can be interpreted as to those in Theorem 3.1, except the second is also affected by the block size p . Meanwhile, the decomposition of $\hat{\Sigma}_n - \Sigma$ in Theorem 3.3 has an extra error of order $O_p(1/p)$. The additional term, which emerges from the computation of the conditional variance of $v_i(f, g)^n$, has an intuitive interpretation, if we recall that in the current setting of bipower variation, the summands in Eq. (3.12) (or Eq. (2.5)) are asymptotically 1-dependent.

Consider the following stylized example. Assume that $(Z_i)_{i \geq 1}$ is a sequence of stationary 1-dependent random variates. Then,

$$\text{var} \left(\frac{1}{\sqrt{p}} \sum_{i=1}^p Z_i \right) = \text{var}(Z_1) + 2 \frac{(p-1)}{p} \text{cov}(Z_1, Z_2) \rightarrow \text{var}(Z_1) + 2 \text{cov}(Z_1, Z_2), \quad (3.14)$$

as $p \rightarrow \infty$.

This calculation shows that the finite sample variance on the left-hand side is not equal to, but converges towards, the asymptotic variance. The difference, i.e. the bias, is the term $-2 \text{cov}(Z_1, Z_2)/p$, which has order $O(1/p)$. This example also helps to illustrate that Theorem 3.3 does not change, and in particular the convergence rate of $\hat{\Sigma}_n$ is unaffected, if we were to compute a higher order multipower variation statistic. Then there would be more covariance terms in Eq. (3.14), but the bias in each of them would still be $O(1/p)$.

The fastest rate is again found by balancing the errors, which means taking:

$$L = O(n^{2/5}), \quad p = O(n^{1/5}), \quad (3.15)$$

for which

$$\hat{\Sigma}_n - \Sigma = O_p(n^{-1/5}). \quad (3.16)$$

While we do not offer a formal proof, we again believe that this rate is optimal within the framework of subsampling high-frequency data. Moreover, we note again that the consistency of $\hat{\Sigma}_n$ holds under the weaker assumptions and do not require Assumption **(K)** and **(M)**. We note that the condition $L/p \rightarrow \infty$ is necessary to deal with an additional bias term.

Theorem 3.4 *Assume that X is a continuous Itô semimartingale as in Eq. (2.1), where Assumption **(V)** is true, as are Assumption **(H³)** and **(K³)** from BGJPS6. As $n \rightarrow \infty$, $p \rightarrow \infty$, $L/p \rightarrow \infty$, and $n/Lp^2 \rightarrow \infty$, it holds that*

$$\hat{\Sigma}_n \xrightarrow{p} \Sigma. \quad (3.17)$$

Proof. See Appendix.

To end this section, we should point out that for the power variation estimator covered by Theorem 3.1 in the previous subsection, it follows from the work of BGJPS6 that there exists another consistent, positive semi-definite estimator of Σ :

$$\hat{S}_n = \frac{1}{2n} \sum_{i=1}^{n-1} \left(f(\sqrt{n}\Delta_i^n X) - f(\sqrt{n}\Delta_{i+1}^n X) \right) \left(f(\sqrt{n}\Delta_i^n X) - f(\sqrt{n}\Delta_{i+1}^n X) \right)'. \quad (3.18)$$

\hat{S}_n has a better rate of convergence $n^{-1/2}$ compared to $n^{-1/3}$ derived in the previous section for $\hat{\Sigma}_n$. \hat{S}_n is therefore more efficient for power variation, but it does not work for bi- or multipower variation.

3.3. Microstructure noise

In practice, assets are not traded within a frictionless market. The recorded data constitute a discrete sample of transactions or bid-ask quotes, whose prices are affected by common sources of market imperfections, such as bid-ask spreads, price discreteness, and so forth (e.g., Black, 1986; Niederhoffer and Osborne, 1966; Roll, 1984). Even if these were small enough to be ignored, high-frequency data are also corrupted by outliers (due to bugs in the data transmission, fat-finger errors, etc.) and subject to other irregularities (e.g., quote stuffing, screen fighting, etc.). The combination of these effects leads to marked differences between real data and those generated by a diffusion model.

To accommodate this, we need to modify the setup. We are going to take the observed price as the true, underlying price perturbed by an additive noise term, i.e.

$$Y_{i/n} = X_{i/n} + \epsilon_{i/n}, \quad (3.19)$$

where X is defined as in Eq. (2.1), while $\epsilon = (\epsilon_t)_{t \geq 0}$ is a noise process. We impose the following:

Assumption (N): (i) ϵ is i.i.d. with $\mathbb{E}[\epsilon_t] = 0$ and $\text{var}(\epsilon_t) = \omega^2$ for all $t \geq 0$, (ii) ϵ is independent of X , (iii) the distribution of ϵ is symmetric around 0, and (iv) ϵ has a finite 8th moment.

3.3.1. Pre-averaging

To alleviate the impact of noise, we make use of the notion that as X is continuous and ϵ is i.i.d., we can locally smooth $Y_{i/n}$ in the vicinity of i/n to retrieve an estimate, say $\bar{Y}_{i/n}$, which tends to be close to $X_{i/n}$, because the noise is largely averaged away (e.g., Jacod, Li, Mykland, Podolskij, and Vetter, 2009; Podolskij and Vetter, 2009a,b). Averaging our discrete sample of noisy high-frequency data this way leads to a new set of increments, $\Delta\bar{Y}_i^n$, based on pre-averaged prices.⁴

To implement pre-averaging, we need some extra notation. We choose a sequence k_n of integers (the pre-averaging window) and a scalar $\theta > 0$, such that

$$k_n = \theta\sqrt{n} + o_p(n^{-1/4}). \quad (3.20)$$

We also need a weight function $w : \mathbb{R} \mapsto \mathbb{R}$ to do averaging. We assume w is continuous on $[0, 1]$ and piecewise continuously differentiable with a piecewise Lipschitz derivative w' . Moreover, we assume that $w(0) = w(1) = 0$ and $\int_0^1 (w(t))^2 dt > 0$. The following numbers and functions are associated with w :

$$\begin{aligned} \phi_1(s) &= \int_s^1 w'(u)w'(u-s)du, & \phi_2(s) &= \int_s^1 w(u)w(u-s)du, & \text{for } s \in [0, 1], \\ \psi_1 &= \phi_1(0), & \psi_2 &= \phi_2(0), & \Phi_{ij} &= \int_0^1 \phi_i(s)\phi_j(s)ds, & \text{for } i, j = 1, 2, \\ \psi_1^n &= k_n \sum_{j=0}^{k_n} (w_{j+1}^n - w_j^n)^2, & \psi_2^n &= \frac{1}{k_n} \sum_{j=1}^{k_n} (w_j^n)^2, \end{aligned} \quad (3.21)$$

where $w_j^n = w(j/k_n)$. In passing, we note that

$$\psi_1^n = \psi_1 + O_p(n^{-1/2}) \quad \text{and} \quad \psi_2^n = \psi_2 + O_p(n^{-1/2}), \quad (3.22)$$

which means that in the asymptotic theory only ψ_1 and ψ_2 appear. Still, it is recommendable to use ψ_1^n and ψ_2^n for simulations and empirical work, as it entails better finite sample properties.

⁴There are several other approaches available to handle microstructure noise, including the realized kernel; based on auto-covariance corrections (see, e.g., Barndorff-Nielsen, Hansen, Lunde, and Shephard, 2008, 2011), and the two- or multi-scale realized variance; based on price subsampling (see, e.g., Zhang, 2006; Zhang, Mykland, and Ait-Sahalia, 2005).

The return series, following pre-averaging, is:

$$\Delta \bar{Y}_i^n = \sum_{j=1}^{k_n} w_j^n \Delta_{i+j}^n Y = - \sum_{j=0}^{k_n} (w_{j+1}^n - w_j^n) Y_{\frac{i+j}{n}}, \quad \text{for } i = 1, \dots, n - k_n + 2. \quad (3.23)$$

3.3.2. Pre-averaged bipower variation

The addition of microstructure noise creates additional complications for inference procedures from high-frequency data. To make our framework analytically tractable, yet practically relevant, we therefore restrict attention to the class of “pure” bipower variations. That is, in this section we are going to assume that $f_i(x) = |x|^{q_i}$ and $g_i(x) = |x|^{r_i}$, where $q_i, r_i \geq 0$. To reflect this change and separate the following results from the previous ones, we write the pre-averaged bipower variation as $V^*(q, r)^n$, where $q = (q_1, \dots, q_m)'$ and $r = (r_1, \dots, r_m)'$ are m -dimensional vectors, whose coordinates index the powers.

The i th coordinate of $V^*(q, r)^n$ is defined as:

$$V^*(q_i, r_i)^n = \frac{1}{n - 2k_n + 2} \sum_{i=1}^{n-2k_n+2} |n^{1/4} \Delta \bar{Y}_i^n|^{q_i} |n^{1/4} \Delta \bar{Y}_{i+k_n}^n|^{r_i}. \quad (3.24)$$

The intuition behind this construction is that pre-averaging induces some autocorrelation (of order k_n) in the pre-averaged price series, which is broken by multiplying pre-averaged returns that are k_n terms apart. In essence, this leads to a lower, effective sample of size $n - 2k_n + 2$.

Podolskij and Vetter (2009a) show that

$$n^{1/4} \left(V^*(q, r)^n - V^*(q, r) \right) \xrightarrow{d_s} \text{MN}(0, \Sigma^*), \quad (3.25)$$

where

$$V^*(q_i, r_i) = \mu_{q_i} \mu_{r_i} \int_0^1 \left(\theta \psi_2 \sigma_s^2 + \frac{1}{\theta} \psi_1 \omega^2 \right)^{\frac{q_i + r_i}{2}} ds. \quad (3.26)$$

Thus, pre-averaging slows down the rate of convergence, but $n^{-1/4}$ is nonetheless the fastest rate in noisy diffusion models (Gloter and Jacod, 2001a,b).

In the above, Σ^* is the $m \times m$ conditional covariance matrix of $V^*(q, r)^n$.⁵ Podolskij and Vetter (2009a) go on to develop a consistent estimator of Σ^* (defined in Eq. (4.4) in the simulation section), but it is based

⁵ We do not state the expression of Σ^* here, which can instead be found in Podolskij and Vetter (2009a). Σ^* has a complicated structure (even with i.i.d. noise) and is typically not known in closed-form. An exception, which is used in the simulations, is the case of pre-averaged realized variance, i.e. $(q, r) = (2, 0)$. It has $\Sigma^*(2, 0) = \int_0^1 4 \left(\theta^3 \Phi_{22} \sigma_s^4 + 2\theta \Phi_{12} \sigma_s^2 \omega^2 + \frac{1}{\theta} \Phi_{11} \omega^4 \right) ds$.

on element-by-element estimation. The disadvantage of this approach is that it does not ensure that the whole covariance matrix estimate is positive definite in finite samples and, as our simulations and empirical analysis show, a large fraction of such estimates in fact fail to be positive definite. Moreover, even if this property does hold, the estimate is often near-singular, resulting in an ill-conditioned and highly unstable covariance matrix.

3.3.3. Subsampling noisy high-frequency data

To construct our estimator in the noisy setting, we follow the procedure from before by splitting the full sample of noisy high-frequency data into subsamples using a blocking approach.

We redefine:

$$B_i(p) = \left\{ j : (i-1)pk_n \leq j \leq ipk_n \right\}, \quad (3.27)$$

where $p \geq 3$ is an integer and $i \geq 1$.

$B_i(p)$ is now the i th block of noisy high-frequency data. As readily seen, the only change compared to the noiseless setting is that $B_i(p)$ uses a larger block size. This implies we can do a sufficient amount of averaging within each block in order to diminish the noise, while still preserving enough of an effective sample size to estimate the correlation structure of $V^*(q, r)^n$.

Then, we set

$$\hat{\Sigma}_n^* = \frac{1}{L} \sum_{l=1}^L \left(\frac{n^{1/4}}{\sqrt{L}} \left(V_l^*(q, r)^n - V^*(q, r)^n \right) \right) \left(\frac{n^{1/4}}{\sqrt{L}} \left(V_l^*(q, r)^n - V^*(q, r)^n \right) \right)', \quad (3.28)$$

where, assuming Lpk_n divides n ,

$$V_l^*(q_k, r_k)^n = \frac{Lpk_n}{n} \sum_{i=1}^{n/Lpk_n} v_{(i-1)L+l}(q_k, r_k)^n, \quad (3.29)$$

$$v_i(q_k, r_k)^n = \frac{1}{pk_n - 2k_n + 2} \sum_{j, j+k_n-1 \in B_i(p)} |n^{1/4} \Delta \bar{Y}_j^n|^{q_k} |n^{1/4} \Delta \bar{Y}_{j+k_n}^n|^{r_k}.$$

As in the above, we should note that the summands $v_i(q_k, r_k)^n$ in the subsample estimates $V_l^*(q_k, r_k)^n$ exploit data solely from $B_i(p)$. Therefore, pre-averaging has to be done locally within the block, so that there is no overlap in the pre-averaged returns across the various blocks.

Theorem 3.5 Assume that $Y_t = X_t + \epsilon_t$ is a noisy diffusion model, where X_t is given by Eq. (2.1) and fulfills Assumption **(H)**. Also, Assumption **(M)** and **(N)** are true. Let $q = (q_1, \dots, q_m)'$ and $r = (r_1, \dots, r_m)'$ be vectors of even non-negative integers. Then, as $n \rightarrow \infty$, $p \rightarrow \infty$, $L \rightarrow \infty$ and $\sqrt{n}/Lp^2 \rightarrow \infty$, it holds that

$$\hat{\Sigma}_n^* - \Sigma^* = \underbrace{O_p\left(\frac{1}{\sqrt{L}}\right)}_{\text{CLT}} + \underbrace{O_p\left(\frac{Lp^2}{\sqrt{n}}\right)}_{\text{blocking}} + \underbrace{O_p\left(\frac{1}{p}\right)}_{\text{HAC}} \quad (3.30)$$

Proof. See Appendix.

As in the previous subsection, the minimal assumptions we need to prove consistency are $n \rightarrow \infty$, $p \rightarrow \infty$, $L \rightarrow \infty$ and $\sqrt{n}/Lp^2 \rightarrow \infty$. The last condition again ensures that a bias term of the statistic $V_l^*(q, r)^n - V^*(q, r)^n$ is negligible with respect to its martingale part.

Now, we achieve the best rate

$$\hat{\Sigma}_n^* - \Sigma^* = O_p(n^{-1/10}), \quad (3.31)$$

by choosing

$$L = O(n^{1/5}) \quad \text{and} \quad p = O(n^{1/10}). \quad (3.32)$$

Thus, the existence of microstructure frictions also adversely affects the speed of convergence of $\hat{\Sigma}_n^*$.

Theorem 3.5 is binding, because it restricts the choice of the powers q_i and r_i to even non-negative integers. But we can actually prove a weaker consistency result for any pre-averaged bipower variation, which is useful for practical work. First, we recall that allowing for general powers $q_i, r_i \geq 0$ by itself requires some stronger assumptions to prove the underlying central limit theorem in Eq. (3.25). In particular, Podolskij and Vetter (2009a) impose that the noise distribution is (i) symmetric with (ii) $\mathbb{E}[|\epsilon|^a] < \infty$ for $a \in (-1, 0)$ (their Assumption **(A)**) and that the noise distribution fulfills Cramer's condition, i.e. $\limsup_{|t| \rightarrow \infty} \chi(t) < 1$, where χ is the characteristic function of ϵ (their Assumption **(A')**). Of course, we also need this. Note, however, that we can again dispense with Assumption **(M)** for consistency.

Theorem 3.6 Assume that $Y_t = X_t + \epsilon_t$ is a noisy diffusion model, where X_t is given by Eq. (2.1) and fulfills Assumption **(V)**, while the noise fulfills Assumption **(N)** and also Assumption **(A)** and **(A')** from Podolskij and Vetter (2009a). Then, as $n \rightarrow \infty$, $p \rightarrow \infty$, $L/p \rightarrow \infty$ and $\sqrt{n}/Lp^2 \rightarrow \infty$, it holds for any $q, r \geq 0$ that

$$\hat{\Sigma}_n^* \xrightarrow{p} \Sigma^*. \quad (3.33)$$

Proof. See Appendix.

Remark An intriguing ability of the subsampling estimator $\hat{\Sigma}_n^*$ is that it tends to be robust against model misspecification of the microstructure noise process, as long as the associated central limit theorem holds. Hautsch and Podolskij (2013) prove a central limit theorem for pre-averaging under m -dependent noise. Here, the form of Σ^* changes, but our estimator $\hat{\Sigma}_n^*$ is still consistent (proof omitted, but available at request). This is because $\hat{\Sigma}_n^*$ imitates the underlying covariation, irrespective of the true microstructure model, and it therefore automatically adapts to the new environment. Kalnina (2011) studies more formally both autocorrelated and heteroscedastic noise for the two-scale realized variance, as we refer to her work for further details.

Remark The subsampling idea can be applied to other estimators, which admit asymptotic mixed normality, such as the two-scale and kernel-based realized variance of Zhang, Mykland, and Ait-Sahalia (2005) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008). The consistency of the subsample estimator of the asymptotic conditional covariance matrix of these estimators has already been proved in prior work (i.e., Ikeda, 2015; Kalnina, 2011, 2012; Kalnina and Linton, 2007; Varneskov, 2014). The optimal choice of the tuning parameters for pre-averaged bipower variation—derived in this paper—should continue to hold in the context of these other estimators, but it needs to be verified formally. We leave this for future work.

4. Simulations

In this section, we conduct a small Monte Carlo study. It takes a closer look at the finite sample properties of covariance matrix estimation by subsampling. Throughout, we restrict attention to the noisy setting and estimation of Σ^* . We examine the ability of our proposed estimator $\hat{\Sigma}_n^*$ to assist in drawing feasible inference about the pre-averaged bipower variation.

The efficient log-price X is simulated as:

$$\begin{aligned} dX_t &= \sigma_t dW_t, \\ d\sigma_t^2 &= \kappa(\sigma^2 - \sigma_t^2)dt + \xi\sigma_t(\rho dW_t + \sqrt{1 - \rho^2}dB_t), \end{aligned} \tag{4.1}$$

where W_t and B_t are independent standard Brownian motions, while κ, σ^2, ξ and ρ are parameters. The process adopted for σ_t^2 is a Heston (1993) model, which is mean-reverting; features square-root volatility; and

accommodates a leverage effect.⁶

To get a version of the model from which we can actually simulate data, we apply a standard Euler approximation to the continuous time formulation in Eq. (4.1). We then simulate 10,000 independent sample path realizations of the discretized system of bivariate equations.⁷ We use two different sample sizes of $n = 2,340$ and $23,400$. In our empirical investigation, we look at high-frequency equity data from NYSE. With a US stock exchange trading session running from 9:30am to 4:00pm—or 6.5 hours—these sample sizes translate into receiving a new price update every ten and one second(s). Our sample sizes are therefore representative of more frequently traded securities.

We assume that the parameter values in the volatility equation are $\kappa = 5$, $\sigma^2 = 0.04$, $\xi = 0.50$ and $\rho = -0.50$, which is broadly consistent with prior work (e.g., Ait-Sahalia, Mykland, and Zhang, 2011; Kalnina, 2011; Wang and Mykland, 2014). This implies that σ_t is about 20% on an average, annualized basis, but the configuration of the model adopted here can generate a substantial degree of intraday variation in volatility via ξ . An example simulation is provided in Figure 2.

An i.i.d. noise term is added to X via $\omega = \gamma\sqrt{\int_0^1 \sigma_s^2 ds/n}$, where γ is the noise-ratio parameter (e.g., Oomen, 2006). This ensures that microstructure noise variation is proportional to the integrated variance of the efficient price. We assume that $\gamma = 0.50$, which is a realistic choice for more liquid assets, see, e.g., Christensen, Oomen, and Podolskij (2014). To alleviate the impact of noise, we pre-average using the bandwidth $k_n = \lceil \theta\sqrt{n} \rceil$, and we experiment with two choices of the tuning parameter $\theta = 1/3$ and 1 . We set the weight function $w(x) = \min(x, 1 - x)$, which has been shown to deliver nearly efficient estimates of the integrated variance, when further parametric assumptions are imposed (see, e.g., Podolskij and Vetter, 2009a).

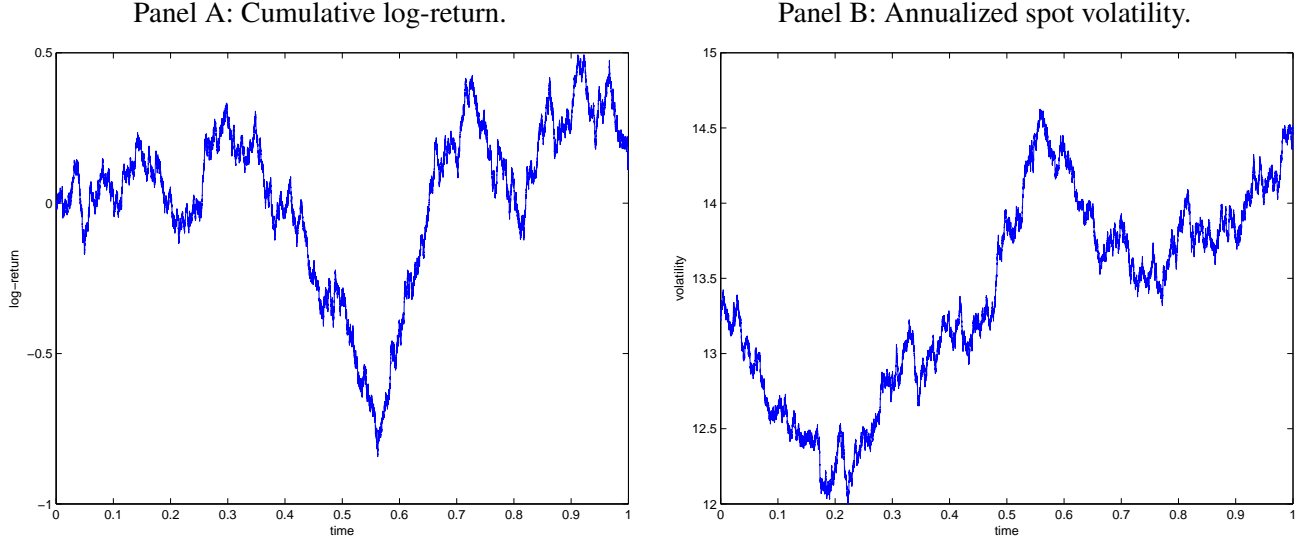
4.1. A preliminary analysis

We begin by verifying the conjecture made in the introduction of this paper, namely that the finite sample properties of existing estimators of Σ^* are poor, which renders a significant fraction of such estimates either

⁶The leverage effect describes a negative correlation between an asset’s return and volatility (e.g., Black, 1976; Christie, 1982). Thus, if a leverage effect is present, one would expect ρ to be negative.

⁷To avoid a systematic effect from an assumed initial condition of volatility, σ_0^2 , we restart the variance process in each simulation by drawing at random from its stationary distribution, $\sigma_t^2 \sim \text{Gamma}(2\kappa\sigma^2\xi^{-2}, 2\kappa\xi^{-2})$.

Figure 2: An illustration of a simulation from the Heston model.



nonpositive definite or at least ill-conditioned. To do this, we implement the estimator suggested by Podolskij and Vetter (2009a). It is constructed by first defining

$$\tilde{Y}_{i,m}^n = \frac{1}{\sqrt{n}} |n^{1/4} \Delta \bar{Y}_m^n|^{q_i} |n^{1/4} \Delta \bar{Y}_{m+k_n}^n|^{r_i}, \quad (4.2)$$

and setting

$$\chi_{ml}^n = \frac{1}{2} \left[\tilde{Y}_{i,m}^n \left(\tilde{Y}_{j,m+l}^n - \tilde{Y}_{j,m+2k_n}^n \right) + \tilde{Y}_{j,m}^n \left(\tilde{Y}_{i,m+l}^n - \tilde{Y}_{i,m+2k_n}^n \right) \right], \quad (4.3)$$

for any $0 \leq m \leq n - 4k_n + 1$ and $0 \leq l \leq 2k_n$. Then,

$$\tilde{\Sigma}_{ij,n}^* = \frac{2}{\sqrt{n}} \sum_{m=0}^{n-4k_n+1} \sum_{l=0}^{2k_n-1} \chi_{ml}^n \xrightarrow{p} \Sigma_{ij}^*, \quad (4.4)$$

and it follows that $\tilde{\Sigma}_n^* = (\tilde{\Sigma}_{ij,n}^*) \xrightarrow{p} \Sigma^*$.

Table 1 shows several diagnostics that highlight the properties of $\tilde{\Sigma}_n^*$ based on $q = (2, 1)'$ and $r = (0, 1)'$, i.e. pre-averaged realized variance and $(1, 1)$ -bipower variation. The results are obtained for a scaled Brownian motion (BM), in which volatility has been fixed at its steady-state value of σ^2 . The column SV is for the Heston stochastic volatility model, while SPY represents some real high-frequency data that are further commented on in Section 5.

Table 1: Proportion of ill-conditioned covariance matrix estimates, $\tilde{\Sigma}_n^*$.

		BM		SV		SPY
$n =$		2,340	23,400	2,340	23,400	n_{actual}
2-dimensional setting						
<i>Panel A: Nonpositive definite</i>						
$\theta =$	0.33	0.178	0.099	0.181	0.111	0.111
	1.00	0.348	0.241	0.347	0.249	0.233
<i>Panel B: Negative variance</i>						
$\theta =$	0.33	0.140	0.070	0.146	0.077	0.064
	1.00	0.292	0.201	0.289	0.206	0.184
<i>Panel C: Condition number ≥ 20</i>						
$\theta =$	0.33	0.069	0.057	0.071	0.062	0.045
	1.00	0.079	0.095	0.081	0.100	0.088
4-dimensional setting						
<i>Panel D: Nonpositive definite</i>						
$\theta =$	0.33	0.453	0.312	0.458	0.314	0.312
	1.00	0.809	0.568	0.806	0.589	0.544

Note. We show the proportion of ill-conditioned covariance matrix estimates, when the estimator $\tilde{\Sigma}_n^*$ from Podolskij and Vetter (2009a) is used. $\tilde{\Sigma}_n^*$ is defined in Eq. (4.4). The columns report results from a Brownian motion (BM) and stochastic volatility (SV) model, where n is the sample size. The simulation design appears in Section 4. SPY is based on real high-frequency data, which are further analyzed in Section 5. n_{actual} denotes the actual sample size, which varies across days, cf. Table 2. In Panel A, we report the fraction of $\tilde{\Sigma}_n^*$ estimates that are nonpositive definite, i.e. with a minimum eigenvalue $\min(\lambda_i) \leq 0$. In Panel B, we report how often the linear combination $\omega = [1, -\mu_1^{-2}]'$ of the pre-averaged bipower variation $V^*(q, r)^n$ leads to a negative variance estimate $\omega' \tilde{\Sigma}_n^* \omega \leq 0$. $V^*(q, r)^n$ is defined in Eq. (3.24). Note that B implies A, but not vice versa. In Panel C, we compute the fraction of the positive definite $\tilde{\Sigma}_n^*$ estimates that return a condition number $\text{cond}(\tilde{\Sigma}_n^*) \geq 20$. Throughout Panels A – C, the table is based on $q = (2, 1)'$ and $r = (0, 1)'$, while Panel D reports the updated numbers from Panel A, after changing the estimation problem to $q = (2, 1, 4, 0)'$ and $r = (0, 1, 0, 2)'$.

In Panel A, we report the fraction of the computed $\tilde{\Sigma}_n^*$, which fail to be positive definite, i.e. which have a minimum eigenvalue $\min(\lambda_i) \leq 0$. It suggests that for a small sample of $n = 2,340$ and depending on θ , between 18% – 35% of $\tilde{\Sigma}_n^*$ are nonpositive definite. As expected, these numbers decrease as the sample size

increases, but even with a fairly large sample of $n = 23,400$ the failure rate is far from negligible. Moreover, it increases if a longer pre-averaging horizon is employed. As such, this issue therefore has substantial bite in practice, because larger values of θ are typically preferred, when the noise is suspected to violate the i.i.d. assumption (e.g., Christensen, Oomen, and Podolskij, 2014; Hautsch and Podolskij, 2013). Note that going from constant to stochastic volatility changes the numbers only slightly, so allowing volatility to be time-varying has no discernable impact on the failure rate.

Turning next to Panel B, we investigate how often the linear combination $\omega'V^*(q, r)^n$ with $\omega = (1, -\mu_1^{-2})'$ results in a negative variance estimate $\omega'\tilde{\Sigma}_n^*\omega \leq 0$. The difference $V^*(2, 0)^n - \mu_1^{-2}V^*(1, 1)^n$ is often used in applied work, as it provides information about presence of jumps in the price process and permits a statistical test of this hypothesis. Even when the covariance matrix estimate is not positive definite, it could still result in a positive variance estimate $\omega'\tilde{\Sigma}_n^*\omega > 0$, thereby allowing the t-statistic to be computed. While the numbers in Panel B are lower compared to Panel A, they are still high.

In Panel C, we look at those $\tilde{\Sigma}_n^*$ estimates that are positive definite. We compute the percentage of these, which return a condition number $\text{cond}(\tilde{\Sigma}_n^*) \geq 20$.⁸ The condition number measures the numerical accuracy of a matrix, and a value above $20 = 10 \times \dim(\tilde{\Sigma}_n^*)$ is generally taken as a sign of an ill-conditioned and nearly singular matrix (e.g., Greene, 2011; Hautsch, Kyj, and Oomen, 2012). As readily seen, we find that about 6% – 10% of the $\tilde{\Sigma}_n^*$ that are deemed ok by a definiteness criteria show signs of being badly scaled.

Lastly, in Panel D we attempt to estimate the joint asymptotic covariance matrix of a 4-dimensional parameter by using $q = (2, 1, 4, 2)'$ and $r = (0, 1, 0, 2)'$. We then report the percentage of the $\tilde{\Sigma}_n^*$ estimates, which are not positive definite, i.e. the numbers can be compared to Panel A. Not surprisingly, increasing the complexity of the problem has a detrimental impact on the estimation errors, and up to 80% of the estimates are now non-positive definite, making this a devastating issue for inference (e.g., Dette, Podolskij, and Vetter, 2006; Dette and Podolskij, 2008; Vetter and Dette, 2012, employ a 3-dimensional statistic to test for the parametric form of volatility in diffusion models (both with or without noise)).

⁸The condition number of an invertible matrix A is defined as $\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$, where $\|\cdot\|$ is the L_2 matrix norm. $\text{cond}(A)$ can be shown to be the ratio between the largest and smallest singular value of A . It can be interpreted as saying how much small changes in the input matrix get amplified under matrix inversion.

4.2. Implementation of the subsampler

We now turn to the subsampler, where we again base our investigation on $V^*(q, r)^n$ using the parameters $q = (2, 1)'$ and $r = (0, 1)'$. As $\hat{\Sigma}_n^*$ depends on two tuning parameters, p and L , we compute it by varying these across a broad range of values in order to gauge the sensitivity of our estimator to specific choices. We set $p = 3, 5$ and 10 , so that the block length of noisy returns before pre-averaging goes from three to ten times the pre-averaging horizon k_n . Moreover, we slice the sample into $L = 5, 10$ and 15 subsamples, yielding a total of nine combinations of p and L .⁹

Our initial set of simulations suggested that, for small p and L , the raw estimator defined by Eq. (3.28) is downward biased, thereby leading to a systematic underestimation of Σ^* . We therefore start by briefly outlining a few corrections that are important in finite samples.

First, in order to center the $V_l^*(q, r)^n$, we should in theory use the unobserved $V^*(q, r)$, which we are forced to replace by a feasible, consistent estimator, i.e. $V^*(q, r)^n$. While this has no impact asymptotically, because $V^*(q, r)^n$ converges much faster than $V_l^*(q, r)^n$, a closer inspection of $\hat{\Sigma}_n^*$ shows that the substitution does entail a standard small sample correction. This implies that $\hat{\Sigma}_n^*$ should be divided by $1 - 1/L$, i.e. the “right” normalization in Eq. (3.28) is $L - 1$ and not L .

Second, there is a HAC error associated with p , which—in contrast to the L correction—is more subtle to deal with. The problem originates from the estimation of the autocovariances of the pre-averaged returns, $|n^{1/4}\Delta\bar{Y}_i^n|^{q_i}|n^{1/4}\Delta\bar{Y}_{i+k_n}^n|^{r_i}$, as exemplified by Eq. (3.14). As such, it depends on the covariance structure of this series, which, in turn, is a function of several parameters and variables, including spot volatility, the weight function and the variance of the noise process (see, e.g., the proof of Theorem 3.5 in the Appendix around Eq. (A.29), or Podolskij and Vetter, 2009a). If we approximate this function, e.g., by assuming that volatility is constant, a detailed calculation (which is omitted here, but available upon request) shows that for the pre-averaged bipower variation estimator $V^*(q_i, r_i)^n$ defined by Eq. (3.24), we can roughly correct for the p error by dividing $\hat{\Sigma}_n^*$ with $1 - 1/p$.

⁹This implies that for the sample size $n = 2, 340$, there are some combinations of p and L , for which there is not enough data to compute the subsampler. Therefore, we restrict attention to $n = 23, 400$ in the following. The results for $n = 2, 340$, when attainable, are not materially worse, reflecting the slow rates of convergence, and all are available by request.

It turns out, however, that this is too much, if either q_i or r_i is zero, as it happens for the pre-averaged realized variance, $V^*(2, 0)^n$. This is because the summands in Eq. (3.24) are, asymptotically, $2k_n$ -dependent for non-zero values of both q_i and r_i , while they are only k_n -dependent, if either is zero. Thus, the bias induced by p in the latter is, loosely speaking, half that of the former. This indicates that we ought to divide the elements of $\hat{\Sigma}_n^*$ involving the variance of $V^*(2, 0)^n$ and its covariance with $V^*(1, 1)^n$ only by $1 - 0.5/p$. This is less appealing, because it would break the positive semi-definiteness property of $\hat{\Sigma}_n^*$. We therefore proceed by using a constant scaling for the entire matrix, and—to strike a balance between the two alternatives—we propose to meet in the middle and rescale $\hat{\Sigma}_n^*$ by $(1 - 0.75/p)$. This choice produces excellent results for the values of p considered in this paper, as corroborated by our numerical experiments below, while for $p \geq 10$ the correction is of limited importance and can be ignored.

Third, our theoretical results hinge on n being a multiple of Lpk_n . In practice, where n varies randomly over time and can be very odd, this is an unrealistic assumption, which is almost never satisfied. Instead, for a given choice of k_n , p and L , the maximum number of blocks of length pk_n that can be assigned to each of the L subsample estimates $V_l^*(q_i, r_i)^n$ is:

$$n_{\text{block}} = \left\lfloor \frac{\lfloor n/pk_n \rfloor}{L} \right\rfloor. \quad (4.5)$$

The effective amount of data used to construct the subsampler is therefore often less than the total sample size, i.e. $n_{\text{block}}Lpk_n \leq n$.

We compute $\hat{\Sigma}_n^*$ from the data that fall within the window $[0, n_{\text{block}}Lpk_n/n]$ and subsequently inflate this estimate to cover the whole unit interval. While this entails some loss of information about the underlying variation of the process towards the end of the sample, in our experience this has a very limited influence on the results, unless the data, from which $\hat{\Sigma}_n^*$ is computed, is not representative of the overall level of volatility. In practice, one can minimize this effect by choosing the parameters, such that $n_{\text{block}}Lpk_n$ is close to n .

4.3. Results

In Figure 3, we plot some kernel smoothed density estimates of the standardized pre-averaged bipower variation, i.e. $n^{1/4}(V^*(q_i, r_i)^n - V^*(q_i, r_i))/\sqrt{\hat{\Sigma}_{ii,n}^*}$, where $\hat{\Sigma}_{ii,n}^*$ is the i th diagonal element of our subsampling covariance matrix estimate $\hat{\Sigma}_n^*$. Here, we use $\theta = 1$. The results in Panels A – B are for $V^*(1, 1)^n$, while Panels

C – D are for $V^*(2, 0)^n$. In addition, the left-hand portion of the figure is for $L = 15$ and p changing, while the right-hand part is based on $p = 10$ and for different L . The infeasible result for $V^*(2, 0)^n$ replaces the subsampler with the true variance, which is known here (cf. footnote 5).

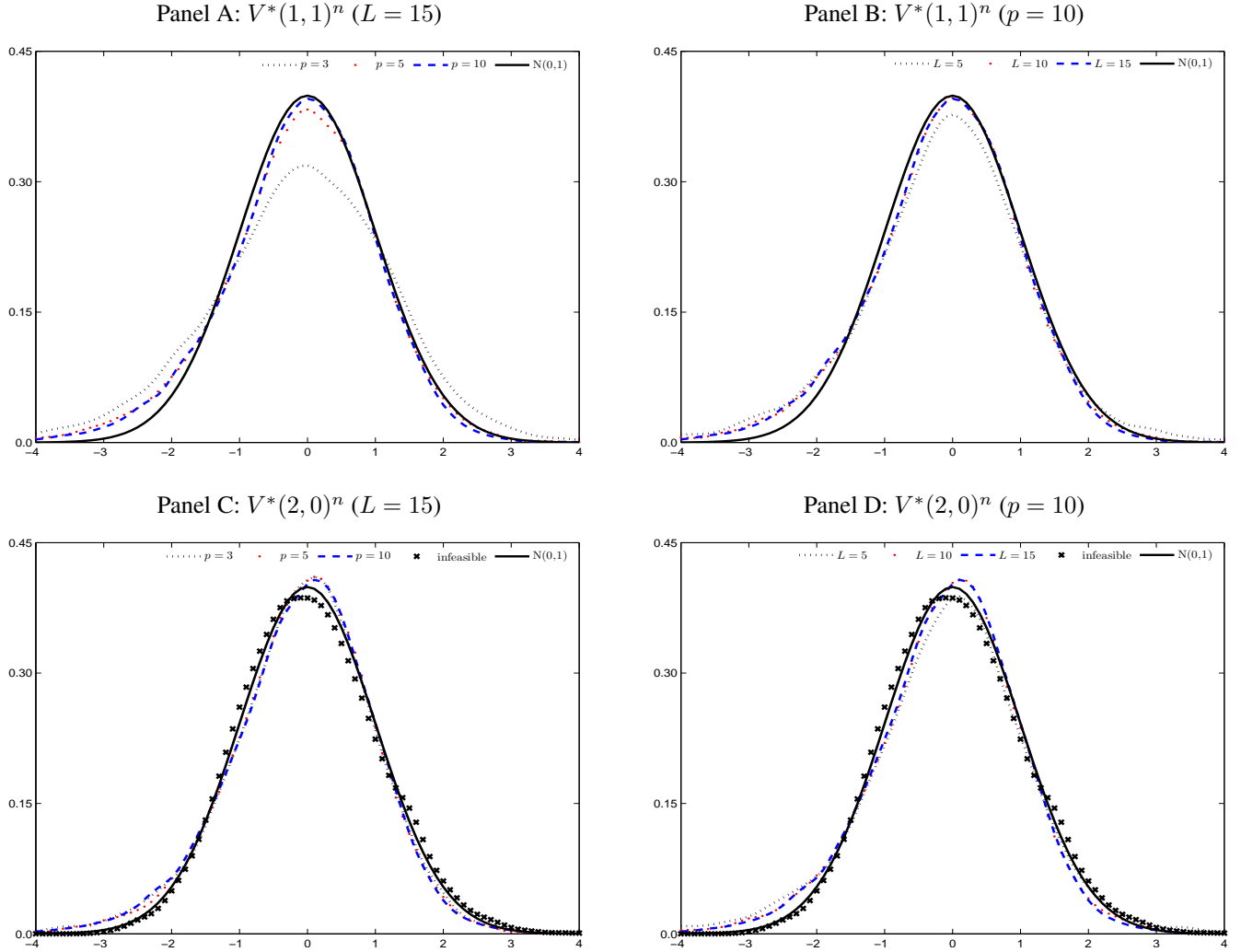
As the figure shows, the studentized pre-averaging estimators tend to track the asymptotic normal approximation closely across combinations of p and L . The sole exception, appearing in Panel A, is for $V^*(1, 1)^n$, when $\hat{\Sigma}_n^*$ is implemented using $p = 3$. Note that if $p = 3$, the effective sample size within a block of noisy high-frequency data is $k_n + 2$ after pre-averaging, whereas the summands of $V^*(1, 1)^n$ are $2k_n$ -dependent. With such a small value of p , the block length is therefore inadequate to permit estimation of all the required autocovariances. As $|n^{1/4}\Delta\bar{Y}_i^n|^{q_i}|n^{1/4}\Delta\bar{Y}_{i+k_n}^n|^{r_i}$ is strongly positively autocorrelated, this leads to a severe underestimation of the true variation of $V^*(1, 1)^n$ and, hence, a pronounced overdispersion of the estimated density. As a practical guide, one should therefore avoid computing $\hat{\Sigma}_n^*$ with $p = 3$, if both q_i and r_i are different from zero, for any $i = 1, \dots, m$. In comparison, the corresponding graph for $V^*(2, 0)^n$ in Panel C is much better scaled, reflecting the lesser dependence inherent in this estimator. Apart from that, the fit tends to improve for larger values of p and L , as expected.¹⁰ Lastly, comparing with the infeasible result in Panels C – D, we note that the estimated densities appear slightly negatively skewed, owing to a modest, positive correlation between $V^*(q_i, r_i)^n$ and $\hat{\Sigma}_{ii,n}^*$.

We turn next to Figure 4, where we explore how sensitive our findings are to the choice of θ . In this figure, and throughout the remainder of this section, we fix the parameters of $\hat{\Sigma}_n^*$ to $p = 10$ and $L = 15$. As apparent from both panels, the relatively large change in θ has only a minuscule effect on the shape of the estimated density for $V^*(1, 1)^n$ and $V^*(2, 0)^n$.

At last, we look at an application that requires one to use information about the full covariance matrix estimate. In Figure 5, using $\theta = 1$, we report some results for the linear combination $\omega'V^*(q, r)^n$ with $\omega = (1, -\mu_1^{-2})'$, i.e. $V^*(2, 0)^n - \mu_1^{-2}V^*(1, 1)^n$. In Table 1, we noted it was problematic to standardize this difference with the covariance matrix estimator put forth by Podolskij and Vetter (2009a), which was often found to be nonpositive definite. The subsampler does not suffer from this issue. In Panel A of Figure 5, we therefore plot

¹⁰Of course, with a fixed sample size n and pre-averaging window k_n , the parameters p , L , and n_{block} are not free. Thus, everything else is not “fixed”, because n_{block} is decreasing for larger values of either p or L (while holding the other fixed).

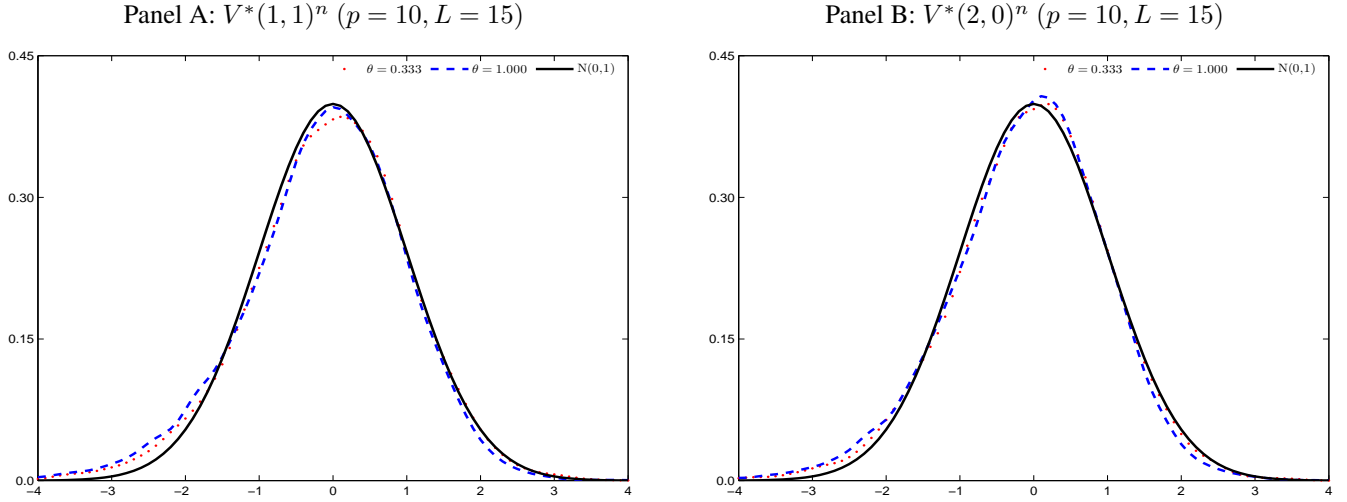
Figure 3: Kernel density estimate of the standardized $V^*(q_i, r_i)^n$: changing p and L .



Note. We show the kernel smoothed density estimates of the standardized pre-averaged bipower variation estimator: $n^{1/4}(V^*(q_i, r_i)^n - V^*(q_i, r_i))/\sqrt{\hat{\Sigma}_{ii,n}^*}$, where $\hat{\Sigma}_{ii,n}^*$ is the i th diagonal element of $\hat{\Sigma}_n^*$. Throughout this figure, $n = 23,400$, $\theta = 1$ and we set $k_n = \lceil \theta\sqrt{n} \rceil$ to implement pre-averaging. The simulation data is from a Heston stochastic volatility model, as described in the main text. In the left panel, the subsampler is based on $L = 15$ subsamples, while varying the block length at $p = 3, 5$ or $10 \times k_n$. The right panel is based on $p = 10$, while changing the number of subsamples at $L = 5, 10$ or 15 . The $n_{\text{sim}} = 10,000$ simulated t-statistics are smoothed using a Gaussian kernel with optimal bandwidth selection $h = 1.06\hat{\sigma}n_{\text{sim}}^{-1/5}$, where $\hat{\sigma}$ is the sample standard deviation of the data. The infeasible result for $V^*(2, 0)^n$ replaces the subsampler with the true variance (cf., footnote 5). The density function of a standard normal random variable (the solid black line) is superimposed as a visual reference.

the time series of the studentized statistic across the simulations runs, using the delta method to conclude that $n^{1/4}\omega'(V^*(q, r)^n - V^*(q, r))/\sqrt{\omega'\hat{\Sigma}_n^*\omega} \xrightarrow{d} N(0, 1)$. Panel B inspects the kernel smoothed density estimate of

Figure 4: Kernel density estimate of the standardized $V^*(q_i, r_i)^n$: changing θ .

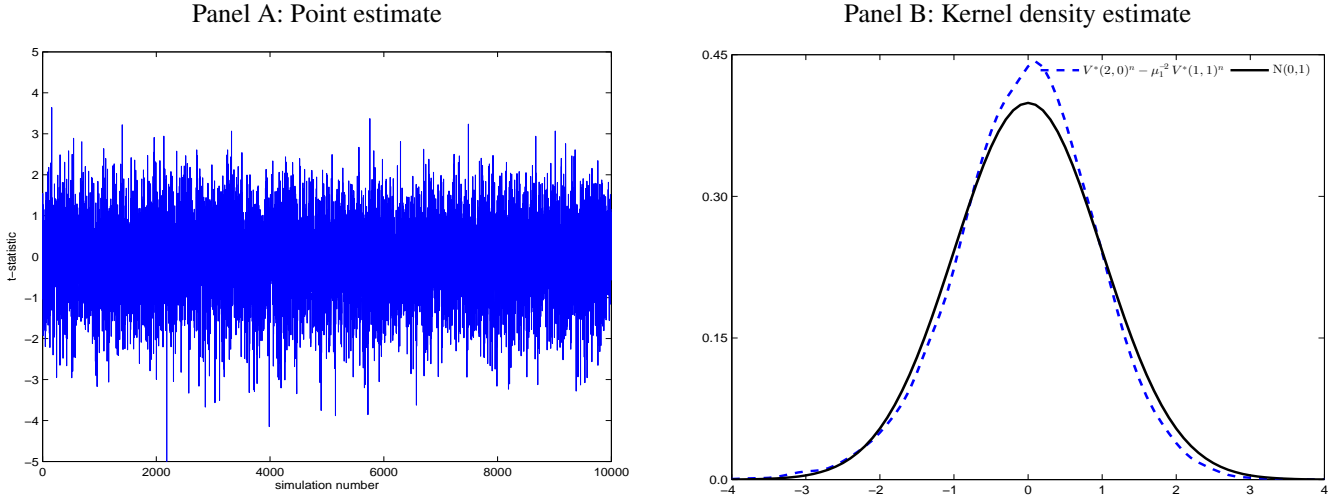


Note. We show the kernel smoothed density estimates of the standardized pre-averaged bipower variation estimator: $n^{1/4}(V^*(q_i, r_i)^n - V^*(q_i, r_i))/\sqrt{\hat{\Sigma}_{ii,n}^*}$, where $\hat{\Sigma}_{ii,n}^*$ is the i th diagonal element of $\hat{\Sigma}_n^*$. Throughout this figure, $n = 23,400$, $L = 15$, $p = 10$, and we set $k_n = \lceil \theta\sqrt{n} \rceil$ to implement pre-averaging using $\theta = 1/3$ and $\theta = 1$. The simulation data is from a Heston stochastic volatility model, as described in the main text. The left panel holds the results for $V^*(1, 1)^n$, while the right panel is for $V^*(2, 0)^n$. The $n_{\text{sim}} = 10,000$ simulated t-statistics are smoothed using a Gaussian kernel with optimal bandwidth selection $h = 1.06\hat{\sigma}n_{\text{sim}}^{-1/5}$, where $\hat{\sigma}$ is the sample standard deviation of the data. The density function of a standard normal random variable (the solid black line) is superimposed as a visual reference.

the t-statistic. As evident, the asymptotic distribution theory is a decent description of the actual finite sample variation, although the fit is not perfect. As explained above, $V^*(2, 0)^n - \mu_1^{-2}V^*(1, 1)^n$ provides information about the presence of jumps in asset prices, and significant positive values would lend support to this hypothesis. Here, the shape of the estimated density implies that the one-sided coverage probabilities of the t-statistic are slightly too large in the right tail. This would render such a hypothesis test mildly conservative, which is preferable in practice (e.g., using the 95% quantile from the standard normal distribution gives a coverage rate of 96.6% in the above figure). We shall explore this further in a companion paper, as it requires additional properties of $\hat{\Sigma}_n^*$ for such a test to retain any power under the alternative (i.e., $\hat{\Sigma}_n^*$ is not jump-robust in its current form, as remarked in Section 3.1). The result could of course change with a different, albeit related, estimator of Σ^* .

Overall, the simulation results suggest that inference based on $\hat{\Sigma}_n^*$ is fairly robust and delivers excellent outcomes, even for modest values of its tuning parameters.

Figure 5: Properties of the standardized $V^*(2, 0)^n - \mu_1^{-2}V^*(1, 1)^n$.



Note. We plot $\omega'V^*(q, r)^n$ with $\omega = (1, -\mu_1^{-2})'$, after it has been standardized by $\hat{\Sigma}_n^*$ based on $p = 10$ and $L = 15$, i.e. $n^{1/4}(V^*(2, 0)^n - \mu_1^{-2}V^*(1, 1)^n)/\sqrt{\omega'\hat{\Sigma}_n^*\omega}$. In Panel A, we plot the point estimates of this t-statistic across simulations, while Panel B displays the corresponding kernel smoothed density estimate. Throughout the figure, $n = 23,400$, and we set $k_n = \lceil \theta\sqrt{n} \rceil$ to implement pre-averaging using $\theta = 1$. The simulation data is from a Heston stochastic volatility model, as described in the main text. In Panel B, the $n_{\text{sim}} = 10,000$ simulated t-statistics are smoothed using a Gaussian kernel with optimal bandwidth selection $h = 1.06\hat{\sigma}n_{\text{sim}}^{-1/5}$, where $\hat{\sigma}$ is the sample standard deviation of data. The density function of a standard normal random variable (the solid black line) is superimposed as a visual reference.

5. Empirical work

Here, we provide a brief illustration of the subsample estimator in the context of some real financial high-frequency data. We analyze tick-data from the Standard and Poor's depository receipts, which is an exchange-traded fund that tracks the performance of the S&P 500 stock index. The shares are listed on several U.S. stock exchanges and trade under the ticker symbol SPY. It is a highly liquid security and provides a good starting point for the subsampler, which is data-intensive. We extracted a transaction price series of the SPY from the TAQ database. The data are recorded at milli-second precision and our complete sample covers the time period from January, 2007 to March, 2011; a total of 1,169 business days. The raw data was filtered for outliers using the recommendations of Christensen, Oomen, and Podolskij (2014).¹¹ We also restrict attention to the NYSE

¹¹The Christensen, Oomen, and Podolskij (2014) filter is to a large extent based on the cleaning routines of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009). The former use a backward-forward matching algorithm to compare a trade to the quote conditions prevailing in the market around the time of the transaction. The latter evaluate each trade against a single preceding

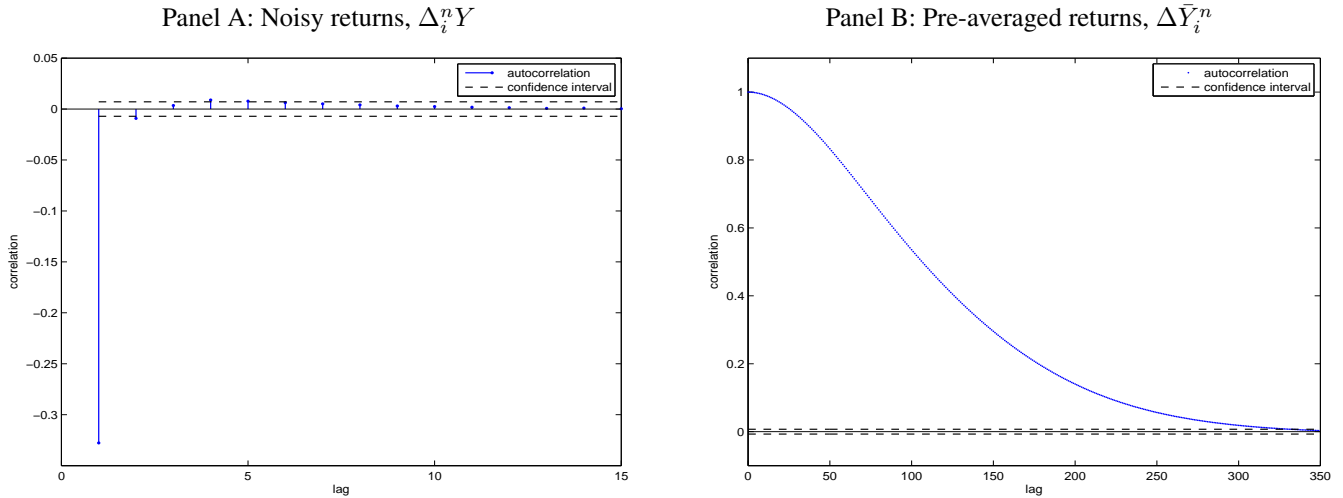
trading session, which runs from 9:30am till 4:00pm Eastern Time. Table 2 provides a few descriptive statistics that summarize key features of the dataset.

Table 2: Summary statistics of the SPY high-frequency data.

Statistic	Sample average	[Min; Max]
r_{oc}	0.003	[-8.254;7.349]
$\hat{\sigma}_{r_{oc}}$	17.037	[3.474;127.923]
n	113.769	[13.127;533.203]
K	320	[115;730]

Note. We report some descriptive statistics of the SPY high-frequency data. r_{oc} is the open-to-close return (in percent), i.e. the difference between the log-price of the last and first transaction of the day. $\hat{\sigma}_{r_{oc}}$ is a realized measure of the standard deviation of r_{oc} . We set $\hat{\sigma}_{r_{oc}} = 100 \times \sqrt{250 \times \widehat{IV}}$, where \widehat{IV} is defined in Eq. (5.1). n is the sample size (in 1,000s), while K is the pre-averaging window. The sample period is January, 2007 through March, 2011.

Figure 6: Autocorrelation function of SPY return series.

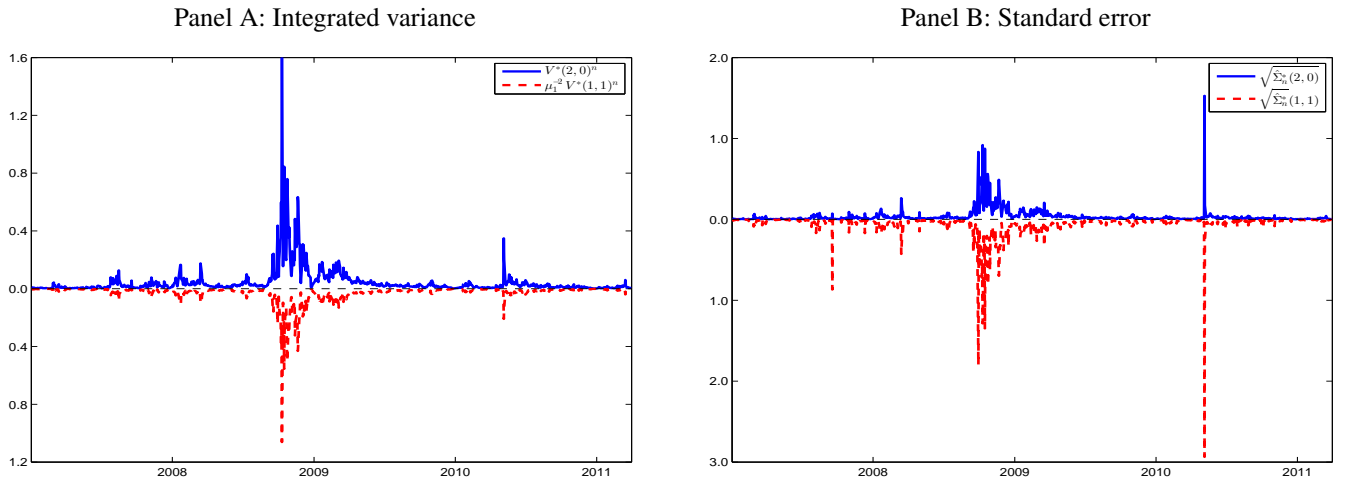


Note. We compute the empirical autocorrelation function (acf) of the SPY returns. Panel A is for the noisy returns (defined in Eq. (3.19)), while Panel B is for the pre-averaged returns (defined in Eq. (3.23)). The acf is estimated daily and then averaged over time. The sample period covers January, 2007 through March, 2011. The dashed line represents a 95% confidence interval for assessing the white noise null hypothesis.

In Panel A of Figure 6, we plot the sample autocorrelation function (acf) of the noisy return series, $\Delta_i^n Y$, up to lag 15. There is a pronounced, significantly negative first-order autocorrelation of about -0.35, which is due to the bid-ask quote, which may lead to excessive removal of data in fast-moving markets. Apart from that, the filters are identical.

is consistent with a bid-ask bounce interpretation of microstructure noise. The acf then increases and turns positive at lag three. The fourth and fifth autocorrelation actually fall outside the 95% confidence bands based on a white noise null hypothesis. Together with the subsequent monotonic decay of the acf, this indicates that noise operating at the tick-level is not i.i.d., as consistent with prior work (e.g., Hansen and Lunde, 2006b). We therefore proceed using a pre-averaging window based on $\theta = 1$, which should be a robust choice in light of the empirical evidence. The acf of the corresponding pre-averaged returns, $\Delta\bar{Y}_i^n$, is presented in Panel B of the figure. As expected, there is a strong dependence in this series up to lag k_n .

Figure 7: Time series of integrated variance estimates and standard error.



Note. In Panel A, we report the time series of the daily $V^*(2,0)^n$ and $\mu_1^{-2}V^*(1,1)^n$ estimates. The series were transformed into measures of the daily integrated variance, as detailed in Eq. (5.1). In Panel B, we plot the associated standard errors, based on $\sqrt{\hat{\Sigma}_n^*(2,0)}$ and $\sqrt{\hat{\Sigma}_n^*(1,1)}$. We use $p = 10$ and $L = 15$ to implement the subsampler. The sample period covers January, 2007 through March, 2011. To facilitate the readability of the figure, the series based on $\mu_1^{-2}V^*(1,1)^n$ and $\sqrt{\hat{\Sigma}_n^*(1,1)}$ are reflected in the x -axis.

We report the resulting time series of $V^*(2,0)^n$ and $\mu_1^{-2}V^*(1,1)^n$ in Panel A of Figure 7. Note that the graph for $\mu_1^{-2}V^*(1,1)^n$ has been reflected in the x -axis. The statistics are first computed day-by-day across the whole sample and subsequently updated using Eq. (3.26) to provide annualized measures of the integrated variance (assuming 250 trading days p.a., on average), i.e.:

$$\widehat{\text{IV}} = \frac{V^*(2,0)^n}{\theta\psi_2^{k_n}} - \frac{\psi_1^{k_n}\hat{\omega}^2}{\theta\psi_2^{k_n}} \xrightarrow{p} \int_0^1 \sigma_s^2 ds, \quad (5.1)$$

with an identical transformation of $\mu_1^{-2}V^*(1, 1)^n$.

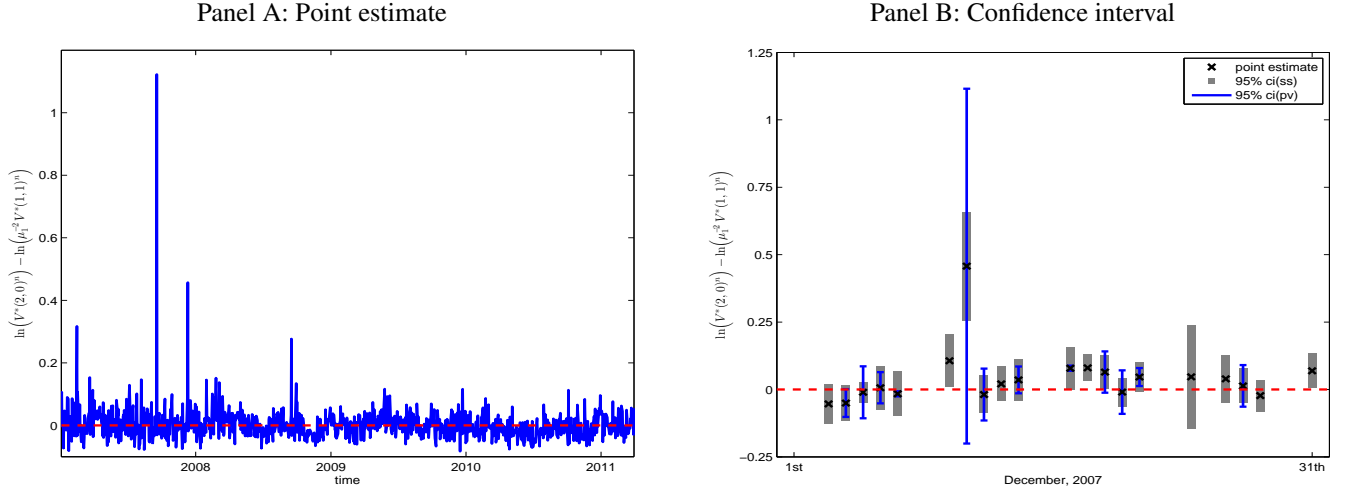
The term $\frac{\psi_1^{k_n} \hat{\omega}^2}{\theta \psi_2^{k_n}}$ is a small bias correction that compensates the pre-averaged bipower variation for the residual effect of microstructure noise. $\hat{\omega}^2$ is an estimate of the noise variance, ω^2 . There are several estimators, which can serve the role of $\hat{\omega}^2$ (see, e.g., Gatheral and Oomen, 2010). Among these, we adopt the one from Oomen (2006), which relies on the first-order autocorrelation of the noisy returns:

$$\hat{\omega}^2 = -\frac{1}{n-1} \sum_{i=1}^{n-1} \Delta_i^n Y \Delta_{i+1}^n Y \xrightarrow{p} \omega^2. \quad (5.2)$$

We find a high degree of time-variation and persistence in the \widehat{IV} series. The onset of the financial crisis and—in particular—the unprecedented volatility surrounding the collapse of Lehman Brothers in 2008 stands out visibly. To attach a measure of uncertainty to these estimates, Panel B charts the associated standard error estimate, $\sqrt{\hat{\Sigma}_{11,n}^*}/\theta \psi_2^{k_n}$ and $\sqrt{\hat{\Sigma}_{22,n}^*}/\theta \psi_2^{k_n}$, where the latter are based on the subsampler with $L = 15$ and $p = 10$. As expected, high levels of volatility spill over into the standard errors and tend to decrease estimation accuracy. The apparent outliers showing up in the standard error series in Q3, 2007 and Q2, 2010, correspond to single days with unusual market activity. The first is September 18, 2007, where the Federal Open Market Committee (FOMC) announced an unexpected reduction of its target for the federal funds rate by 50 basis points, while the second is May 6, 2010; the day of the S&P 500 Flash Crash.

Turn next to Figure 8, where we conduct inference about $V^*(2, 0)^n - \mu_1^{-2}V^*(1, 1)^n$. In Panel A, we compute the difference in the logarithms of these numbers, i.e. $\ln(V^*(2, 0)^n) - \ln(\mu_1^{-2}V^*(1, 1)^n)$, which tends to be more stable over time compared to the raw statistic. As shown, the majority of the point estimates hover around zero, which is the theoretical limit in diffusion models. There are some notable exceptions though, and in Panel B we examine one of these by zooming in on the month of December, 2007. Alongside the statistic, we here report a two-sided 95% confidence interval. Standard errors were found by applying the delta method (for the function $f(x, y) = \ln(y) - \ln(\mu_1^{-2}x)$) to the joint asymptotic distribution in Eq. (3.25) and then replacing the asymptotic variance of the difference by a feasible estimate. In particular, we compare a set of intervals based on the subsampler, $\hat{\Sigma}_n^*$, with those computed from the Podolskij and Vetter (2009a) estimator, $\tilde{\Sigma}_n^*$. If the latter leads to a negative variance estimate, it is excluded. As consistent with Table 1, this is a recurrent problem. Moreover, if both estimates are well-defined, they are often closely aligned, but the subsampler appears notably

Figure 8: Inference about $\ln(V^*(2, 0)^n) - \ln(\mu_1^{-2}V^*(1, 1)^n)$.



Note. We compute the difference $\ln(V^*(2, 0)^n) - \ln(\mu_1^{-2}V^*(1, 1)^n)$, which shows potential violations of the assumed continuous sample path model. $V^*(2, 0)^n$ and $V^*(1, 1)^n$ are defined in Eq. (3.24). In Panel A, we plot the time series of the daily estimates of this number across the sample, which covers January, 2007 through March, 2011. In Panel B, we add a two-sided 95% confidence interval for the log-difference during the month of December, 2007. The standard errors are found by applying the delta method to the joint asymptotic distribution in Eq. (3.25). We replace the asymptotic covariance matrix by the subsampler (box), $\tilde{\Sigma}_n^*$, and the estimator proposed in Podolskij and Vetter (2009a) (whisker), $\tilde{\Sigma}_n^*$. The former is implemented by setting $p = 10$ and $L = 15$. In both panels, the dashed line represents the limiting value in a pure diffusion model.

more stable, while $\tilde{\Sigma}_n^*$ is often very narrow or wide. This is most visible from the big discrepancy on December 11, 2007, marking a day with yet another rate cut by the Fed. On this day, the condition number of $\tilde{\Sigma}_n^*$ is $\text{cond}(\tilde{\Sigma}_n^*) = 452.36$, which suggest that the underlying covariance matrix estimate is very fragile. Meanwhile, the condition number of the subsampler is more modest at $\text{cond}(\tilde{\Sigma}_n) = 15.16$, and it generally appears more stable and reliable over time.

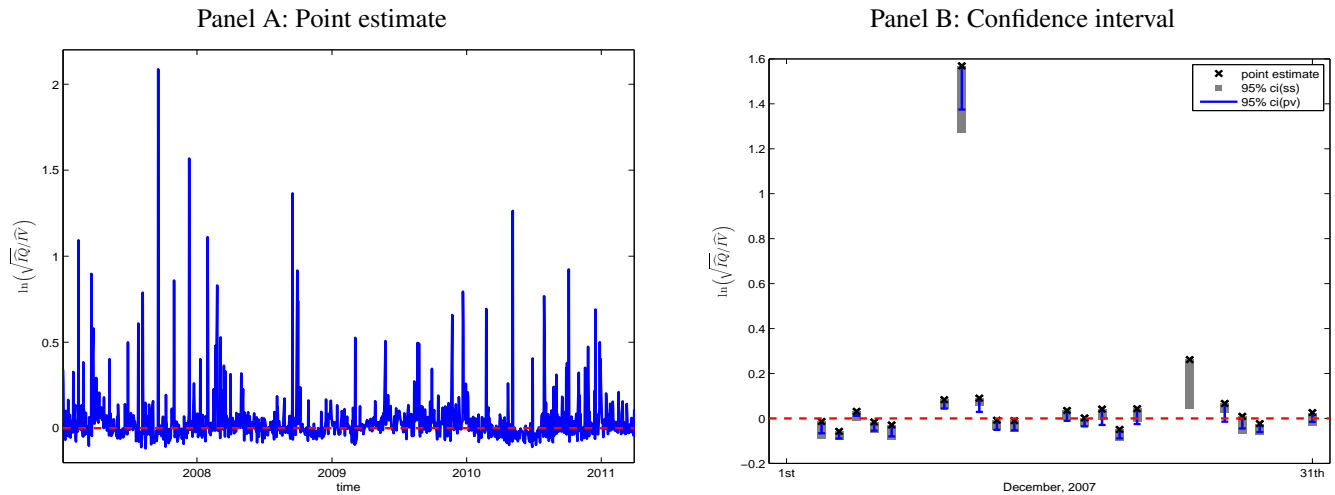
To end the paper, we provide an alternative application, where the subsampler is used to draw inference about the amount of heteroscedasticity in noisy high-frequency data. To do this, we start by computing the statistics $V^*(2, 0)^n$ and $V^*(4, 0)^n$, i.e. the pre-averaged bipower variation based on the parameter $q = (4, 2)'$ (and $r = (0, 0)'$). Taking these as input, we appeal to Eq. (3.26) by forming the estimate:

$$\widehat{\mathbf{IQ}} = \frac{\mu_4^{-1}V^*(4, 0)^n}{(\theta\psi_2^{k_n})^2} - \frac{2\psi_2^{k_n}\psi_1^{k_n}\hat{\omega}^2}{(\theta\psi_2^{k_n})^2}\widehat{\mathbf{IV}} - \frac{(\psi_1^{k_n}\hat{\omega}^2)^2}{(\theta^2\psi_2^{k_n})^2} \xrightarrow{p} \int_0^1 \sigma_s^4 ds, \quad (5.3)$$

which converges to the so-called integrated quarticity. We then exploit that $\sqrt{\widehat{\mathbf{IQ}}/\widehat{\mathbf{IV}}} \xrightarrow{p} \sqrt{\int_0^1 \sigma_s^4 ds / \int_0^1 \sigma_s^2 ds} \geq$

1, with equality if and only if σ is constant. Thus, an estimated ratio far above one suggests there is significant variation in volatility within the day, while a ratio close to one means σ can be regarded, as if it was approximately constant. This type of statistic has been used in earlier work to test for the parametric form of volatility, e.g., Dette, Podolskij, and Vetter (2006) and Vetter and Dette (2012).

Figure 9: Inference about $\ln\left(\sqrt{\widehat{IQ}/\widehat{IV}}\right)$.



Note. We compute the log-ratio $\ln\left(\sqrt{\widehat{IQ}/\widehat{IV}}\right)$, which measures the degree of heteroscedasticity in σ within the day. \widehat{IV} and \widehat{IQ} are defined in Eqs. (5.1) and (5.3). In Panel A, we plot the time series of the daily estimates of this number across the sample, which covers January, 2007 through March, 2011. In Panel B, we add a left-sided 95% confidence interval for the log-ratio during the month of December, 2007 (the upper end of the interval extending to $+\infty$ is not shown). The standard errors are found by applying the delta method to the joint asymptotic distribution in Eq. (3.25). We replace the asymptotic covariance matrix by the subsampler (box), $\widehat{\Sigma}_n^*$, and the estimator proposed in Podolskij and Vetter (2009a) (whisker), $\widetilde{\Sigma}_n^*$. The former is implemented by setting $p = 10$ and $L = 15$. In both panels, the dashed line represents the limiting value in a constant volatility model.

The outcome of this exercise is collected in Figure 9. In Panel A, we plot the time series of the estimated log-ratio, i.e. $\ln\left(\sqrt{\widehat{IQ}/\widehat{IV}}\right)$. We again use a log-transformation in order to improve the scaling of the results and facilitate interpretation of the graphs. The log-ratio should cluster around zero, if volatility is constant. We observe an extreme degree of fluctuation in this statistic over time, and, as anticipated, there are many days, where the log-ratio is large. Still, we also find a decent portion of estimates, which are close to zero. Small negative numbers can be observed as a result of sampling variation. In Panel B, we complement the analysis by looking at the month of December, 2007. We add a left-sided 95% confidence interval for $\ln\left(\sqrt{\widehat{IQ}/\widehat{IV}}\right)$,

where standard errors are again retrieved via the delta method and two estimates of the asymptotic covariance matrix. The interpretation is that on some days, such as the day of the FOMC meeting, volatility is changing a lot, while on others it is not moving much, which is consistent with the findings of Mykland and Zhang (2014). Of course, the latter finding can also arise, if the high-frequency data is not informative enough to discriminate random sampling errors from genuine parameter variation in σ , which could be difficult in times of severe stress in financial markets. In this respect, it is important to acknowledge the limitations of the subsampler, which, albeit consistent, is itself subject to a substantial degree of sampling uncertainty in practice.

6. Conclusion

In this paper, we propose a subsample estimator of the asymptotic conditional covariance matrix of bipower variation. The theory is developed for diffusion models both with and without microstructure noise. We show our estimator is consistent and, under suitable conditions, we find an error decomposition of our statistic, from which we are able to derive its rate of convergence. To complement the theory, we conduct a Monte Carlo study, which documents how the subsampler can be used to draw feasible inference about pre-averaged bipower variation in the presence of noise. The results are compelling and show that the subsampler delivers accurate inference and that it is robust to the choice of its tuning parameters, i.e. the number of subsamples and the block length. We concluded with an empirical analysis that provides an illustration of a few of the directions, in which the subsampler can be applied to real high-frequency data.

In future research, several extensions of the current paper are possible. First, as we noted here, we can use the subsampler as an ingredient to test for the presence of jumps in noisy high-frequency data. This requires the subsampler to be robust against such jumps, which the current implementation is not. This is a line of research that we are currently looking into ourselves in a companion paper. Second, there is some work to be done on the optimal selection of tuning parameters. Finally, as we stressed in the main text, pre-averaging can not only be used to consistently estimate integrated variance and quarticity, but also more general functionals of volatility. Meanwhile, subsampling facilitates estimation of the asymptotic covariance matrix of such pre-averaged high-frequency statistics under mild conditions on the noise process. The combination is therefore potent, and we envision that a lot of empirical papers can benefit from and follow in the aftermath of our work.

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A. Appendix of proofs

In this appendix, we prove the theoretical results that are found in the main text. Without loss of generality, we assume that the processes $a, \sigma, \tilde{a}, \tilde{\sigma}$ and \tilde{v} are bounded. This follows from a standard localization procedure, e.g., BGJPS6. Also, C (or C_p , if it depends on a parameter p) denotes a generic constant, whose value changes from line to line. And, due to the polarization identity, we can (and shall) assume throughout that $m = 1$, so that all statistics are 1-dimensional.

A.1. Proof of Theorem 3.1

We start by introducing some notation. We define:

$$\alpha_i^n = \sqrt{n}\sigma_{\frac{i-1}{n}}\Delta_i^n W \quad \text{and} \quad \chi_i^n = f(\alpha_i^n) - \mathbb{E}\left[f(\alpha_i^n) \mid \mathcal{F}_{\frac{i-1}{n}}\right]. \quad (\text{A.1})$$

We note that α_i^n is a first-order approximation of $\sqrt{n}\Delta_i^n X$. The next lemma is shown in BGJPS6.

Lemma A.1 *Assume that $p \geq 2$ and let h be any function of polynomial growth. Then,*

$$\mathbb{E}[|\alpha_i^n|^p] + \mathbb{E}[|\sqrt{n}\Delta_i^n X|^p] + \mathbb{E}[|h(\alpha_i^n)|^p] + \mathbb{E}[|\chi_i^n|^p] \leq C_p, \quad (\text{A.2})$$

and

$$\mathbb{E}[|\sqrt{n}\Delta_i^n X - \alpha_i^n|^p] \leq C_p n^{-p/2}. \quad (\text{A.3})$$

In the proofs, we make use of Burkholder's inequality several times. This inequality basically says that for any process Z of the form in Eq. (2.1) and for any $p \geq 2$:

$$\mathbb{E}[|Z_t - Z_s|^p] \leq C_p |t - s|^{p/2}. \quad (\text{A.4})$$

Following the comments above, the definition of $\hat{\Sigma}_n$ here collapses to:

$$\hat{\Sigma}_n = \frac{1}{L} \sum_{l=1}^L \left(\sqrt{\frac{n}{L}} \left(V_l(f)^n - V(f)^n \right) \right)^2,$$

To estimate $\hat{\Sigma}_n - \Sigma$, we make the following approximations:

$$\Sigma_n = \frac{1}{L} \sum_{l=1}^L \left(\sqrt{\frac{n}{L}} \left(V_l(f)^n - V(f) \right) \right)^2, \quad Q_n = \frac{1}{n} \sum_{l=1}^L \left(\sum_{i=1}^{n/L} \chi_{(i-1)L+l}^n \right)^2,$$

$$U_n = \frac{1}{n} \sum_{l=1}^L \sum_{i=1}^{n/L} (\chi_{(i-1)L+l}^n)^2, \quad R_n = \frac{1}{n} \sum_{l=1}^L \sum_{i=1}^{n/L} \mathbb{E} \left[(\chi_{(i-1)L+l}^n)^2 \mid \mathcal{F}_{\frac{(i-1)L+l-1}{n}} \right].$$

The goal is then to find an optimal upper bound on the error entailed by these approximations. In particular, the proof is completed, if we can show that under the conditions of Theorem 3.1:

$$\mathbb{E}[|\Sigma_n - Q_n|] \leq C \left(\frac{L}{n} + \frac{1}{\sqrt{n}} \right), \quad (\text{i})$$

$$\mathbb{E}[|Q_n - U_n|] \leq \frac{C}{\sqrt{L}}, \quad (\text{ii})$$

$$\mathbb{E}[|U_n - R_n|] \leq \frac{C}{\sqrt{n}}, \quad (\text{iii})$$

$$\mathbb{E}[|R_n - \Sigma|] \leq \frac{C}{n}, \quad (\text{iv})$$

$$\mathbb{E}[|\hat{\Sigma}_n - \Sigma_n|] \leq \frac{C}{L}. \quad (\text{v})$$

We prove these estimates in the order (iii), (v), (ii), (iv), and (i); the last step being the hardest.

Proof of (iii). This part is almost trivial. We note that

$$U_n - R_n = \frac{1}{n} \sum_{i=1}^n (\chi_i^n)^2 - \mathbb{E} \left[(\chi_i^n)^2 \mid \mathcal{F}_{\frac{i-1}{n}} \right],$$

which is a sum of martingale differences. Lemma A.1 implies that $\mathbb{E} \left[(\chi_i^n)^4 \right] \leq C$. Then, we are done, as

$$\mathbb{E}[|U_n - R_n|^2] \leq \frac{C}{n}. \quad (\text{A.5})$$

■

Proof of (v). First, the identity

$$\sum_{l=1}^L V_l(f)^n = LV(f)^n,$$

together with some simple algebra imply that

$$\hat{\Sigma}_n - \Sigma_n = -\frac{n}{L} (V(f)^n - V(f))^2.$$

Then, we finish the proof of (v) by using that

$$\mathbb{E}\left[(V(f)^n - V(f))^2\right] \leq \frac{C}{n},$$

which has already been established in prior work cited above. ■

Proof of (ii). We define $S_l^m = \sum_{i=1}^m \chi_{(i-1)L+l}^n$ and $T_l^m = \sum_{i=1}^m (\chi_{(i-1)L+l}^n)^2$ and observe that

$$Q_n - U_n = \frac{1}{n} \sum_{l=1}^L A_l^n,$$

where, for each l , we use the notation

$$A_l^n = \left(S_l^{n/L}\right)^2 - T_l^{n/L} = \sum_{i,j=1, i \neq j}^{n/L} \chi_{(i-1)L+l}^n \chi_{(j-1)L+l}^n.$$

As $A_{l_1}^n$ and $A_{l_2}^n$ are uncorrelated for every $l_1 \neq l_2$, we find that

$$\begin{aligned} \mathbb{E}[(Q_n - U_n)^2] &= \frac{1}{n^2} \sum_{l=1}^L \mathbb{E}[(A_l^n)^2] \\ &\leq \frac{C}{n^2} \sum_{l=1}^L \left(\mathbb{E}[(S_l^{n/L})^4] + \mathbb{E}[(T_l^{n/L})^2] \right). \end{aligned} \tag{A.6}$$

We note that $(S_l^m)_{m=1}^{n/L}$ is a discrete martingale for each fixed l . Then, the discrete Burkholder and Cauchy-Schwarz inequalities together with Lemma A.1 imply that

$$\mathbb{E}[(S_l^{n/L})^4] \leq C \mathbb{E} \left[\left(\sum_{i=1}^{n/L} (\chi_{(i-1)L+l}^n)^2 \right)^2 \right] \leq C \left(\frac{n}{L} \right)^2. \tag{A.7}$$

We conclude the proof by using Lemma A.1 once again, but for the $T_l^{n/L}$ term. This yields:

$$\mathbb{E}[(Q_n - U_n)^2] \leq \frac{C}{L}. \tag{A.8}$$

Proof of (iv). We start by noting that

$$\mathbb{E} \left[(\chi_{(i-1)L+l}^n)^2 \mid \mathcal{F}_{\frac{(i-1)L+l-1}{n}} \right] = \rho_{\sigma_{\frac{(i-1)L+l-1}{n}}}(f^2) - \rho_{\sigma_{\frac{(i-1)L+l-1}{n}}}^2(f),$$

so R_n is a Riemann approximation of Σ , because

$$R_n = \frac{1}{n} \sum_{i=1}^n \rho_{\sigma_{\frac{i-1}{n}}} (f^2) - \rho_{\sigma_{\frac{i-1}{n}}}^2 (f).$$

In the remainder of the proof, we suppress the dependence on f by defining the function $\tau(x) = \rho_x(f^2) - \rho_x^2(f)$.

Now, the mapping:

$$\phi(x) \equiv \rho_x(f) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi x^2}} \exp\left(-\frac{y^2}{2x^2}\right) f(y) dy,$$

is a smooth function, as f has polynomial growth. Hence, it follows that τ too is smooth. Then, Taylor's theorem and the inequality in Eq. (A.4) applied to σ mean that

$$R_n - \Sigma = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} [\tau_{\sigma_{\frac{i-1}{n}}} - \tau_{\sigma_s}] ds = \sum_{i=1}^n \mu_i^n(1) + \sum_{i=1}^n \mu_i^n(2) + O_p\left(\frac{1}{n}\right),$$

where

$$\mu_i^n(1) = -\tau'(\sigma_{\frac{i-1}{n}}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\int_{\frac{i-1}{n}}^s \tilde{a}_u du \right) ds \quad \text{and} \quad \mu_i^n(2) = -\tau'(\sigma_{\frac{i-1}{n}}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\int_{\frac{i-1}{n}}^s \tilde{\sigma}_u dW_u + \int_{\frac{i-1}{n}}^s \tilde{v}_u dB_u \right) ds.$$

We find that:

$$\mathbb{E}[|\mu_i^n(1)|^2] \leq \frac{C}{n^4} \quad \text{and} \quad \mathbb{E}[|\mu_i^n(2)|^2] \leq \frac{C}{n^3}. \quad (\text{A.8})$$

Using the martingale difference property, this implies that

$$\mathbb{E}\left[\left| \sum_{i=1}^n \mu_i^n(2) \right|^2 \right] = \mathbb{E}\left[\sum_{i=1}^n |\mu_i^n(2)|^2 \right] \leq \frac{C}{n^2}. \quad (\text{A.9})$$

The result then follows from Eqs. (A.8) – (A.9) and the Cauchy-Schwarz inequality. ■

To prove Eq. (i), we need a bit of preparation. We let

$$\tilde{V}_l(f)^n = \frac{1}{n/L} \sum_{i=1}^{n/L} f(\alpha_{(i-1)L+l}^n) \quad \text{and} \quad \hat{V}_l(f)^n = \frac{1}{n/L} \sum_{i=1}^{n/L} \mathbb{E}\left[f(\alpha_{(i-1)L+l}^n) \mid \mathcal{F}_{\frac{(i-1)L+l-1}{n}} \right].$$

With the decomposition

$$V_l(f)^n - V(f) = \left(V_l(f)^n - \tilde{V}_l(f)^n \right) + \left(\tilde{V}_l(f)^n - \hat{V}_l(f)^n \right) + \left(\hat{V}_l(f)^n - V(f) \right),$$

we get that

$$\Sigma_n - Q_n = D_n^{(1)} + D_n^{(2)} + D_n^{(3)} + D_n^{(4)},$$

where

$$D_n^{(1)} = \frac{2n}{L^2} \sum_{l=1}^L \left(V_l(f)^n - \tilde{V}_l(f)^n \right) \left(\tilde{V}_l(f)^n - V(f) \right),$$

$$D_n^{(2)} = \frac{2n}{L^2} \sum_{l=1}^L \left(\hat{V}_l(f)^n - V(f) \right) \left(\tilde{V}_l(f)^n - \hat{V}_l(f)^n \right),$$

$$D_n^{(3)} = \frac{n}{L^2} \sum_{l=1}^L \left(V_l(f)^n - \tilde{V}_l(f)^n \right)^2,$$

$$D_n^{(4)} = \frac{n}{L^2} \sum_{l=1}^L \left(\hat{V}_l(f)^n - V(f) \right)^2.$$

To provide an estimate of these terms, we exploit the following preliminary result.

Lemma A.2 *Assume that the conditions of Theorem 3.1 are fulfilled. Then, uniformly in l :*

$$\mathbb{E} \left[|\tilde{V}_l(f)^n - \hat{V}_l(f)^n|^2 \right] \leq C \frac{L}{n}, \quad (\text{A.2a})$$

$$\mathbb{E} \left[|\hat{V}_l(f)^n - V(f)|^2 \right] \leq C \frac{L^2}{n^2}, \quad (\text{A.2b})$$

$$\mathbb{E} \left[|V_l(f)^n - \tilde{V}_l(f)^n|^2 \right] \leq C \frac{L}{n^2}. \quad (\text{A.2c})$$

Proof of Lemma A.2. Part (A.2a) is shown by using the discrete Burkholder inequality as in Eq. (A.7). The proof of part (A.2b) follows along the lines of the proof of Eq. (iv). To prove part (A.2c), we recall condition (V) and make the decomposition

$$\xi_i^n = \sqrt{n} \Delta_i^n X - \alpha_i^n = \sqrt{n} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} a_s \, ds + \int_{\frac{i-1}{n}}^{\frac{i}{n}} (\sigma_s - \sigma_{\frac{i-1}{n}}) \, dW_s \right) \equiv \xi_i^n(1) + \xi_i^n(2),$$

where

$$\xi_i^n(1) = \sqrt{n} \left(\frac{1}{n} a_{\frac{i-1}{n}} + \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left[\tilde{\sigma}_{\frac{i-1}{n}} \left(W_s - W_{\frac{i-1}{n}} \right) + \tilde{v}_{\frac{i-1}{n}} \left(B_s - B_{\frac{i-1}{n}} \right) \right] \, dW_s \right),$$

$$\xi_i^n(2) = \sqrt{n} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(a_s - a_{\frac{i-1}{n}} \right) \, ds + \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left[\int_{\frac{i-1}{n}}^s \tilde{a}_u \, du \right] \, dW_s \right)$$

$$+ \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left[\int_{\frac{i-1}{n}}^s (\tilde{\sigma}_u - \tilde{\sigma}_{\frac{i-1}{n}}) dW_u + \int_{\frac{i-1}{n}}^s (\tilde{v}_u - \tilde{v}_{\frac{i-1}{n}}) dB_u \right] dW_s.$$

By the assumptions of Theorem 3.1, the Burkholder and Cauchy-Schwarz inequalities yield that

$$\mathbb{E}[|\xi_i^n(1)|^4] \leq \frac{C}{n^2}, \quad (\text{A.10})$$

$$\mathbb{E}[|\xi_i^n(2)|^4] \leq \frac{C}{n^4}. \quad (\text{A.11})$$

Then, using Taylor's theorem, we may write $V_l(f)^n - \tilde{V}_l(f)^n = S_l^n(1) + S_l^n(2) + O_p(1/n)$, where

$$S_l^n(1) = \frac{1}{n/L} \sum_{i=1}^{n/L} f'(\alpha_{(i-1)L+l}^n) \xi_{(i-1)L+l}^n(1) \quad \text{and} \quad S_l^n(2) = \frac{1}{n/L} \sum_{i=1}^{n/L} f'(\alpha_{(i-1)L+l}^n) \xi_{(i-1)L+l}^n(2).$$

As f is even, f' is odd. This implies a martingale difference property

$$\mathbb{E} \left[f'(\alpha_{(i-1)L+l}^n) \xi_{(i-1)L+l}^n(1) \mid \mathcal{F}_{\frac{(i-1)L+l-1}{n}} \right] = 0.$$

Then, the Cauchy-Schwarz inequality, f' being of polynomial growth, Lemma A.1 and Eq. (A.10) imply

$$\mathbb{E} \left[|S_l^n(1)|^2 \right] = \frac{L^2}{n^2} \sum_{i=1}^{n/L} \mathbb{E} \left[|f'(\alpha_{(i-1)L+l}^n) \xi_{(i-1)L+l}^n(1)|^2 \right] \leq \frac{L}{n^2}. \quad (\text{A.12})$$

From the Cauchy-Schwarz inequality, Lemma A.1 and Eq. (A.11), we also get that

$$\mathbb{E} \left[|S_l^n(2)|^2 \right] \leq \frac{L}{n} \sum_{i=1}^{n/L} \mathbb{E} \left[|f'(\alpha_{(i-1)L+l}^n) \xi_{(i-1)L+l}^n(2)|^2 \right] \leq \frac{1}{n^2}. \quad (\text{A.13})$$

Then, we finish the proof via Eqs. (A.12) – (A.13). ■

The next result then implies Eq. (i), and the entire proof is complete.

Lemma A.3 *Assume that the conditions of Theorem 3.1 are fulfilled. Then,*

$$\mathbb{E} [|D_n^{(4)}|] \leq C \frac{L}{n}, \quad (\text{A.3a})$$

$$\mathbb{E} [|D_n^{(3)}|] \leq C \frac{1}{n}, \quad (\text{A.3b})$$

$$\mathbb{E} [|D_n^{(1)}|] \leq C \frac{1}{\sqrt{n}}, \quad (\text{A.3c})$$

$$\mathbb{E} [|D_n^{(2)}|] \leq C \left(\frac{L}{n} + \frac{1}{\sqrt{n}} \right). \quad (\text{A.3d})$$

Proof of Lemma A.3. We observe that part (A.3a) is a direct consequence of Lemma A.2 Eq. (A.2b). Regarding parts (A.3b) and (A.3c), we note that Lemma A.2 implies that

$$\mathbb{E} \left[\left(\tilde{V}_l(f)^n - V(f) \right)^2 \right] \leq C \frac{L}{n}.$$

Then, the Cauchy-Schwarz inequality and the above equation yield

$$\left(\mathbb{E} [|D_n^{(1)}|] \right)^2 \leq C \frac{n}{L^2} \sum_{l=1}^L \mathbb{E} \left[|V_l(f)^n - \tilde{V}_l(f)^n|^2 \right] = C \mathbb{E} [|D_n^{(3)}|].$$

Hence, it suffices to show part (A.3b), which follows from Lemma A.2 Eq. (A.2c).

We proceed to the proof of part (A.3d). In this part of the proofs, we adopt the notation $t_{i,l} = (iL + l)/n$. Applying Taylor's theorem and Eq. (A.4) for σ , we can rewrite

$$D_n^{(2)} = E_n + F_n + O_p(L/n) + O_p(1/\sqrt{n}),$$

with

$$E_n = \frac{2n}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/L} \phi'(\sigma_{t_{i-1,l-1}}) \int_{t_{i-1,l-1}}^{t_{i,l-1}} [\sigma_{t_{i-1,l-1}} - \sigma_s] ds \right) \times \left(\tilde{V}_l(f)^n - \hat{V}_l(f)^n \right),$$

$$F_n = \frac{-n}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/L} \phi''(\sigma_{t_{i-1,l-1}}) \int_{t_{i-1,l-1}}^{t_{i,l-1}} [\sigma_{t_{i-1,l-1}} - \sigma_s]^2 ds \right) \times \left(\tilde{V}_l(f)^n - \hat{V}_l(f)^n \right).$$

We note that the $O_p(1/\sqrt{n})$ error is due to the boundary integral term around 0. If we then recall Assumption **(H)**, an application of Eq. (A.4) for \tilde{a} , $\tilde{\sigma}$ and \tilde{v} implies that

$$E_n = E_n(1) + E_n(2) + O_p(L/n),$$

where

$$E_n(1) = \frac{2n}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/L} \phi'(\sigma_{t_{i-1,l-1}}) \frac{L^2}{2n^2} \tilde{a}_{t_{i-1,l-1}} \right) \left(\tilde{V}_l(f)^n - \hat{V}_l(f)^n \right),$$

$$E_n(2) = \frac{2n}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/L} \phi'(\sigma_{t_{i-1,l-1}}) \int_{t_{i-1,l-1}}^{t_{i,l-1}} \left[\tilde{\sigma}_{t_{i-1,l-1}} (W_s - W_{t_{i-1,l-1}}) + \tilde{v}_{t_{i-1,l-1}} (B_s - B_{t_{i-1,l-1}}) \right] ds \right) \times \left(\tilde{V}_l(f)^n - \hat{V}_l(f)^n \right).$$

Now, to deal with the term $E_n(1)$, we first define:

$$Q_l^n = \frac{L}{n} \sum_{i=1}^{n/L} \phi'(\sigma_{t_{i-1,l-1}}) \tilde{a}_{t_{i-1,l-1}},$$

which has the limit $Q = \int_0^1 \phi'(\sigma_s) \tilde{a}_s ds$. We find that:

$$\begin{aligned} E_n(1) &= \frac{1}{L} \sum_{l=1}^L Q (\tilde{V}_l(f)^n - \hat{V}_l(f)^n) + \frac{1}{L} \sum_{l=1}^L (Q_l^n - Q) (\tilde{V}_l(f)^n - \hat{V}_l(f)^n) \\ &\equiv E_n(1.1) + E_n(1.2). \end{aligned}$$

We note that

$$E_n(1.1) = \frac{Q}{n} \sum_{i=1}^n \chi_i^n \quad \text{and} \quad \mathbb{E}[|Q_l^n - Q|^2] \leq C \frac{L}{n},$$

uniformly in l . Then, using the Cauchy-Schwarz inequality, the martingale difference property of the χ_i^n 's and Lemma (A.2a), we find that

$$\mathbb{E}[|E_n(1)|] \leq \mathbb{E}[|E_n(1.1)|] + \mathbb{E}[|E_n(1.2)|] \leq C \left(\frac{1}{\sqrt{n}} + \frac{L}{n} \right).$$

If we also apply these techniques to the term F_n , we get that

$$\mathbb{E}[|F_n|] \leq C \left(\frac{1}{\sqrt{n}} + \frac{L}{n} \right).$$

So in the rest of the proof, we are left with the term $E_n(2)$ only. Here we assume that $\tilde{v}_s = 0$. Apart from expositional purposes, this is without loss of generality, as the terms involving the product of \tilde{v} and B are much simpler to handle, because W and B are independent.

We define:

$$G_{i,l}^n = \phi'(\sigma_{t_{i-1,l-1}}) \int_{t_{i-1,l-1}}^{t_{i,l-1}} \tilde{\sigma}_{t_{i-1,l-1}} (W_s - W_{t_{i-1,l-1}}) ds. \quad (\text{A.14})$$

Then,

$$E_n(2) = \frac{2n}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/L} G_{i,l}^n \right) (\tilde{V}_l(f)^n - \hat{V}_l(f)^n). \quad (\text{A.15})$$

We find that

$$(E_n(2))^2 = E_n(2.1) + E_n(2.2), \quad (\text{A.16})$$

where

$$E_n(2.1) = \frac{4n^2}{L^4} \sum_{l=1}^L \left(\sum_{i=1}^{n/L} G_{i,l}^n \right)^2 \left(\tilde{V}_l(f)^n - \hat{V}_l(f)^n \right)^2,$$

$$E_n(2.2) = \frac{8n^2}{L^4} \sum_{l_a \neq l_b}^L \left(\sum_{i=1}^{n/L} G_{i,l_a}^n \right) \left(\tilde{V}_{l_a}(f)^n - \hat{V}_{l_a}(f)^n \right) \left(\sum_{i=1}^{n/L} G_{i,l_b}^n \right) \left(\tilde{V}_{l_b}(f)^n - \hat{V}_{l_b}(f)^n \right).$$

As, for each fixed l , $G_{i,l}^n$ and $G_{j,l}^n$ are uncorrelated for $i \neq j$, the Cauchy-Schwarz inequality and Lemma (A.2a) imply that

$$\mathbb{E}[E_n(2.1)] \leq \frac{C}{n}. \quad (\text{A.17})$$

Recalling Eq. (A.16), we will be done with the proof of Lemma (A.3d), if we show

$$\mathbb{E}[E_n(2.2)] \leq \frac{C}{n}. \quad (\text{A.18})$$

It turns out that proving Eq. (A.18) is rather complicated. The way we handle it involves tools from Malliavin calculus (refer to Section A.6). We start by noting that

$$E_n(2.2) = \frac{8}{L^2} \sum_{l_a \neq l_b}^L \sum_{i_1, i_2, i_3, i_4=1}^{n/L} G_{i_1, l_a}^n \chi_{(i_2-1)L+l_a}^n G_{i_3, l_b}^n \chi_{(i_4-1)+l_b}^n. \quad (\text{A.19})$$

We fix l_a and l_b in Eq. (A.19). Then, for several choices of i_1, i_2, i_3 and i_4 , the expectation of the summand is zero. To see this, recall that expected value of each G_i^n and χ_i^n is zero. Hence, if we take the conditional expectation on the left endpoint of the largest interval, and if the other three terms are measurable with respect to this point in time, then the expectation vanishes. Thus, we need to study the expectation of the summands, only when the two largest indexes are equal, which for a fixed pair l_a, l_b reduces the effective number of summands to Cn^3/L^3 . Without loss of generality, we assume in the following that $i_2 < i_4 \leq i_1 = i_3$.

Next, we find an upper bound on the expected values of all the nonzero terms. We will rely heavily on the theory of Malliavin calculus, as presented in Section A.6. The Malliavin derivative introduced in Eq. (A.46) implies that χ_i^n is in $\mathbb{D}_{1,2}$ for each i . Then, it follows from the Clark-Ocone formula in Eq. (A.49) that

$$\chi_{(i-1)L+l}^n = \sqrt{n} \int_{t_{i-1, l-1}}^{t_{i-1, l}} \zeta_t^{n, i, l} dW_t,$$

with $\zeta_s^{n,i,l} = n^{-1/2} \mathbb{E}[D_s(\chi_{(i-1)L+l}) \mid \mathcal{F}_s]$ and $t_{i-1,l-1} \leq s \leq t_{i-1,l}$. Hence,

$$E_n(2.2) = \frac{8}{L^2} \sum_{l_a \neq l_b}^L \sum_{i_1, i_2, i_3, i_4=1}^{n/L} \int_{t_{i_3-1, l_b-1}}^{t_{i_3, l_b-1}} \int_{t_{i_1-1, l_a-1}}^{t_{i_1, l_a-1}} M_n(s, u) \mathbf{d}s \mathbf{d}u, \quad (\text{A.20})$$

where $M_n = M_n^{l_a, l_b, i_1, i_2, i_3, i_4}$ is defined by

$$M_n(s, u) = \tau_n^{i_1, l_a, i_3, l_b} (W_s - W_{t_{i_1-1, l_a-1}}) (W_u - W_{t_{i_3-1, l_b-1}}) n \int_{t_{i_4-1, l_b-1}}^{t_{i_4-1, l_b}} \zeta_t^{n, i_2, l_a} \mathbf{d}W_t \zeta_r^{n, i_4, l_b} \mathbf{d}W_r,$$

$$\tau_n^{i_1, l_a, i_3, l_b} = \phi'(\sigma_{t_{i_1-1, l_a-1}}) \tilde{\sigma}_{t_{i_1-1, l_a-1}} \phi'(\sigma_{t_{i_3-1, l_b-1}}) \tilde{\sigma}_{t_{i_3-1, l_b-1}}.$$

In view of Eq. (A.20), finding a bound on $\mathbb{E}[E_n(2.2)]$ amounts to estimating

$$I \equiv \mathbb{E}[M_n^{l_a, l_b, i_1, i_2, i_3, i_4}(s, u)].$$

To accomplish this, we use the notation $C_a = [t_{i_2-1, l_a-1}, t_{i_2-1, l_a}]$ and $C_b = [t_{i_4-1, l_b-1}, t_{i_4-1, l_b}]$. If we apply the integration by parts formula of Malliavin calculus as in Eq. (A.50) twice, we find that

$$\begin{aligned} I &= n \mathbb{E} \left[\int_{C_b} D_r \left(\tau_n^{i_1, l_a, i_3, l_b} (W_s - W_{t_{i_1-1, l_a-1}}) (W_u - W_{t_{i_3-1, l_b-1}}) \right) \left(\int_{C_a} \zeta_t^{n, i_2, l_a} \mathbf{d}W_t \right) \zeta_r^{n, i_4, l_b} \mathbf{d}r \right] \\ &= n \mathbb{E} \left[\int_{C_b} \int_{C_a} D_t \left(D_r \left(\tau_n^{i_1, l_a, i_3, l_b} (W_s - W_{t_{i_1-1, l_a-1}}) (W_u - W_{t_{i_3-1, l_b-1}}) \right) \zeta_r^{n, i_4, l_b} \right) \zeta_t^{n, i_2, l_a} \mathbf{d}t \mathbf{d}r \right] \\ &= n \int_{C_b} \int_{C_a} \mathbb{E}[H_{t,r}^n] \mathbf{d}t \mathbf{d}r. \end{aligned} \quad (\text{A.21})$$

The chain rule for the Malliavin derivative from Eq. (A.48) implies that the random variable $H_{t,r}^n$ has the form

$$\begin{aligned} H_{t,r}^n &= Z_1 (W_s - W_{t_{i_1-1, l_a-1}}) (W_u - W_{t_{i_3-1, l_b-1}}) \\ &\quad + Z_2 \left((W_s - W_{t_{i_1-1, l_a-1}}) \mathbb{1}_{[t_{i_3-1, l_b-1}, u]}(t) + (W_u - W_{t_{i_3-1, l_b-1}}) \mathbb{1}_{[t_{i_1-1, l_a-1}, s]}(t) \right) \\ &\quad + Z_3 \left((W_s - W_{t_{i_1-1, l_a-1}}) \mathbb{1}_{[t_{i_3-1, l_b-1}, u]}(r) + (W_u - W_{t_{i_3-1, l_b-1}}) \mathbb{1}_{[t_{i_1-1, l_a-1}, s]}(r) \right) \\ &\quad + Z_4 \left(\mathbb{1}_{[t_{i_3-1, l_b-1}, u]}(r) \mathbb{1}_{[t_{i_1-1, l_a-1}, s]}(t) + \mathbb{1}_{[t_{i_3-1, l_b-1}, u]}(t) \mathbb{1}_{[t_{i_1-1, l_a-1}, s]}(r) \right) \\ &= \sum_{k=1}^4 H_{t,r}^{n,k}, \end{aligned} \quad (\text{A.22})$$

where Z_i are random variables with uniformly bounded second moment. At this point, we invoke Assumption **(M)** and the calculation rules from Malliavin calculus introduced in Section A.6. We readily deduce that

$$\mathbb{E}[|H_{t,r}^{n,1}|] \leq C \frac{L}{n}.$$

On the other hand, the term $\mathbb{1}_{[t_{i_1-1}, t_{a-1}, s]}(r)$ is different from 0 only for one index i_4 (this consideration also hold for all other indicator functions). Hence, plugging $\mathbb{E}[H_{t,r}^{n,k}]$ back into Eq. (A.21), and observing the identity in Eq. (A.20), we find that $\mathbb{E}[|H_{t,r}^{n,1}|]$ is the dominating term. We therefore conclude that

$$\mathbb{E}[E_n(2.2)] \leq \frac{C}{n}.$$

This completes the proof of Eq. (A.18), Eq. (i), and Theorem 3.1. ■

A.2. Proof of Theorem 3.2

We need to show that the quantities introduced on the left side of Eq. (i)-(v) of the last subsection all converge to 0 under the weaker assumptions of Theorem 3.2. By observing the proof of Theorem 3.1, we realise that the steps behind Eq. (ii)-(v) do not depend on the stronger Assumptions **(H)** and **(M)** nor the differentiability of the function f . Hence, we can immediately deduce that

$$\mathbb{E}[|Q_n - U_n|] \rightarrow 0, \quad \mathbb{E}[|U_n - R_n|] \rightarrow 0, \quad \mathbb{E}[|R_n - \Sigma|] \rightarrow 0, \quad \mathbb{E}[|\hat{\Sigma}_n - \Sigma_n|] \rightarrow 0.$$

On the other hand, from Section 8 in BGJPS6, we also know that

$$\sup_{1 \leq l \leq L} \mathbb{E} \left[\left| \sqrt{\frac{n}{L}} \left(V_l(f)^n - V(f) \right) - \sqrt{\frac{L}{n}} \sum_{i=1}^{n/L} \chi_{(i-1)L+l}^n \right|^{1+\epsilon} \right] \rightarrow 0.$$

for $\epsilon > 0$ small enough. Hence, we find by Hölder inequality that

$$\mathbb{E}[|\Sigma_n - Q_n|] \rightarrow 0,$$

which corresponds to part (i) of the previous subsection and finishes the proof. ■

A.3. Proof of Theorem 3.5

We begin by introducing some notation. For $m \geq i$, we define

$$\Delta \bar{Y}_{m,i}^n = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right) \left(\sigma_{\frac{i}{n}} \Delta_{m+j}^n W + \Delta_{m+j}^n \epsilon \right). \quad (\text{A.23})$$

We note that $\Delta \bar{Y}_{m,i}^n$ approximates $\Delta \bar{Y}_m^n$ by evaluating σ at the point i/n . Moreover, we state two auxiliary results from Podolskij and Vetter (2009a), which provide the stochastic order of the statistics $\Delta \bar{S}_i^n$ for the processes, $S = W, X, \epsilon$ or Y (Lemma A.4), and which permits to substitute the limit ψ_i for ψ_i^n for $i = 1, 2$ without altering the consistency statements (Lemma A.5).

Lemma A.4 *Assume that s is a non-negative real number, such that $\mathbb{E}[|\epsilon|^s] < \infty$. Then, for any i and n ,*

$$\mathbb{E}\left[|\Delta \bar{Y}_{m,i}^n|^s \mid \mathcal{F}_{\frac{i}{n}}\right] + \mathbb{E}\left[|\Delta \bar{Y}_i^n|^s \mid \mathcal{F}_{\frac{i}{n}}\right] \leq C n^{-s/4}. \quad (\text{A.24})$$

Lemma A.5 *Let $s \geq 0$. Then,*

$$\int_0^1 \left(\theta \psi_2^n \sigma_u^2 + \frac{1}{\theta} \psi_1^n \omega^2 \right)^s \mathrm{d}u - \int_0^1 \left(\theta \psi_2 \sigma_u^2 + \frac{1}{\theta} \psi_1 \omega^2 \right)^s \mathrm{d}u = o_p(n^{-1/4}). \quad (\text{A.25})$$

We again use the short form notation $t_{i,l} = (iL + l)pk_n/n$ and introduce some approximations of $\hat{\Sigma}_n^*$ and Σ^* :

$$\begin{aligned} \Sigma_n^* &= \frac{1}{L} \sum_{l=1}^L \left(\frac{n^{1/4}}{\sqrt{L}} \left(V_l^*(q, r)^n - V^*(q, r) \right) \right)^2, & Q_n &= \frac{k_n^2 p^2}{n^{3/2}} \sum_{l=1}^L \left(\sum_{i=1}^{n/Lpk_n} \chi_{(i-1)L+l}^n \right)^2, \\ U_n &= \frac{k_n^2 p^2}{n^{3/2}} \sum_{l=1}^L \sum_{i=1}^{n/Lpk_n} \left(\chi_{(i-1)L+l}^n \right)^2, & R_n &= \frac{k_n^2 p^2}{n^{3/2}} \sum_{l=1}^L \sum_{i=1}^{n/Lpk_n} \mathbb{E}\left[\left(\chi_{(i-1)L+l}^n \right)^2 \mid \mathcal{F}_{t_{i-1,l-1}} \right], \end{aligned}$$

where

$$\eta_i^n = \frac{n^{\frac{q+r}{4}}}{pk_n - 2k_n + 2} \sum_{m \in B_i(p)} |\Delta \bar{Y}_{m,(i-1)pk_n}^n|^q |\Delta \bar{Y}_{m+k_n,(i-1)pk_n}^n|^r,$$

and

$$\chi_i^n = \eta_i^n - \mathbb{E}\left[\eta_i^n \mid \mathcal{F}_{\frac{(i-1)pk_n}{n}} \right].$$

There exists a $C > 0$, independent of i , such that

$$\mathbb{E}\left[(\eta_i^n)^4 \right] \leq C \quad \text{and} \quad \mathbb{E}\left[(\chi_i^n)^4 \right] \leq \frac{C}{p^2}, \quad (\text{A.26})$$

where the last inequality holds as, for fixed i , the terms $\Delta \bar{Y}_{m,(i-1)pk_n}^n$ are k_n -dependent.

As in the no noise setting, we complete the proof by showing the following results and the relationship $p/\sqrt{n} \ll \sqrt{p}/n^{1/4} \ll 1/\sqrt{L}$ (which follow from $\sqrt{n}/Lp^2 \rightarrow \infty$):

$$\mathbb{E}[|\Sigma_n^* - Q_n|] \leq C \left(\frac{Lp^2}{\sqrt{n}} + \frac{1}{\sqrt{L}} \right), \quad (\text{i})$$

$$\mathbb{E}[|Q_n - U_n|] \leq C \frac{1}{\sqrt{L}}, \quad (\text{ii})$$

$$\mathbb{E}[|U_n - R_n|] \leq C \frac{\sqrt{p}}{n^{1/4}}, \quad (\text{iii})$$

$$\mathbb{E}[|R_n - \Sigma^*|] \leq C \left(\frac{p}{\sqrt{n}} + \frac{1}{p} \right), \quad (\text{iv})$$

$$\mathbb{E}[|\hat{\Sigma}_n^* - \Sigma_n^*|] \leq C \frac{1}{\sqrt{L}}. \quad (\text{v})$$

We proceed as in the part absent of microstructure noise. First, we prove the steps (ii), (iii), (iv), (v), which are relatively simple and short. Then, we offer the proof of (i), which is much longer and more complicated.

Proof of (iii). This part is again easy. We observe that

$$U_n - R_n = \frac{k_n^2 p^2}{n^{3/2}} \sum_{i=1}^{n/pk_n} (\chi_i^n)^2 - \mathbb{E} \left[(\chi_i^n)^2 \mid \mathcal{F}_{\frac{(i-1)pk_n}{n}} \right] \quad \text{and} \quad \mathbb{E}[|U_n - R_n|^2] \leq C \frac{p}{\sqrt{n}},$$

where the last part is due to the martingale difference property and Eq. (A.26). ■

Proof of (v). This follows by using a difference of squares and that $\frac{n^{1/4}}{\sqrt{L}} \left(V_l^*(q, r)^n - V^*(q, r) \right) = O_p(1)$ and $\frac{n^{1/4}}{\sqrt{L}} \left(V^*(q, r)^n - V^*(q, r) \right) = O_p \left(\frac{1}{\sqrt{L}} \right)$, where the first term is due to Lemma A.6. We skip the details. ■

Proof of (ii). We define $S_l^m = \sum_{i=1}^m \chi_{(i-1)L+l}^n$ and $T_l^m = \sum_{i=1}^m (\chi_{(i-1)L+l}^n)^2$ and note that

$$Q_n - U_n = \frac{k_n^2 p^2}{n^{3/2}} \sum_{l=1}^L A_l^n$$

where

$$A_l^n = \sum_{i,j=1}^{n/Lpk_n} \chi_{(i-1)L+l}^n \chi_{(j-1)L+l}^n = \left(S_l^{n/Lpk_n} \right)^2 - T_l^{n/Lpk_n}.$$

Since $A_{l_1}^n$ and $A_{l_2}^n$ are uncorrelated for every $l_1 \neq l_2$, we find that

$$\mathbb{E}[|Q_n - U_n|^2] = \frac{k_n^4 p^4}{n^3} \sum_{l=1}^L \mathbb{E}[(A_l^n)^2] \leq C \frac{k_n^4 p^4}{n^3} \sum_{l=1}^L \left(\mathbb{E}[(S_l^{n/Lpk_n})^4] + \mathbb{E}[(T_l^{n/Lpk_n})^2] \right). \quad (\text{A.27})$$

Proceeding as in the noiseless case, Eq. (A.26), the Cauchy-Schwarz and the Burkholder inequalities yield:

$$\mathbb{E}[(S_l^{n/Lpk_n})^4] \leq C \frac{n}{L^2 p^4} \quad \text{and} \quad \mathbb{E}[(T_l^{n/Lpk_n})^2] \leq C \frac{n}{L^2 p^4}. \quad (\text{A.28})$$

Then, we finish the proof using Eqs. (A.27) and (A.28). ■

Proof of (iv). We draw upon the proof of Lemma 8 in Podolskij and Vetter (2009a). We find that

$$\begin{aligned} \Sigma^* &= 2\theta \int_0^1 \int_0^2 h(\sigma_u, t, f(s)) \, ds du \\ &= \frac{2}{\sqrt{n}} \frac{pk_n}{n} \sum_{i=1}^{n/pk_n} \sum_{j=0}^{2k_n-1} h\left(\sigma_{\frac{(i-1)pk_n}{n}}, t_n, f^n\left(\frac{j}{k_n}\right)\right) + O_p\left(\frac{pk_n}{n}\right) \\ &\equiv R'_n + O_p\left(\frac{pk_n}{n}\right) \end{aligned} \quad (\text{A.29})$$

where the function h and its associated notation were introduced in Eq. (3.6) in Podolskij and Vetter (2009a).

To estimate the term $R_n - R'_n$, we recall that

$$R_n = \frac{k_n^2 p^2}{n^{3/2}} \sum_{i=1}^{n/pk_n} \mathbb{E}\left[(\chi_i^n)^2 \mid \mathcal{F}_{\frac{(i-1)pk_n}{n}}\right].$$

For $m \geq l \geq i$, we get

$$\begin{aligned} h\left(\sigma_{\frac{i}{n}}, t_n, f^n\left(\frac{m-l}{k_n}\right)\right) &= \mathbb{E}\left[|n^{1/4} \Delta \bar{Y}_{m,i}^n|^q |n^{1/4} \Delta \bar{Y}_{m+k_n,i}^n|^r \times |n^{1/4} \Delta \bar{Y}_{l,i}^n|^q |n^{1/4} \Delta \bar{Y}_{l+k_n,i}^n|^r \mid \mathcal{F}_{\frac{i}{n}}\right] \\ &\quad - \mathbb{E}\left[|n^{1/4} \Delta \bar{Y}_{m,i}^n|^q |n^{1/4} \Delta \bar{Y}_{m+k_n,i}^n|^r \mid \mathcal{F}_{\frac{i}{n}}\right] \times \mathbb{E}\left[|n^{1/4} \Delta \bar{Y}_{l,i}^n|^q |n^{1/4} \Delta \bar{Y}_{l+k_n,i}^n|^r \mid \mathcal{F}_{\frac{i}{n}}\right]. \end{aligned}$$

Note that the above term vanishes for $m-l \geq 2k_n$. Then, by denoting $N = pk_n - 2k_n + 2$, we find that

$$N \mathbb{E}\left[(\chi_i^n)^2 \mid \mathcal{F}_{\frac{(i-1)pk_n}{n}}\right] = h\left(\sigma_{\frac{(i-1)pk_n}{n}}, t_n, f^n(0)\right) + \frac{2}{N} \sum_{j=1}^{2k_n-1} (N-j) h\left(\sigma_{\frac{(i-1)pk_n}{n}}, t_n, f^n\left(\frac{j}{k_n}\right)\right)$$

$$= 2 \sum_{j=0}^{2k_n-1} h \left(\sigma_{\frac{(i-1)pk_n}{n}}, t_n, f^n \left(\frac{j}{k_n} \right) \right) + O_p(1) + O_p \left(\frac{k_n}{p} \right).$$

This yields that:

$$\frac{pk_n}{\sqrt{n}} \mathbb{E} \left[(\chi_i^n)^2 \mid \mathcal{F}_{\frac{(i-1)pk_n}{n}} \right] = \frac{2}{\sqrt{n}} \sum_{j=0}^{2k_n-1} h \left(\sigma_{\frac{(i-1)pk_n}{n}}, t_n, f^n \left(\frac{j}{k_n} \right) \right) + O_p \left(\frac{1}{\sqrt{n}} \right) + O_p \left(\frac{1}{p} \right)$$

uniformly in i . As a result,

$$\mathbb{E}[|R_n - R'_n|] \leq C \left(\frac{1}{\sqrt{n}} + \frac{1}{p} \right). \quad (\text{A.30})$$

In view of Eqs. (A.29) – (A.30), the proof is complete. ■

As in the noiseless setting, we need to prepare a bit to show Eq. (i). We denote by:

$$\tilde{V}_l^*(q, r)^n = \frac{Lpk_n}{n} \sum_{i=1}^{n/Lpk_n} \eta_{(i-l)L+l}^n, \quad \hat{V}_l^*(q, r)^n = \frac{Lpk_n}{n} \sum_{i=1}^{n/Lpk_n} \mathbb{E} \left[\eta_{(i-l)L+l}^n \mid \mathcal{F}_{t_{i-1, l-1}} \right].$$

Then, from the decomposition

$$V_l^*(q, r)^n - V^*(q, r) = \left(V_l^*(q, r)^n - \tilde{V}_l^*(q, r)^n \right) + \left(\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n \right) + \left(\hat{V}_l^*(q, r)^n - V^*(q, r) \right)$$

we find that

$$\Sigma_n^* - Q_n = D_n^{(1)} + D_n^{(2)} + D_n^{(3)} + D_n^{(4)},$$

where

$$D_n^{(1)} = \frac{2\sqrt{n}}{L^2} \sum_{l=1}^L \left(V_l^*(q, r)^n - \tilde{V}_l^*(q, r)^n \right) \left(\tilde{V}_l^*(q, r)^n - V^*(q, r) \right),$$

$$D_n^{(2)} = \frac{2\sqrt{n}}{L^2} \sum_{l=1}^L \left(\hat{V}_l^*(q, r)^n - V^*(q, r) \right) \left(\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n \right),$$

$$D_n^{(3)} = \frac{\sqrt{n}}{L^2} \sum_{l=1}^L \left(V_l^*(q, r)^n - \tilde{V}_l^*(q, r)^n \right)^2,$$

$$D_n^{(4)} = \frac{\sqrt{n}}{L^2} \sum_{l=1}^L \left(\hat{V}_l^*(q, r)^n - V^*(q, r) \right)^2.$$

To bound these terms, we exploit the following auxiliary Lemma.

Lemma A.6 Assume that the conditions of Theorem 3.5 are fulfilled. Then, uniformly in l :

$$\mathbb{E} \left[|\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n|^2 \right] \leq C \frac{L}{\sqrt{n}}, \quad (\text{A.6a})$$

$$\mathbb{E} \left[|\hat{V}_l^*(q, r)^n - V^*(q, r)|^2 \right] \leq C \left(\frac{L^2 p^2}{n} + \frac{1}{\sqrt{n}} \right), \quad (\text{A.6b})$$

$$\mathbb{E} \left[|V_l^*(q, r)^n - \tilde{V}_l^*(q, r)^n|^2 \right] \leq C \frac{L p^2}{n}. \quad (\text{A.6c})$$

Proof of Lemma A.6. Part (A.6a) follows by exploiting the martingale difference property with Eq. (A.26).

To prove part (A.6b), we start with the decomposition

$$\hat{V}_l^*(q, r)^n - V^*(q, r) = \left(\hat{V}_l^*(q, r)^n - \check{V}_l^*(q, r)^n \right) + \left(\check{V}_l^*(q, r)^n - V^*(q, r) \right),$$

where

$$\check{V}_l^*(q, r)^n = \mu_q \mu_r \frac{L p k_n}{n} \sum_{i=1}^{n/L p k_n} \left(\theta \psi_2 \sigma_{t_{i-1, l-1}}^2 + \frac{1}{\theta} \psi_1 \omega^2 \right)^{\frac{q+r}{2}}. \quad (\text{A.31})$$

To deal with the second term, we recall Lemma A.2 Eq. (A.2b). Hence, the Riemann approximation yields

$$\mathbb{E} \left[|\check{V}_l^*(q, r)^n - V^*(q, r)|^2 \right] \leq C \frac{L^2 p^2}{n}. \quad (\text{A.32})$$

To estimate the first term, let $m \in B_i(p)$. Taking q and r to be even non-negative integers, we invoke Assumption (N) and the binomial expansion theorem to conclude that

$$\mathbb{E} \left[|n^{1/4} \Delta \bar{Y}_{m, (i-1) p k_n}^n|^q |n^{1/4} \Delta \bar{Y}_{m+k_n, (i-1) p k_n}^n|^r \mid \mathcal{F}_{\frac{(i-1) p k_n}{n}} \right] = \mu_q \mu_r \left(\theta \psi_2 \sigma_{\frac{(i-1) p k_n}{n}}^2 + \frac{1}{\theta} \psi_1 \omega^2 \right)^{\frac{q+r}{2}} + o_p(n^{-1/4}),$$

uniformly in i and m . Hence, we find that

$$\mathbb{E} \left[\eta_i^n \mid \mathcal{F}_{t_{i-1, l-1}} \right] = \mu_q \mu_r \left(\theta \psi_2 \sigma_{\frac{(i-1) p k_n}{n}}^2 + \frac{1}{\theta} \psi_1 \omega^2 \right)^{\frac{q+r}{2}} + o_p(n^{-1/4}),$$

uniformly in i and m . Using these insights, we can finish part (A.6b) by deducing that

$$\mathbb{E} \left[|\hat{V}_l^*(q, r)^n - \check{V}_l^*(q, r)^n|^2 \right] \leq \frac{C}{\sqrt{n}}. \quad (\text{A.33})$$

As for the proof of Eq. (A.6c), we proceed as in the proof of Eq. (A.2c); see also page 2818 in Podolskij and Vetter (2009a). Thus, we just provide a sketch of the main steps for $r = 0$. For any $m \geq i$, we define

$$\begin{aligned}\xi_{m,i}^n(1) &\equiv \sum_{j=1}^{k_n} w\left(\frac{j}{k_n}\right) \left(\frac{1}{n} a_{\frac{i}{n}} + \int_{\frac{m+j-1}{n}}^{\frac{m+j}{n}} \left[\tilde{\sigma}_{\frac{i}{n}}(W_s - W_{\frac{i}{n}}) + \tilde{v}_{\frac{i}{n}}(B_s - B_{\frac{i}{n}}) \right] dW_s \right) \\ \xi_{m,i}^n(2) &\equiv \sum_{j=1}^{k_n} w\left(\frac{j}{k_n}\right) \left(\int_{\frac{m+j-1}{n}}^{\frac{m+j}{n}} (a_s - a_{\frac{i}{n}}) ds + \int_{\frac{m+j-1}{n}}^{\frac{m+j}{n}} \int_{\frac{i}{n}}^s \tilde{a}_u du dW_s \right. \\ &\quad \left. + \int_{\frac{m+j-1}{n}}^{\frac{m+j}{n}} \left(\int_{\frac{i}{n}}^s (\tilde{\sigma}_u - \tilde{\sigma}_{\frac{i}{n}}) dW_u + \int_{\frac{i}{n}}^s (\tilde{v}_u - \tilde{v}_{\frac{i}{n}}) dB_u \right) dW_s \right).\end{aligned}$$

We note that $\Delta \bar{Y}_m^n - \Delta \bar{Y}_{m,i}^n = \xi_{m,i}^n(1) + \xi_{m,i}^n(2)$. Assumption **(H)**, the Hölder and Burkholder inequalities imply

$$\mathbb{E}[|\xi_{m,i}^n(1)|^4] \leq C \frac{p^2}{n^2}, \quad (\text{A.34})$$

$$\mathbb{E}[|\xi_{m,i}^n(2)|^4] \leq C \frac{p^4}{n^3}. \quad (\text{A.35})$$

Now, we let $f(x) = |x|^q$. Taylor's theorem then yields that

$$V_l^*(q, r) - \tilde{V}_l^*(q, r)^n = S_l^n(1) + S_l^n(2) + O_p\left(\frac{p}{\sqrt{n}}\right),$$

where

$$\begin{aligned}S_l^n(1) &= \frac{Lpk_n}{n} \frac{n^{q/4}}{pk_n - 2k_n + 2} \sum_{i=1}^{n/Lpk_n} \sum_{m \in B_{(i-1)L+l}(p)} f'(\Delta \bar{Y}_{m,((i-1)L+(l-1))pk_n}^n) \xi_{m,((i-1)L+(l-1))pk_n}^n(1), \\ S_l^n(2) &= \frac{Lpk_n}{n} \frac{n^{q/4}}{pk_n - 2k_n + 2} \sum_{i=1}^{n/Lpk_n} \sum_{m \in B_{(i-1)L+l}(p)} f'(\Delta \bar{Y}_{m,((i-1)L+(l-1))pk_n}^n) \xi_{m,((i-1)L+(l-1))pk_n}^n(2).\end{aligned}$$

In order to bound these terms, we note that Assumption **(N)** implies that $(W, B, \epsilon) \stackrel{d}{=} -(W, B, \epsilon)$. Also, since $f'(\Delta \bar{Y}_{m,i}^n)$ is an odd function and $\xi_{m,i}^n(1)$ is an even function of (W, B, ϵ) , it follows that

$$\mathbb{E}\left[f'(\Delta \bar{Y}_{m,ipk_n}^n) \xi_{m,ipk_n}^n(1) \mid \mathcal{F}_{\frac{ipk_n}{n}} \right] = 0.$$

This property—together with the Cauchy-Schwarz inequality, Eqs. (A.24) and (A.34)—means that

$$\mathbb{E}[|S_l^n(1)|^2] \leq C \frac{Lp^2}{n}. \quad (\text{A.36})$$

Applying the Cauchy-Schwarz inequality again, combined with Eqs. (A.24) and (A.35), we also find that

$$\mathbb{E}[|S_l^n(2)|^2] \leq C \frac{p^2}{n}, \quad (\text{A.37})$$

and with Eqs. (A.36) – (A.37) at hand, the proof is complete. \blacksquare

The following results are then sufficient to complete the proof of Theorem 3.5.

Lemma A.7 *Assume that the conditions of Theorem 3.5 are fulfilled. Then, it holds that*

$$\mathbb{E}[|D_n^{(4)}|] \leq C \left(\frac{Lp^2}{\sqrt{n}} + \frac{1}{L} \right), \quad (\text{A.7a})$$

$$\mathbb{E}[|D_n^{(3)}|] \leq C \frac{p^2}{\sqrt{n}}, \quad (\text{A.7b})$$

$$\mathbb{E}[|D_n^{(1)}|] \leq C \left(\frac{p}{n^{1/4}} + \frac{\sqrt{L}p^2}{\sqrt{n}} \right). \quad (\text{A.7c})$$

Lemma A.8 *Assume that the conditions of Theorem 3.5 are fulfilled. Then, it holds that*

$$\mathbb{E}[|D_n^{(2)}|] \leq C \left(\frac{1}{\sqrt{L}} + \frac{Lp^{3/2}}{\sqrt{n}} \right). \quad (\text{A.8a})$$

Proof of Lemma A.7. Again, part (A.7a) follows from Eq. (A.6b). Concerning parts (A.7b) – (A.7c), we apply Lemma A.6 and find that

$$\mathbb{E} \left[\left(\tilde{V}_l^*(q, r)^n - V^*(q, r) \right)^2 \right] \leq C \left(\frac{L}{\sqrt{n}} + \frac{L^2 p^2}{n} \right). \quad (\text{A.38})$$

Then, the Cauchy-Schwarz inequality and Eq. (A.38) yield

$$\left(\mathbb{E}[|D_n^{(1)}|] \right)^2 \leq C \left(1 + \frac{Lp^2}{\sqrt{n}} \right) \frac{\sqrt{n}}{L^2} \sum_{l=1}^L \mathbb{E}[|V_l^*(q, r)^n - \tilde{V}_l^*(q, r)^n|^2] = C \left(1 + \frac{Lp^2}{\sqrt{n}} \right) \mathbb{E}[|D_n^{(3)}|].$$

Hence, it is enough to show part (A.7b), which follows from Eq. (A.6c). \blacksquare

Proof of Lemma A.8. We start by denoting $\phi(x) = \mu_q \mu_r \left(\theta \psi_2 x^2 + \frac{1}{\theta} \psi_1 \omega^2 \right)^{\frac{q+r}{2}}$. Note that $\phi(x)$ is a smooth function of x , because both q and r are even. After recalling Eq. (A.31), an application of Taylor's theorem, the Cauchy-Schwarz inequality and Eq. (A.4) for σ then implies that

$$D_n^{(2)} = E_n + F_n + G_n + O_p \left(\frac{Lp^{3/2}}{\sqrt{n}} \right) + O_p \left(\frac{p}{n^{1/4}} \right),$$

where the last error term comes from the boundary integral around 0 and

$$\begin{aligned}
E_n &= \frac{2\sqrt{n}}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/Lpk_n} \phi'(\sigma_{t_{i-1,l-1}}) \int_{t_{i-1,l-1}}^{t_{i,l-1}} (\sigma_{t_{i-1,l-1}} - \sigma_s) ds \right) \times \left(\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n \right), \\
F_n &= -\frac{\sqrt{n}}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/Lpk_n} \phi''(\sigma_{t_{i-1,l-1}}) \int_{t_{i-1,l-1}}^{t_{i,l-1}} (\sigma_{t_{i-1,l-1}} - \sigma_s)^2 ds \right) \times \left(\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n \right), \\
G_n &= \frac{2\sqrt{n}}{L^2} \sum_{l=1}^L \left(\hat{V}_l^*(q, r)^n - \check{V}_l^*(q, r)^n \right) \left(\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n \right).
\end{aligned}$$

From the Cauchy-Schwarz inequality, Lemma A.6, Eq. (A.6a) and Eq. (A.33), we get

$$\mathbb{E}[|G_n|] \leq \frac{C}{\sqrt{L}}.$$

At this stage, we assume $\tilde{v}_s = 0$ as in the noiseless setting. We recall Assumption **(H)**, apply Eq. (A.4) to $\tilde{a}, \tilde{\sigma}$, and subsequently use Taylor's theorem to conclude that

$$E_n = E_n(1) + E_n(2) + O_p\left(\frac{Lp^{3/2}}{\sqrt{n}}\right),$$

where

$$\begin{aligned}
E_n(1) &= \frac{2\sqrt{n}}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/Lpk_n} \phi'(\sigma_{t_{i-1,l-1}}) \frac{L^2 k_n^2 p^2}{2n^2} \tilde{a}_{t_{i-1,l-1}} \right) \left(\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n \right), \\
E_n(2) &= \frac{2\sqrt{n}}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/Lpk_n} \phi'(\sigma_{t_{i-1,l-1}}) \int_{t_{i-1,l-1}}^{t_{i,l-1}} \tilde{\sigma}_{t_{i-1,l-1}} (W_s - W_{t_{i-1,l-1}}) ds \right) \left(\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n \right).
\end{aligned}$$

For the term $E_n(1)$, we proceed as in the noiseless case. After recalling Eqs. (A.26) and (A.6a), we find that

$$\mathbb{E}[|E_n(1)|] \leq C \left(\frac{p}{n^{1/4}} + \frac{Lp^{3/2}}{\sqrt{n}} \right).$$

Next, the F_n -term can be handled in a similar fashion. Thus, we get the estimate

$$\mathbb{E}[|F_n|] \leq C \left(\frac{p}{n^{1/4}} + \frac{Lp^{3/2}}{\sqrt{n}} \right).$$

So we will be done, if we can show that

$$\mathbb{E}[|E_n(2)|] \leq C \frac{p}{n^{1/4}}. \tag{A.39}$$

Throughout the remainder of the proof, we assume that $r = 0$, so that q is an even integer. Again, this is without loss of generality. We then appeal to the binomial theorem in order to find an expansion of

$$|\Delta \bar{Y}_{m,i}^n|^q = (\Delta \bar{Y}_{m,i}^n)^q = \left(\sigma_{\frac{i}{n}} \Delta \bar{W}_m^n + \Delta \bar{\epsilon}_m^n \right)^q,$$

whereby we can separate $\tilde{V}_l^*(q, r)^n - \hat{V}_l^*(q, r)^n$, and hence $E_n(2)$, into $q + 1$ terms of the form

$$E_n(2) = \sum_{s=0}^q E_n^{(s)}(2),$$

where

$$E_n^{(s)}(2) = \frac{2\sqrt{n}}{L^2} \sum_{l=1}^L \left(\sum_{i=1}^{n/Lpk_n} \phi'(\sigma_{t_{i-1,l-1}}) \int_{t_{i-1,l-1}}^{t_{i,l-1}} \tilde{\sigma}_{t_{i-1,l-1}} (W_s - W_{t_{i-1,l-1}}) \mathbf{d}s \right) \left(\frac{Lpk_n}{n} \sum_{i=1}^{n/Lpk_n} \chi_{(i-1)L+l}^n(s) \right),$$

$$\chi_i^n(s) = \frac{q!}{s!(q-s)!} \frac{n^{q/4}}{pk_n - k_n + 2} \sum_{m \in B_i(p)} \left(\sigma_{\frac{(i-1)pk_n}{n}} \right)^{q-s} \left((\Delta \bar{W}_m^n)^{q-s} (\Delta \bar{\epsilon}_m^n)^s - \mathbb{E}[(\Delta \bar{W}_m^n)^{q-s} (\Delta \bar{\epsilon}_m^n)^s] \right).$$

As a result, it is sufficient to show that

$$\mathbb{E}[|E_n^{(s)}(2)|] \leq C \frac{p}{n^{1/4}}, \quad (\text{A.40})$$

where s is an arbitrary integer chosen from $0 \leq s \leq q$. Note the equality:

$$\begin{aligned} (\Delta \bar{W}_m^n)^{q-s} (\Delta \bar{\epsilon}_m^n)^s - \mathbb{E}[(\Delta \bar{W}_m^n)^{q-s} (\Delta \bar{\epsilon}_m^n)^s] &= \mathbb{E}[(\Delta \bar{\epsilon}_m^n)^s] \left((\Delta \bar{W}_m^n)^{q-s} - \mathbb{E}[(\Delta \bar{W}_m^n)^{q-s}] \right) \\ &\quad + (\Delta \bar{W}_m^n)^{q-s} \left((\Delta \bar{\epsilon}_m^n)^s - \mathbb{E}[(\Delta \bar{\epsilon}_m^n)^s] \right). \end{aligned}$$

We then divide $\chi_i^n(s)$, and hence $E_n^{(s)}(2)$, into two parts and denote (by preserving the above order)

$$E_n^{(s)}(2) = \bar{E}_n^{(s)}(2) + \tilde{E}_n^{(s)}(2). \quad (\text{A.41})$$

The term $\bar{E}_n^{(s)}(2)$ can be handled using a decomposition as in Eq. (A.16) in the no noise proof:

$$(\bar{E}_n^{(s)}(2))^2 = \bar{E}_n^{(s)}(2.1) + \bar{E}_n^{(s)}(2.2),$$

where $\bar{E}_n^{(s)}(2.1)$ and $\bar{E}_n^{(s)}(2.2)$ are, respectively, composed of squared and mixed terms. We recall that the sequence $n^{s/4} \mathbb{E}[(\Delta \bar{\epsilon}_m^n)^s]$ is uniformly bounded in m and n . Then, proceeding as in Eq. (A.17), we find that

$$\mathbb{E}[\bar{E}_n^{(s)}(2.1)] \leq C \frac{p^2}{\sqrt{n}}. \quad (\text{A.42})$$

Assumption **(M)**, and the steps in the Malliavin calculus also applied in Eqs. (A.21) – (A.22), mean that

$$\mathbb{E}\left[\bar{E}_n^{(s)}(2.2)\right] \leq C \frac{p^2}{\sqrt{n}}. \quad (\text{A.43})$$

For the last term $\tilde{E}_n^{(s)}(2)$, we recall that $X \perp\!\!\!\perp \epsilon$ and $(\Delta \bar{\epsilon}_m^n)^s - \mathbb{E}[(\Delta \bar{\epsilon}_m^n)^s]$ has mean zero. Then, we decompose $(\tilde{E}_n^{(s)}(2))^2$ as in Eq. (A.16), and since the mixed terms for different l 's are mean zero, we find that

$$\mathbb{E}\left[(\tilde{E}_n^{(s)}(2))^2\right] \leq C \frac{p^2}{\sqrt{n}}. \quad (\text{A.44})$$

Hence, Eqs. (A.42) – (A.44) lead to Eq. (A.40). ■

A.4. Proof of Theorem 3.6

The proof is reminiscent to the proof of Theorem 3.2. A careful inspection of the proof of Theorem 3.5 implies that the following steps also hold under the weaker assumptions of Theorem 3.6:

$$\mathbb{E}[|Q_n - U_n|] \rightarrow 0, \quad \mathbb{E}[|U_n - R_n|] \rightarrow 0, \quad \mathbb{E}[|R_n - \Sigma^*|] \rightarrow 0, \quad \mathbb{E}[|\hat{\Sigma}_n^* - \Sigma_n^*|] \rightarrow 0.$$

Hence, given the rates on p and L , it suffices to show that $\mathbb{E}[|\Sigma_n^* - Q_n|] \rightarrow 0$. Following Lemma 4 – 5 of Podolskij and Vetter (2009a), we can prove Lemma A.6 under the assumptions of Theorem 3.6. However, we note that the rate of Lemma A.6(c) changes from Lp^2/n to p/\sqrt{n} . Then, we finish the proof by applying the Cauchy-Schwarz inequality and an additional condition $L/p \rightarrow \infty$. ■

A.5. Proofs of Theorems 3.3 and 3.4

We can prove Theorem 3.3 by employing the techniques that are used in the proof of Theorem 3.1 and 3.5. Hence, here we only sketch the main parts that enable us to find the convergence rate. We define:

$$\begin{aligned} \Sigma_n &= \frac{1}{L} \sum_{l=1}^L \left(\sqrt{\frac{n}{L}} (V_l(f, g) - V(f, g)) \right)^2, & Q_n &= \frac{p^2}{n} \sum_{l=1}^L \left(\sum_{i=1}^{n/Lp} \chi_{(i-1)L+l}^n \right)^2, \\ U_n &= \frac{p^2}{n} \sum_{l=1}^L \sum_{i=1}^{n/Lp} (\chi_{(i-1)L+l}^n)^2, & R_n &= \frac{p^2}{n} \sum_{l=1}^L \sum_{i=1}^{n/Lp} \mathbb{E}\left[(\chi_{(i-1)L+l}^n)^2 \mid \mathcal{F}_{\frac{(i-1)L+l-1}{n}}\right], \end{aligned}$$

with

$$\eta_i^n = \frac{1}{p-1} \sum_{m \in B_i(p)} f\left(\sqrt{n}\sigma_{\frac{(i-1)p}{n}} \Delta_m^n W\right) g\left(\sqrt{n}\sigma_{\frac{(i-1)p}{n}} \Delta_{m+1}^n W\right) \text{ and } \chi_i^n = \eta_i^n - \mathbb{E}\left[\eta_i^n \mid \mathcal{F}_{\frac{(i-1)p}{n}}\right].$$

There exists a $C > 0$, independent of i , such that

$$\mathbb{E}[(\eta_i^n)^4] \leq C \quad \text{and} \quad \mathbb{E}[(\chi_i^n)^4] \leq \frac{C}{p^2}, \quad (\text{A.45})$$

where the last inequality holds, because χ_i^n is a sum of 1-dependent random variates. Then, we are done due to the relationship $p/n \ll \sqrt{p/n} \ll p/\sqrt{n} \ll 1/\sqrt{L}$ (the latter follows from $n/Lp^2 \rightarrow \infty$) and the following Lemma. We omit the proof for the sake of brevity.

Lemma A.9 *Assume that the conditions of Theorem 3.3 are fulfilled. Then, we get that*

$$\mathbb{E}[|\Sigma_n - Q_n|] \leq C \left(\frac{Lp^2}{n} + \frac{p}{\sqrt{n}} \right), \quad (\text{A.9a})$$

$$\mathbb{E}[|Q_n - U_n|] \leq C \frac{1}{\sqrt{L}}, \quad (\text{A.9b})$$

$$\mathbb{E}[|U_n - R_n|] \leq C \sqrt{\frac{p}{n}}, \quad (\text{A.9c})$$

$$\mathbb{E}[|R_n - \Sigma|] \leq C \left(\frac{p}{n} + \frac{1}{p} \right), \quad (\text{A.9d})$$

$$\mathbb{E}[|\hat{\Sigma}_n - \Sigma_n|] \leq C \frac{1}{\sqrt{L}}. \quad (\text{A.9e})$$

Lastly, the proof of Theorem 3.4 is analogous to the proofs of Theorems 3.2 and 3.6 and is omitted.

A.6. Selected elements of Malliavin calculus

Here, we introduce some concepts from Malliavin calculus. The interested readers are referred to Ikeda and Watanabe (1989) and Nualart (2006) for more thorough textbooks on this subject. Set $\mathbb{H} = \mathbb{L}^2([0, 1], dx)$ and let $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ denote the scalar product on \mathbb{H} and $W = \{W(h) : h \in \mathbb{H}\}$ be an isonormal Gaussian family indexed by \mathbb{H} , i.e. the random variables $W(h)$ are centered Gaussian with a covariance structure determined via

$$\mathbb{E}[W(g)W(h)] = \langle g, h \rangle_{\mathbb{H}}.$$

In our setting, $W(h) = \int_0^1 h_s dW_s$, where W is a Brownian motion. The set of smooth random variables is introduced with

$$\mathcal{S} = \left\{ F = f(W(h_1), \dots, W(h_n)), n \geq 1, h_i \in \mathbb{H} \right\},$$

where $f \in C_p^\infty(\mathbb{R}^n)$ (i.e., the space of infinitely often differentiable functions such that all derivatives exhibit polynomial growth). The k th order Malliavin derivative of $F \in \mathcal{S}$, denoted by $D^k F$, is defined by

$$D^k F = \sum_{i_1, \dots, i_k=1}^n \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} f(W(h_1), \dots, W(h_n)) h_{i_1} \otimes \cdots \otimes h_{i_k}. \quad (\text{A.46})$$

The space $\mathbb{D}_{k,q}$ is the completion of the set \mathcal{S} with respect to the norm

$$\|F\|_{k,q} \equiv \left(\mathbb{E}[|F|^q] + \sum_{m=1}^k \mathbb{E}[\|D^m F\|_{\mathbb{H}^{\otimes m}}^q] \right)^{1/q}.$$

If $(X_t)_{t \in [0,1]}$ is a solution of a stochastic differential equation (SDE)

$$dX_t = a(X_t)dt + \sigma(X_t)dW_t,$$

and $a, \sigma \in C^1(\mathbb{R})$, then DX_t is given as the solution of the SDE

$$D_s X_t = \sigma(X_s) + \int_s^t a'(X_u) D_s(X_u) du + \int_s^t \sigma'(X_u) D_s(X_u) dW_u, \quad (\text{A.47})$$

for $s \leq t$, and $D_s X_t = 0$, if $s > t$.

The Malliavin derivative satisfies a chain rule: If $F \in \mathbb{D}_{1,2}$ and $g \in C^1(\mathbb{R})$, then it holds that

$$D(g(F)) = g'(F)DF. \quad (\text{A.48})$$

Another application of Malliavin calculus is a refinement of the Clark-Ocone formula. Let $F \in \mathbb{D}_{1,2}$, then

$$F = \mathbb{E}[F] + \int_0^1 \mathbb{E}[D_t F | \mathcal{F}_t] dW_t. \quad (\text{A.49})$$

The operator D^k possesses an unbounded adjoint denoted by δ^k (also called a multiple Skorokhod integral).

The following integration by parts formula holds: if $u \in \text{Dom}(\delta^k)$ and $F \in \mathbb{D}_{k,2}$, then

$$\mathbb{E}[F \delta^k(u)] = \mathbb{E}[\langle D^k F, u \rangle_{\mathbb{H}^{\otimes k}}]. \quad (\text{A.50})$$