Affine Term Structure Modeling and Macroeconomic Risks at the Zero Lower Bound

Guillaume ROUSSELLET*

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Abstract

This paper introduces the first affine term structure model able to include observable macroeconomic variables while being consistent with the zero lower bound. Combining gamma-zero and linear-quadratic Gaussian processes, the model produces nominal and real interest rates as closed-form functions of the macroeconomy. The short-term nominal rate is non-negative and can stay extensively at zero, and is shown to follow a Taylor-type rule. Estimation on U.S. data is performed incorporating inflation and three latent factors, and fitting errors are only of a few basis points on both yield curves. The decomposition of nominal rates in real and inflation risk premia is produced and shown to be consistent. A last section focuses on the effect of lifting off on inflation, and provides evidence of time-varying and sign-changing liftoff risk premia.

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*NYU Stern School of Business – Volatility Institute, groussel@stern.nyu.edu

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1 Introduction

While a prolonged period of low nominal interest rates has been a common state in major economies, it has been of crucial importance for policymakers and central bankers to assess market expectations about future macroeconomic outcomes and monetary policy. In particular, recent attention has been directed towards the so-called *liftoff event*, when the central bank will escape from the zero lower bound (ZLB) and raise the short-term nominal interest rate. Recent FOMC statements has made clear that the liftoff occurrence would be related to macroeconomic perspectives, but its effect on the path of future macroeconomic states, specifically inflation, is largely unknown. In addition, the existing literature has been considerably silent about the evolution of the risk premium associated with the liftoff during the zero lower bound period. Market expectations and risk premia estimates are usually extracted from the longer-end of the nominal yield curve using an asset pricing model. However, most existing models that are consistent with the zero lower bound do not belong to the *affine* class and are not easy to manipulate, both for estimation or for the study of interactions between the macroeconomy and monetary policy.

This paper follows the tradition of macro-finance models *à la* Ang and Piazzesi (2003) and is the first to introduce an affine term structure model (ATSM) able to include observable macroeconomic variables while being consistent with the zero lower bound.[2] There is an inherent technical difficulty in including observable variables in an affine ZLB-consistent model: observable variables usually take values on the entire real axis whereas most of the non-negative affine processes must have strictly positive inputs. Creating a link between the two hence requires a non-linear positive transformation of the observables that destroys the closed-formedness of the pricing formulas in most existing models.

In this paper, the model builds on two blocks. First, endogenous risk factors are assumed to be divided in observable macroeconomic variables and latent yield factors and to follow a Gaussian VAR(1) process. Second, the conditional distribution of the short-term nominal interest rate is specified as a *gamma-zero*, and its intensity is a positive quadratic combination of current macroeconomic and yield factors. The gamma-zero dis-

[2] Macro-finance asset pricing models have been widely used since their introduction as they help relating the macroeconomic fluctuations to asset prices. Notable examples include among others Diebold, Piazzesi, and Rudebusch (2005), Diebold, Rudebusch, and Aruoba (2006), Ang, Piazzesi, and Wei (2006), Hordahl, Tristani, and Vestin (2006), Dewachter and Lyrio (2006), Rudebusch and Wu (2008), and Bikbov and Chernov (2010).

tribution was introduced by Monfort, Pegoraro, Renne, and Roussellet (2014) extending the class of positive CIR-type distribution pioneered by Cox, Ingersoll, and Ross (1985), and further developed in discrete-time by Gouriéroux and Jasiak (2006) or Dai, Le, and Singleton (2010). It is an affine process which is able to mimic the behavior of the short-term nominal interest rate during the zero lower bound. The short-term interest rate is therefore to be non-negative, can reach zero and stay there for extended periods of time (its model-implied distribution has a zero point-mass). Using the conditional moments of the gamma-zero distribution, I show that it is possible to interpret this specification as a quadratic Taylor-type rule where the loadings on macroeconomic variables are varying over time as in Ang, Boivin, Dong, and Loo-Kung (2011).

The nominal pricing kernel is specified as an exponential-affine function of all factors, with time-varying prices of risk *à la* Duffee (2002). As a result, the macroeconomic and yield factors also follow a VAR(1) under the risk-neutral measure, and the short-term nominal rate is also conditionally gamma-zero distributed, with shifted parameters. In particular, the pricing kernel specification authorizes the conditional variance of the VAR(1) to be different under both physical and risk-neutral measures, an absent feature of most term structure models. Despite the inherent non-linearities in the specification, an important property is that the model belongs to the class of ATSM.

Combining CIR-type processes with linear-quadratic processes allows to get the best of both worlds. The affine property under the risk-neutral measure provides closed-form nominal interest rates at all maturities. Model-implied nominal rates are consistently positive and can be expressed as linear-quadratic combinations of both macroeconomic and latent variables. Thus the model belongs to the class of quadratic term structure models as in e.g. Constantinides (1992), Ahn, Dittmar, and Gallant (2002), Leippold and Wu (2002), Leippold and Wu (2007), Cheng and Scaillet (2007), Andreasen and Meldrum (2011), or Dubecq, Monfort, Renne, and Roussellet (2014). As long as the inflation rate is included in the set of observables, the model also produces closed-form interest rate formulas for inflation-indexed bonds at all maturities. Using the real yield curve in a term structure model improves the estimation of inflation expectations and makes it easy to isolate the inflation risk components in the nominal interest rates.[3] The affine property

[3]Since the introduction of inflation-indexed securities in 1982 in the U.K. and in 1996 in the U.S., a large literature has been focused on trying to isolate inflation risk in nominal interest rates. Fama (1976, 1990) constitutes its first attempts with linear regressions, Campbell and Shiller (1996) study the properties of inflation-linked securities before they were introduced in the U.S. and Wright (2011)

under the physical measure allows to compute macroeconomic forecasts, nominal and real interest rate forecasts, and impulse-response functions with closed-form formulas. Last, the gamma-zero distribution produces liftoff probabilities available as closed-form functions of macroeconomic variables under both the physical and the risk-neutral measure. The liftoff risk premium is easily obtained as a by-product.

I study the empirical performance of the model using monthly U.S. data from January 1990 to March 2015. The model is estimated to fit both the nominal and the real (TIPS) term structures of interest rates, expectations of inflation and expectations on long-term nominal interest rates at different horizons as measured by surveys of professional forecasters. Even though the latter are only measured quarterly, the recent quadratic Kalman filter (see Monfort, Renne, and Roussellet (2015)) treats missing data easily and is fitted to estimate the model expressed in linear-quadratic state-space form. With the year-on-year inflation rate and only three latent variables as endogenous factors, the model shows an impressive average fit of 5bps on the nominal and 12bps on the real term structures respectively. I use the estimated model to provide the decomposition of nominal interest rates in expected real rates, expected inflation, real term premia and inflation risk premia during the whole sample. Consistently with the recent literature, I show that the short-term inflation risk premium changes sign over time and reaches −200bps when the zero lower bound binds, as the fear of deflation arises. A lot of these fluctuations are offset by the short-term real term premium, producing a very low nominal risk premium component. In comparison, the long-term inflation risk premium is low and slowly fluctuating between −30bps and 30bps, emphasizing economic agents' confidence in monetary policy effectiveness, even during the zero lower bound period. Long-term nominal risk premia are therefore driven mostly by real term premia, while the 10-year nominal expected component stagnates at 100bps during the ZLB.

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uses panel data regressions to assess the size of international inflation premia. Asset pricing models have also been developed by numerous authors, including Barr and Campbell (1997), Evans (1998), Anderson and Sleath (2001), Joyce, Lildholdt, and Sorensen (2010), Garcia and Werner (2010), Zinna (2014) or Carriero, Mouabbi, and Vangelista (2015) on European data, and Campbell and Viceira (2001), Buraschi and Jiltsov (2005), Ang, Bekaert, and Wei (2008), Adrian and Wu (2009), Campbell, Shiller, and Viceira (2009), Chen, Liu, and Cheng (2010), Hordahl and Tristani (2012), Grischenko and Huang (2013), Chernov and Mueller (2012), Haubrich, Pennacchi, and Ritchken (2012), Campbell, Sunderam, and Viceira (2013) Abrahams, Adrian, Crump, and Moench (2013), Hsu, Li, and Palomino (2014), or D'Amico, Kim, and Wei (2014) use U.S. data (TIPS). Other studies are focused on exploiting the TIPS specificities to back out inflation pdfs or deflation probabilities, see among others Grischenko, Vanden, and Zhang (2011), Christensen and Gillan (2012), Christensen, Lopez, and Rudebusch (2012), Fleckenstein, Longstaff, and Lustig (2013), Kitsul and Wright (2013) or Christensen, Lopez, and Rudebusch (2014). Last, inflation expectation data can be modeled to obtain a partial decomposition of nominal interest rates, as in e.g. Ang, Bekaert, and Piazzesi (2007) or Duffee (2014).

To test the reliability of this decomposition, I extend the classical Campbell and Shiller (1991) regressions considering jointly nominal and real term structures. While excess returns of nominal rates are often expressed in nominal terms and excess returns of real rates are expressed in real terms, it is possible to formulate real excess returns of nominal rates and nominal excess returns of real rates. I obtain two new regressions, in addition to the two existing ones. Using the LPY-I and LPY-II conditions as formulated by Dai and Singleton (2002), I provide evidence that the estimated factor dynamics under both the physical and the risk-neutral measure are able to reproduce the observed deviations from the expectation hypothesis, and that the model produces consistent expected excess returns. This result might seem at odds with the previous literature on CIR-type models where they are shown to have difficulties getting reliable risk premia estimates. However, the rich linear-quadratic structure of the model allows to correct for this bias.

The last part of the paper is devoted to quantify the interactions between monetary policy and the inflation rate during the zero lower bound. I find that as the central bank is trying to restore positive short-term inflation when the economy hits the ZLB, the loading on inflation in the Taylor-type rule formulation drops dramatically to virtually zero. Whereas positive inflation shocks do not have any impact on the short-term interest rates at the ZLB, a 10bps persistent positive monetary policy shock translates into -7bps on the inflation rate at the ZLB, and the inflation risk premia falls by about -20bps. I compute both objective and risk-neutral liftoff probabilities, and the impact of inflation shocks on the liftoff. I document that the liftoff risk premium becomes significantly negative between 2011 and 2013 as the difference between physical and risk-neutral probabilities reaches -0.2, while being consistently positive during the rest of the ZLB period. Though the inflation rate drives most of the liftoff probabilities' fluctuations, the liftoff risk premium is less related to developments in inflation risk premia. One implication of these computations is that the liftoff is likely to be strongly influenced by inflation outcomes, and the liftoff event can be a desirable outcome for economic agents.

This paper hence contribute to the fast-growing literature on modeling the term structure at the ZLB and extracting liftoff probabilities. A large number of authors have focused on the so-called shadow-rate (or Black (1995) model), such as e.g. Kim and Singleton (2012), Bauer and Rudebusch (2013), Kim and Priebsch (2013), Krippner (2013), Wu and Xia

(2013), Priebsch (2013), Christensen and Rudebusch (2013), Jackson (2014), Lemke and Vladu (2014), Pericoli and Taboga (2015), or Andreasen and Meldrum (2015).[4] However, the shadow-rate model does not produce closed-form pricing formulas. This often leads to computational complexity in terms of estimation when numerous securities are included. The absence of closed-form liftoff probabilities does not allow for the decomposition of the liftoff probabilities.

The remainder of the paper is organized as follows. Section 2 presents the formulation and the properties of the term structure model. Section 3 details the data and the estimation strategy, while Section 4 focuses on a first analysis of the results in terms of interest rate decomposition. Section 5 is devoted to the study of the monetary policy during the zero lower bound, and produces impulse-response functions and the liftoff probabilities. Section 6 concludes.

2 The Model

2.1 Macroeconomic and yields joint dynamics

Consider an economy where the risks are driven by n risk factors X_t. These factors gather both observable macroeconomic variables denoted by $M_t \in \mathbb{R}^{n_M}$ and latent yield factors denoted by $Z_t \in \mathbb{R}^{n_Z}$, such that $X_t = [M_t', Z_t]'$. The joint dynamics of X_t is given by a standard Gaussian VAR(1) of the following form:

$$\begin{pmatrix} M_t \\ Z_t \end{pmatrix} = \begin{pmatrix} \mu_M \\ 0 \end{pmatrix} + \begin{pmatrix} \Phi_M & \Phi_{M,Z} \\ \Phi_{Z,M} & \Phi_Z \end{pmatrix} \begin{pmatrix} M_{t-1} \\ Z_{t-1} \end{pmatrix} + \begin{pmatrix} \Sigma_M & 0 \\ 0 & I_{n_Z} \end{pmatrix}^{1/2} \varepsilon_t \, , \tag{1}$$

where ε_t is a zero-mean unit-variance Gaussian white noise of size n. This dynamics can be written in compact form as:

$$X_t = \mu + \Phi X_{t-1} + \Sigma^{1/2} \varepsilon_t \, , \tag{2}$$

[4]Note that alternative approaches have been developed to enforce the zero lower bound. Filipovic, Larsson, and Trolle (2013) develop the linear-rational term structure model and Feunou, Fontaine, and Le (2015) model directly the price of bonds in a nearly arbitrage-free framework. Renne (2014) uses a term structure model where the short-rate can reach discrete positive states.

with according sizes for μ, Φ and Σ.

At time t, economic agents can invest in a one-period risk-less zero-coupon bond providing a known interest rate r_t at $t+1$. The dynamics of r_t is designed to be consistent with three main properties. First, consistently with standard monetary policy rules, r_t depends on the current macroeconomic state M_t. Second, due to no-arbitrage with holding cash (including possible storage costs), the short-term interest rate r_t should have a lower bound. Last, empirical consistency imposes that r_t should be able to stay at its lower bound for extended periods to reproduce the high persistence of very low interest rates. The gamma-zero distribution introduced by Monfort, Pegoraro, Renne, and Roussellet (2014) is well-fitted to gather the previous properties. Formally:

$$r_t|(\underline{X_t}, \underline{r_{t-1}}) \sim \gamma_0\left(\theta_0 + \theta' X_t + X_t'\Theta X_t, \varsigma\right) , \tag{3}$$

where $(\underline{X_t}, \underline{r_{t-1}}) = \{X_t, (r_{t-1}, X_{t-1}, r_{t-2}, X_{t-2}, \ldots)\}$ is the set of present and past risk factors and of the past short-term interest rates, θ_0 is a constant, θ is a vector of size n and Θ is a positive symmetric $(n \times n)$ matrix.

The gamma-zero distribution defines a non-negative random variable. The first argument $(\theta_0 + \theta' X_t + X_t'\Theta X_t)$ must also be non-negative for the distribution to be well-defined, and it is sufficient that $\theta_0 = \frac{1}{4}\theta'\Theta^{-1}\theta$. This argument is the intensity of an underlying Poisson mixing variable controlling the shape of the gamma distribution. When this mixing variable is equal to zero, the short-rate distribution collapses to a Dirac mass at zero. Conversely, the higher the mixing variable, the more the short-rate probability density function is shifted to the right. The second argument ς is a positive scaling parameter. As such, the short-term interest rate gathers both a non-linear dependence to macroeconomic risks, and the ability to reach zero and to stay there for several periods. Details on the gamma-zero distribution are provided in Appendix A.1.

This formulation also allows a more conventional interpretation in terms of the underlying monetary policy reaction function. From the gamma-zero distribution properties (see

Appendix A.1), I have:

$$\begin{aligned} r_t &= \varsigma\left(\theta_0 + \theta' X_t + X_t'\Theta X_t\right) + \upsilon_t \quad \text{with} \quad \mathbb{E}(\upsilon_t | \underline{X_t}, \underline{r_{t-1}}) = 0 \\ &= \varsigma\left[\theta_0 + \left(\theta_M + 2\Theta_{M,Z} Z_t + \Theta_M M_t\right)' M_t + Z_t'\Theta_Z Z_t\right] + \upsilon_t \\ &=: \alpha + \beta_t' M_t + \delta_t + \upsilon_t \,, \end{aligned} \tag{4}$$

where the subscripts $(.)_M$ and $(.)_Z$ are explicit notations for the partitions of θ and Θ. Equation (4) can be seen as a Taylor-type rule – in which case, M_t would typically be composed at least of the inflation rate – where the loadings on macroeconomic variables are time-varying Gaussian variables, as in e.g. Ang, Boivin, Dong, and Loo-Kung (2011). β_t therefore represents the central bank's sensitivity to macroeconomic shocks. In contrast, both δ_t and υ_t are monetary policy shocks that are instantaneously uncorrelated with macroeconomic shocks. The former is authorized to be persistent, whereas the latter is non-persistent. Note however that the distributions of the monetary policy shocks δ_t and υ_t are non-Gaussian, resulting in a crucial difference with the standard Taylor-type rule case.[5]

2.2 The nominal pricing kernel and the risk-neutral dynamics

Between $t-1$ and t, economic agents discount payoffs with the nominal pricing kernel (or stochastic discount factor, SDF henceforth) denoted by $m_{t-1,t}$. As standard in asset pricing models, I specify the SDF as an exponential-affine function of (X_t, r_t) with time-varying prices of risk.

$$m_{t-1,t} = \exp\left\{-r_{t-1} + \Lambda_{t-1}' X_t + \Lambda_r r_t - \log \mathbb{E}\left[\exp\left(\Lambda_{t-1}' X_t + \Lambda_r r_t | \underline{X_{t-1}}, \underline{r_{t-1}}\right)\right]\right\} \,, \tag{5}$$

where the last term is the convexity adjustment such that $\mathbb{E}(m_{t-1,t}|\underline{X_{t-1}}, \underline{r_{t-1}}) = \exp(-r_{t-1})$, Λ_r is the price of short-term nominal interest rate risk and Λ_{t-1} is the size n vector gathering the prices of macroeconomic shocks and yield factors. The prices of risk are themselves affine function of the past risk factors X_{t-1}, in the spirit of essentially affine models introduced by Duffee (2002):

$$\Lambda_{t-1} =: \lambda_0 + \lambda X_{t-1} \,. \tag{6}$$

[5] This short-term interest rate specification should not be interpreted as a structural monetary policy reaction function. As noted by Backus, Chernov, and Zin (2015), the identification of the structural Taylor rule parameters can be difficult in the affine framework. I therefore interpret the present specification as a reduced-form for the short-term interest rate dynamics.

The previous SDF specification allows for a very simple derivation of risk-neutral $\mathbb{Q}$-dynamics of (X_t, r_t). In particular, the form of Equations (5) and (6) preserves the same class of probability distributions under the risk-neutral measure. It is shown in Appendix A.2 that under the risk-neutral measure:

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} + \Sigma^{\mathbb{Q}^{1/2}} \varepsilon_t^{\mathbb{Q}} , \tag{7}$$

where $\varepsilon_t^{\mathbb{Q}}$ is a zero-mean unit-variance Gaussian white noise, and $\mu^{\mathbb{Q}}$, $\Phi^{\mathbb{Q}}$ and $\Sigma^{\mathbb{Q}}$ are given by:

$$\begin{cases} \mu^{\mathbb{Q}} &= \Sigma^{\mathbb{Q}} \left(\lambda_0 + \dfrac{\Lambda_r \varsigma}{1 - \Lambda_r \varsigma} \theta + \Sigma^{-1} \mu \right) , \\ \Phi^{\mathbb{Q}} &= \Sigma^{\mathbb{Q}} \left(\lambda + \Sigma^{-1} \Phi \right) , \\ \Sigma^{\mathbb{Q}} &= \left(\Sigma^{-1} - 2 \dfrac{\Lambda_r \varsigma}{1 - \Lambda_r \varsigma} \Theta \right)^{-1} . \end{cases} \tag{8}$$

Equations (7) and (8) state that the macroeconomic variables M_t and the yield factors Z_t also have Gaussian dynamics under the risk-neutral measure, but with shifted parameters. Additional flexibility appears compared to the standard Gaussian ATSM. First, when the price of the short-term interest rate risk Λ_r is different from zero, the conditional variance of the Gaussian VAR is different under the physical and the risk-neutral measure, an absent feature of most ATSMs.[6] When $\Lambda_r = 0$ however, standard transition formulas are obtained and $\Sigma^{\mathbb{Q}} = \Sigma$. Second, the short-term interest rate physical parameters θ and Θ explicitly appear in the risk-neutral dynamics of X_t.

To get intuition on these risk-neutral parameters, let us assume that the only state variable is the inflation rate ($n = n_M = 1$) and that the short-term interest rate risk is priced positively, that is $\Lambda_r > 0$. Then $\Sigma^{\mathbb{Q}} = \frac{1 - \Lambda_r \varsigma \Sigma}{1 - \Lambda_r \varsigma (1 + 2\Sigma\Theta)}$. In that case, a positive Θ increase the discrepancy between $\Sigma^{\mathbb{Q}}$ and Σ, implying a positive premium on the variance of inflation shocks. In turn, this gap increases the difference in persistence under the two measures and in the marginal mean of the process as long as θ is positive, such that there is a positive premium associated with high inflation. In economic terms, keeping the current value of inflation constant, high θ and Θ indicates a more aggressive monetary policy towards high inflation (the parabola $\theta_0 + \theta M_t + \Theta M_t^2$ is shifted to the left, making its

[6] A notable exception is the model developed by Monfort and Pegoraro (2012), where the risk factors follow a standard Gaussian VAR under the physical measure, and the SDF is an exponential-quadratic function of the risk factors. In this case the risk-neutral conditional variance of the VAR is different from the physical one.

first derivative increase in $M_t = M$). This aggressiveness imply a more volatile short-term interest rate, which makes it more difficult for economic agents to obtain smooth aggregates, and turns into a positive risk premium.

Similar transition formulas between the physical and risk-neutral measures can be derived for the short-term interest rate dynamics. Under the risk-neutral measure (see Appendix A.2):

$$r_t|(\underline{X_t}, \underline{r_{t-1}}) \overset{\mathbb{Q}}{\sim} \gamma_0 \left(\theta_0^{\mathbb{Q}} + \theta^{\mathbb{Q}\prime} X_t + X_t' \Theta^{\mathbb{Q}} X_t, \varsigma^{\mathbb{Q}}\right), \tag{9}$$

where the risk-neutral parameters are given by:

$$\theta_0^{\mathbb{Q}} = \frac{\theta_0}{1 - \Lambda_r \varsigma}, \quad \theta^{\mathbb{Q}} = \frac{1}{1 - \Lambda_r \varsigma}\theta, \quad \Theta^{\mathbb{Q}} = \frac{1}{1 - \Lambda_r \varsigma}\Theta, \quad \text{and} \quad \varsigma^{\mathbb{Q}} = \frac{\varsigma}{1 - \Lambda_r \varsigma}. \tag{10}$$

Again, a positive Λ_r drives a positive discrepancy between risk-neutral and physical parameters, shifting all risk-neutral moments of the short-term interest rate upwards. Interestingly, Equations (9) and (10) also imply a different Taylor-type rule under the risk neutral measure. Similarly to Equation (4), I have:

$$\begin{aligned} r_t &= \varsigma^{\mathbb{Q}} \left(\theta_0^{\mathbb{Q}} + \theta^{\mathbb{Q}\prime} X_t + X_t' \Theta^{\mathbb{Q}} X_t\right) + \upsilon_t^{\mathbb{Q}} \quad \text{with} \quad \mathbb{E}^{\mathbb{Q}}(\upsilon_t^{\mathbb{Q}}|\underline{X_t}, \underline{r_{t-1}}) = 0 \\ &= \frac{\varsigma\,(\theta_0 + \theta' X_t + X_t' \Theta X_t)}{(1 - \Lambda_r \varsigma)^2} + \upsilon_t^{\mathbb{Q}} = \frac{\alpha + \beta_t' M_t + \delta_t}{(1 - \Lambda_r \varsigma)^2} + \upsilon_t^{\mathbb{Q}} \\ &= \alpha^{\mathbb{Q}} + \beta_t^{\mathbb{Q}\prime} M_t + \delta_t^{\mathbb{Q}} + \upsilon_t^{\mathbb{Q}}, \end{aligned} \tag{11}$$

where the risk-neutral sensitivity to the macroeconomic variables $\beta_t^{\mathbb{Q}}$ is exactly proportional to the physical one β_t by a factor $\frac{1}{(1-\Lambda_r \varsigma)^2}$. This discrepancy is indeed a reflection of the premium attributed to the aggressiveness of the monetary policy driven by the price of the short-term interest rate risk Λ_r.

2.3 The affine property of the model

Affine asset pricing models are a very convenient class of models since they allow to obtain closed-form price and interest rate formulas for zero-coupon bonds.[7] A model verifies the affine property if the joint process gathering the risk-factors and the short-term interest rate is *affine* under the risk-neutral measure. Mathematically, this means that the conditional Laplace transform of this joint process under the risk-neutral measure given its

[7] see e.g. Duffie and Kan (1996), Dai and Singleton (2000), or Darolles, Gourieroux, and Jasiak (2006).

past is an exponential-affine function of its past values. Though not obvious from the previous sections, this section shows that the present model belongs to the class of affine term structure models (ATSM henceforth).

To prove the affine property of the model, we need to consider the *extended vector of factors* (in the terminology of Cheng and Scaillet (2007)) expressed as follows. Let us consider $f_t = [X_t', \text{Vec}(X_t X_t'), r_t]'$ the vector of size $(n + n^2 + 1)$ gathering linear and quadratic combinations of X_t and the short-term interest rate r_t. The conditional Laplace transform of f_t given its past under the risk-neutral measure is given by:

$$\phi_{t-1}^{\mathbb{Q}}(u) = \mathbb{E}^{\mathbb{Q}}\left[\exp(u'f_t)\big|\underline{f_{t-1}}\right] \quad \text{where} \quad u = [u_x', \text{Vec}(U_x)', u_r]' . \tag{12}$$

The explicit computation of $\phi_{t-1}^{\mathbb{Q}}(u)$ requires the law of iterated expectations, the conditional Laplace transform of a gamma-zero distribution, and the conditional Laplace transform of a linear-quadratic combination of a Gaussian process. I obtain:

$$\phi_{t-1}^{\mathbb{Q}}(u) \quad =: \quad \exp\left\{\mathbb{A}^{\mathbb{Q}}(u) + \mathbb{B}^{\mathbb{Q}'}(u)X_{t-1} + X_{t-1}'\mathbb{C}^{\mathbb{Q}}(u)\,X_{t-1}\right\} ,$$

where the loadings $\mathbb{A}^{\mathbb{Q}}(u)$, $\mathbb{B}^{\mathbb{Q}}(u)$ and $\mathbb{C}^{\mathbb{Q}}(u)$ are detailed in Appendix A.2. Hence $\phi_{t-1}^{\mathbb{Q}}(u)$ is an exponential-quadratic function of X_{t-1}, i.e. an exponential-affine function of f_{t-1}, and (f_t) is an affine process under the risk-neutral measure. With the quadratic combination of X_{t-1} the model is indeed a quadratic term structure model (QTSM), a class included in the broader class of ATSMs. An important corollary for the pricing of bonds is that the multi-horizon conditional Laplace transform of $(f_t, \ldots, f_{t+k})$ given $\underline{f_{t-1}}$ is also an exponential-affine function of f_{t-1}. More specifically, letting $(u_0, \ldots, u_k)$ being vectors of size $n + n^2 + 1$, I have:

$$\begin{aligned}
\phi_{t-1}^{\mathbb{Q}}(u_0, \ldots, u_k) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sum_{i=0}^{k} u_i' f_{t+i}\right)\big|\underline{f_{t-1}}\right] \\
=: \quad &\exp\left(\mathbb{A}_k^{\mathbb{Q}}(u_0, \ldots, u_k) + \mathbb{B}_k^{\mathbb{Q}'}(u_0, \ldots, u_k)X_{t-1} + X_{t-1}'\mathbb{C}_k^{\mathbb{Q}}(u_0, \ldots, u_k)\,X_{t-1}\right) ,
\end{aligned} \tag{13}$$

where $\mathbb{A}_k^{\mathbb{Q}}(u_0, \ldots, u_k)$, $\mathbb{B}_k^{\mathbb{Q}}(u_0, \ldots, u_k)$ and $\mathbb{C}_k^{\mathbb{Q}}(u_0, \ldots, u_k)$ can be derived with closed-form recursions given in Appendix A.3. Note for future record that any object which can be expressed as the multi-horizon conditional Laplace transform $\phi_{t-1}^{\mathbb{Q}}(u_0, \ldots, u_k)$ for some arguments $(u_0, \ldots, u_k)$, such as bond prices, will have a closed-form expression.

Since the class of distributions of (X_t, r_t) are the same under the physical and the risk-neutral measures (see previous Section), it follows naturally that f_t is also an affine process under the physical measure. The conditional Laplace transform of f_t given $\underline{f_{t-1}}$ under the physical measure $\phi_{t-1}(u)$ is thus the same function as the risk-neutral one $\phi^{\mathbb{Q}}_{t-1}(u)$, but plugging in the physical parameters instead of the risk-neutral ones. I obtain trivially:

$$\begin{aligned} \phi_{t-1}(u_0, \dots, u_k) &= \mathbb{E}\left[\exp\left(\sum_{i=0}^{k} u_i' f_{t+i}\right) | \underline{f_{t-1}}\right] \\ =: \quad & \exp\left(\mathbb{A}_k(u_0, \dots, u_k) + \mathbb{B}_k'(u_0, \dots, u_k) X_{t-1} + X_{t-1}' \mathbb{C}_k(u_0, \dots, u_k)\, X_{t-1}\right) . \quad (14) \end{aligned}$$

The properties of affine processes imply that the first-two conditional moments of f_t given its past are affine functions of f_{t-1}, and that the marginal mean and covariance matrix of f_t can be obtained in closed-form. More specifically, the dynamics of f_t can be expressed as:

$$f_t =: \Psi_0 + \Psi f_{t-1} + \left[\mathrm{Vec}^{-1}\left(\Omega_0 + \Omega f_{t-1}\right)\right]^{1/2} \xi_t \,, \qquad (15)$$

where ξ_t is a martingale difference with zero mean and unit variance, and exact formulas for Ψ_0, Ψ, Ω_0 and Ω depend explicitly on $\{\mu, \Phi, \Sigma, \theta_0, \theta, \Theta, \varsigma\}$ and can be found in Appendix A.4. Equation (15) provides a semi-strong affine VAR(1) representation for the entire vector of factors f_t, which will constitute the transition equation of the model.

2.4 Pricing nominal zero-coupon bonds

Nominal zero-coupon bonds are securities that deliver one unit of cash at maturity date only. Let us denote by $B(t, h)$ and $R(t, h)$ respectively the price and continuously compounded interest rate of a nominal zero-coupon bond at time t, with residual maturity h. Standard no-arbitrage arguments imply:

$$B(t, h) = \mathbb{E}^{\mathbb{Q}}\left[\exp(-r_t) B(t+1, h-1) \big| \underline{f_t}\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=0}^{h-1} r_{t+i}\right) | \underline{f_t}\right] . \qquad (16)$$

Since r_t is conditionally gamma-zero distributed under the risk-neutral measure, it admits a zero-lower bound. Formula (16) emphasizes that the non-negativity of r_t imposes that bond prices at all maturities are constrained between 0 and 1, such that the associated interest rates are always positive.

Equation (16) also emphasizes that the price of any nominal bond is the multi-horizon conditional Laplace transform $\phi_t^{\mathbb{Q}}(u_1, \ldots, u_h)$ where $u_i = [0_n', 0_{n^2}', -1]'$. Therefore, the price $B(t,h)$ is given by an exponential-affine function of f_t as is the case for any ATSM. Said differently, the interest rate $R(t,h)$ is a linear-quadratic function of X_t and an affine function of r_t where the loadings are computable recursively in closed-form:

$$
\begin{aligned}
B(t,h) &= \exp\left(A_h + B_h' X_t + X_t' C_h X_t - r_t\right) \\
\text{hence} \quad R(t,h) &= a_h + b_h' X_t + X_t' c_h X_t + \frac{r_t}{h} =: a_h + \mathcal{B}_h' f_t \,,
\end{aligned} \tag{17}
$$

where $a_h = -A_h/h$, $b_h = -B_h/h$ and $c_h = -C_h/h$, and $\mathcal{B}_h = (b_h', \text{Vec}(c_h)', 1/h)'$, and the explicit recursive expressions for computing A_h, B_h and C_h are given by:

$$
\begin{aligned}
A_h &= A_{h-1} + \mathbb{A}^{\mathbb{Q}}\left(\left[B_{h-1}', \text{Vec}\left(C_{h-1}\right)', -1\right]'\right) \\
&= A_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}\theta_0^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}} - \tfrac{1}{2}\log\left|I_n - 2\Sigma^{\mathbb{Q}}\left(C_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right| \\
&\quad + \left(B_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'\left[I_n - 2\Sigma^{\mathbb{Q}}\left(C_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\left[\mu^{\mathbb{Q}} + \tfrac{1}{2}\Sigma^{\mathbb{Q}}\left(B_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)\right] \\
&\quad + \mu^{\mathbb{Q}\prime}\left(C_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n - 2\Sigma^{\mathbb{Q}}\left(C_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\mu^{\mathbb{Q}} \\
B_h &= \mathbb{B}^{\mathbb{Q}}\left(\left[B_{h-1}', \text{Vec}\left(C_{h-1}\right)', -1\right]'\right) \\
&= \left[\left(B_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)' + 2\mu^{\mathbb{Q}\prime}\left(C_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]\left[I_n - 2\Sigma^{\mathbb{Q}}\left(C_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}} \\
C_h &= \mathbb{C}^{\mathbb{Q}}\left(\left[B_{h-1}', \text{Vec}\left(C_{h-1}\right)', -1\right]'\right) \\
&= \Phi^{\mathbb{Q}\prime}\left(C_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n - 2\Sigma^{\mathbb{Q}}\left(C_{h-1} - \tfrac{\varsigma^{\mathbb{Q}}}{1+\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}} \,.
\end{aligned} \tag{18}
$$

Combining the VAR representation of Equation (15) with the interest rate formula of Equation (17), I obtain that the conditional forecasts of nominal interest rates are also affine functions of f_t.

$$
\mathbb{E}\left[R(t+k,h)\middle|\underline{f_t}\right] = a_h + \mathcal{B}_h'\left[(I_{n+n^2+1} - \Psi)^{-1}(I_{n+n^2+1} - \Psi^k)\Psi_0 + \Psi^k f_t\right] \,. \tag{19}
$$

Importantly, nominal bond interest rates and their forecasts are explicit functions of macroeconomic variables M_t, in a quadratic fashion.

2.5 Pricing inflation-indexed zero-coupon bonds

Inflation-indexed zero-coupon bonds (or TIPS) are securities that deliver a payment at maturity which is equal to the compounded inflation between the inception date and the maturity date. As such, they can be seen as inflation hedges in nominal terms, or risk-less investments in real terms. Let us denote by $B^*(t,h)$ the price in dollars of an inflation-indexed zero-coupon bond issued at time t and maturing at $t+h$. The reference price index used to compute inflation-indexed securities payments is denoted by P_t. The one-period inflation rate between t and $t+1$ is denoted by π_{t+1} and is equal to $\log(P_{t+1}/P_t)$. Standard no-arbitrage arguments imply:

$$\begin{aligned} B^*(t,h) &= \mathbb{E}^{\mathbb{Q}}\left[\exp(-r_t+\pi_{t+1})B^*(t+1,h-1)\big|\underline{f_t}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=0}^{h-1}(r_{t+i}-\pi_{t+i+1})\right)\big|\underline{f_t}\right], \end{aligned} \tag{20}$$

that is the current price of an inflation-indexed bond is the conditional risk-neutral expectation of the compounded *ex-post* real interest rate between the inception and the maturity date. For inflation-indexed bonds, the phrasing *interest rate* actually refers to the *ex-ante* real interest rate – denoted by $R_a^*(t,h)$ in this model – and which is given by $R_a^*(t,h) = -\frac{1}{h}\log B^*(t,h)$.

To be able to price inflation-indexed bonds in closed-form, we need to consider models where the inflation rate is included in the set of observable macroeconomic variables M_t.[8] The simplest version would be to include directly the period-to-period inflation rate π_t in M_t, but a larger set of models can be considered using an inflation rate reflecting longer time-variation. In the remainder of the section, I consider models where the inflation rate at longer time horizons $\pi_{t-k,t} = \log(P_t/P_{t-k})$ is the first component of M_t.

The simplest bonds to price are those which maturity is a multiple of k. A specific case of Equation (20) is:

$$B^*(t,kh) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=0}^{kh-1} r_{t+i} + \sum_{i=0}^{h-1}\pi_{t+ki,\,t+k(i+1)}\right)\big|\underline{f_t}\right]. \tag{21}$$

As for nominal bonds, the conditional Laplace transform of Equation (21) is only the conditional multi-horizon Laplace transform $\phi_t^{\mathbb{Q}}(u_1,\ldots,u_{kh})$ where $u_i = [e_1',\, 0_{n^2}',\, -1]'$ with e_1 being the first colmun of the identity matrix I_n if i is a multiple of k, and $u_i = [0_n',\, 0_{n^2}',\, -1]'$ otherwise. Thus, again, the price of inflation-indexed bonds can be expressed as an exponential-affine function of f_t, that is the *ex-ante* real interest rate $R_a^*(t,h)$ can be expressed as a linear-quadratic function of X_t and an affine function of r_t

[8] Another option would be to consider π_t as a quadratic combination of latent variables and filtering it from the data, as in e.g. Abrahams, Adrian, Crump, and Moench (2013).

where the loadings are computable recursively in closed-form.

$$\begin{aligned} B^*(t,kh) &= \exp\left(A^*_{kh} + B^{*\prime}_{kh} X_t + X_t' C_{kh} X_t - r_t\right) \\ \text{hence} \quad R^*_a(t,kh) &= a^*_{kh} + b^{*\prime}_{kh} X_t + X_t' c_{kh} X_t + \frac{r_t}{kh} =: a^*_{kh} + \mathcal{B}^{*\prime}_{kh} f_t \,, \end{aligned} \tag{22}$$

where $a^*_{kh} = -A^*_{kh}/kh$, $b^*_{kh} = -B^*_{kh}/kh$ and $c_{kh} = -C_{kh}/kh$, and $\mathcal{B}^*_{kh} = (b^{*\prime}_{kh}, \text{ Vec}(c_{kh})', \frac{1}{kh})'$, and the explicit recursive expressions for computing A^*_{kh}, B^*_{kh} and C_{kh} are obtained considering directly the loadings $\mathbb{A}^{\mathbb{Q}}_k(u_1,\ldots,u_{kh})$, $\mathbb{B}^{\mathbb{Q}}_k(u_1,\ldots,u_{kh})$ and $\mathbb{C}^{\mathbb{Q}}_k(u_1,\ldots,u_{kh})$ of the multi-horizon Laplace transform of Equation (13) (see Appendix A.3).[9] Again, movements in the term structure of *ex-ante* real interest rates can be directly related to movements not only in inflation, but also in the other possible macroeconomic variables present in M_t.

When $k > 1$, the rest of the real yield curve can also be obtained easily. One just has to notice that the dynamics of the vector of risk factors X_t can be expressed equivalently in a larger form with the extended vector $\widetilde{X}_t$:

$$\widetilde{X}_t = [M_t', \, Z_t', \, \pi_{t-1,t}, \, \ldots, \, \pi_{t-k,t-k+1}]' \,.$$

The probability distribution of this vector is degenerate since $\pi_{t-k,t} = \pi_{t-k,t-k+1} + \ldots + \pi_{t-1,t}$, but its dynamics are easily expressed with a constrained Gaussian VAR using Equation (2). The whole model can be expressed in terms of $\widetilde{X}_t$ in the same spirit as in the previous sections, the affine property still holds and the pricing of inflation-indexed bonds can be performed at all maturities. The entire procedure is detailed in Appendix A.5.

2.6 Decomposing liftoff probabilities

Since the model-implied short-term interest rate r_t is allowed to reach zero and stay there for several periods, the model provides a natural framework for analyzing the so-called liftoff probabilities. These probabilities are associated with the event that the economy goes out of the zero lower bound in the future. Though this economic statement can be represented by different probabilistic events, I choose to focus on the short-term interest rate reaching a strictly positive value at a fixed point in the future, that is $\{r_{t+k} > 0\}$.

[9] Note that the quadratic loadings C_h are the same for nominal and real bonds. This can easily be shown using the recursions presented in the Appendix.

One appealing characteristic of the gamma-zero distribution is that conditional probabilities of the future short-term interest rate reaching zero are available in closed-form using its multi-horizon Laplace transform (see Monfort, Pegoraro, Renne, and Roussellet (2014)). Letting $u = \left[0'_n,\, 0'_{n^2},\, u_r\right]'$, the liftoff probability under the physical measure is given by:

$$\begin{aligned}
&\mathbb{P}\left(r_{t+k} > 0 \middle| \underline{f_t}\right) = 1 - \lim_{u_r \to -\infty} \mathbb{E}\left[\exp\left(u_r r_{t+k}\right) \middle| \underline{f_t}\right] \\
=\ & 1 - \lim_{u_r \to -\infty} \phi_t(0, \ldots, 0, u) \\
=\ & 1 - \exp\left(\mathbb{A}_k(0, \ldots, 0, u^{(-\infty)}) + \mathbb{B}'_k(0, \ldots, 0, u^{(-\infty)}) X_t + X'_t \mathbb{C}_k(0, \ldots, 0, u^{(-\infty)}) X_t\right) \quad (23)
\end{aligned}$$

where $\mathbb{A}_k(0, \ldots, 0, u^{(-\infty)}) = \lim_{u_r \to -\infty} \mathbb{A}_k(0, \ldots, 0, u)$, $\mathbb{B}_k(0, \ldots, 0, u^{(-\infty)}) = \lim_{u_r \to -\infty} \mathbb{B}_k(0, \ldots, 0, u)$ and $\mathbb{C}_k(0, \ldots, 0, u^{(-\infty)}) = \lim_{u_r \to -\infty} \mathbb{C}_k(0, \ldots, 0, u)$. The liftoff probability is therefore an exponential-quadratic function of X_t. It turns out that the limit of the loadings is available with closed-form recursions. Again, since the class of distributions are the same under both the physical and the risk-neutral measure, similar formulas are obtained in the risk-neutral world:

$$\begin{aligned}
\mathbb{Q}(r_{t+k} > 0 | \underline{f_t}) \ = \ & 1 - \exp\left(\mathbb{A}^{\mathbb{Q}}_k(0, \ldots, 0, u^{(-\infty)}) + \mathbb{B}^{\mathbb{Q}'}_k(0, \ldots, 0, u^{(-\infty)}) X_t \right. \\
& + \left. X'_t \mathbb{C}^{\mathbb{Q}}_k(0, \ldots, 0, u^{(-\infty)}) X_t\right). \quad (24)
\end{aligned}$$

Whereas $\mathbb{P}$-probabilities of liftoff are fairly intuitive, interpreting risk-neutral liftoff probabilities is less trivial. They can actually be considered as the cost of lifting off. Consider a *zero lower bound insurance bond* which provides a payoff at maturity if and only if the economy is still at the zero lower bound. Now define the payoff of this bond as indexed on the compounded short-term interest rate up to one date before maturity. The price $B_{zlb}(t, h)$ of such a bond trading at time t and maturing at $t + h$ is given by:

$$\begin{aligned}
B_{zlb}(t, h) \ &= \ \mathbb{E}^{\mathbb{Q}}\left[\overbrace{\exp\left(-\sum_{i=0}^{h-1} r_{t+i}\right)}^{discount} \times \overbrace{\exp\left(\sum_{i=0}^{h-1} r_{t+i}\right) \mathbb{1}\{r_{t+h} = 0\}}^{payoff} \middle| \underline{f_t}\right] \\
&= \ \mathbb{E}^{\mathbb{Q}}\left(\mathbb{1}\{r_{t+h} = 0\} \middle| \underline{f_t}\right) = \mathbb{Q}\left(r_{t+h} = 0 \middle| \underline{f_t}\right),
\end{aligned}$$

that is the risk-neutral probability of still being in the zero lower bound is exactly equal to the price of this synthetic zero lower bound insurance bond. As well as for any bond, a corresponding *ex-ante* interest rate can be calculated as $R_{zlb}(t,h) = -\frac{1}{h}\log B_{zlb}(t,h)$. Since $B_{zlb}(t,h)$ is an exponential-affine quadratic function of X_t, it turns out that $R_{zlb}(t,h)$ has the usual linear-quadratic expression:

$$R_{zlb}(t,h) = \frac{-\mathbb{A}_h^{\mathbb{Q}}(0,\ldots,0,u^{(-\infty)})}{h} - \frac{\mathbb{B}_h^{\mathbb{Q}'}(0,\ldots,0,u^{(-\infty)})}{h}X_t - X_t'\frac{\mathbb{C}_h^{\mathbb{Q}}(0,\ldots,0,u^{(-\infty)})}{h}X_t\,. \quad (25)$$

When concerns about still being in the ZLB at date $t+h$ increase, either due to risk premia or the fundamental state of the economy, the price $B_{zlb}(t,h)$ supposedly increase. Hence the interest rate $R_{zlb}(t,h)$ grows with the liftoff probability up to horizon h.

A nice feature associated with the previous closed-form expressions is that the influence of the macroeconomic variables M_t on the liftoffs can be easily computed. Indeed, I can first calculate the partial first derivative of the liftoff probabilities with respect to the macroeconomic variables using Equations (23) and (24). Second, I can also decompose $R_{zlb}(t,h)$ in its different components, namely all that is a function of the macroeconomic variables M_t, and all that is a function of the rest. I can additionally obtain the usual expected/risk premia decomposition on $R_{zlb}(t,h)$ to observe the evolution of the fear of liftoff. The empirical application presented below explores all these routes.

2.7 Possible extensions

Even though the previous model is already very rich, it is possible to enhance it in several dimensions. I mention a few improvements in this section, and their empirical relevance is left for further research.

Note first that the unspanned macroeconomic factors model of Joslin, Priebsch, and Singleton (2014) is a nested specification of the general model presented above. Two constraints need to be imposed. Macroeconomic variables should not intervene in the short-term nominal interest rate specification. This is easily obtained imposing $\theta_M = 0$, $\Theta_{Z,M} = 0$ and $\Theta_M = 0$ in Equation (3). In that case, the short-term interest rate is only a function of latent factors. The second constraint is that the macroeconomic variables M_t do not Granger-cause the yield factors Z_t under the risk neutral measure. Since $\Phi^{\mathbb{Q}} = \Sigma^{\mathbb{Q}}(\lambda + \Sigma^{-1}\Phi)$, imposing the bottom-left block of $\Phi^{\mathbb{Q}}$ to be equal to zero is easy via linear constraints on λ. As in the unspanned risk literature, the macroeconomic variables would not be priced in nominal interest rates but would help predict and be predicted by yield factors. Interestingly, if the upper-left block of $\Phi^{\mathbb{Q}}$ is allowed to be non-zero, the macroeconomic variables can be spanned in real rates.

Another possibility is to increase the flexibility of the transition between physical and risk-neutral dynamics. If we are mostly interested in variance risk-premia, we might want to loosen the links between $\Sigma^{\mathbb{Q}}$ and Σ. This can be achieved considering an exponential-affine SDF, as in Monfort and Pegoraro (2012). Using the same notations as in Equation (5):

$$m_{t-1,t} = \exp\left\{-r_{t-1} + \Lambda'_{t-1}X_t + X'_t\Delta X_t + \Lambda_r r_t - \log\mathbb{E}\left[\exp\left(\Lambda'_{t-1}X_t + X'_t\Delta X_t + \Lambda_r r_t | \underline{X_{t-1}}, \underline{r_{t-1}}\right)\right]\right\},$$

where Δ is a symmetric matrix. In that case, the risk-neutral dynamics of X_t are still given by the Gaussian VAR(1) of Equation (7), but the conditional covariance matrix is given by:

$$\Sigma^{\mathbb{Q}} = \left(\Sigma^{-1} - 2\Delta - 2\frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma}\Theta\right)^{-1}.$$

Obviously, Δ should be such that the matrix $\Sigma^{\mathbb{Q}}$ has non-negative eigenvalues. Additional degrees of freedom can therefore be gained on the risk-neutral covariance of the macroeconomic variables and the yield factors.

Last, it is possible to include time-varying volatility for the macroeconomic variables. Engle (1982) notes that inflation dynamics are best reproduced with time-varying conditional variance. A simple solution is to consider models where X_t is composed of two sets of latent variables that follow Gaussian VAR(1) dynamics as in Equation (1). Hence $X_t = [W'_t, Z'_t]'$, where W_t is a set of latent variables as well as Z_t. The macroeconomic variables are now quadratic combinations of W_t only, such that:

$$\forall j \in \{1, \dots, n_M\}, \quad M_{j,t} = \omega_{j,0} + \omega'_{j,1}W_t + W'_t\omega_{j,2}W_t,$$

where $\omega_{j,2}$ is symmetric but does not need to be positive. The conditional short-rate dynamics are still given by a gamma-zero distribution with a quadratic intensity. However, the influence of macroeconomic variables is represented by the impact of W_t on the short-term interest rate and the mapping between the macroeconomy and the short-term interest rate is highly non-linear.

3 Empirical application: estimation strategy

I show the performance and the usefulness of the model by considering a monetary policy application. I'm interested in estimating the interdependence between inflation shocks and nominal and real term structures of interest rates, both during normal times and when the economy is at the zero lower bound. As will be shown in Section 4, the model allows for a simple practical calculation of inflation risk premia and real term premia, impulse-response functions from inflation to the yields or from yields to inflation, and a decomposition of liftoff probabilities with respect to inflation risk. In contrast, the present section focuses on data and econometric issues, such as the estimation procedure and identification constraints.

3.1 The empirical exercise and the data

I consider monthly U.S. data from January 1990 to March 2015.[10] The starting date is determined to avoid issues related to the Volcker period. The full set of observable variables is composed of the following subsets.

First, monthly nominal zero-coupon yields are extracted from Gurkaynak, Sack, and Wright (2007) for maturities of 1, 2, 3, 5, 7, and 10 years. I add the one-month nominal interest rate series taken from Bloomberg.[11] Second, following Haubrich, Pennacchi, and Ritchken (2012), I compute liquidity-adjusted synthetic yields for inflation-linked bonds using zero-coupon inflation swap rates obtained from Bloomberg for maturities of 1, 2, 3, 5, 7, and 10 years.[12] Christensen and Gillan (2012) note that though not free from liquidity risk, inflation swaps are less likely to be affected by liquidity issues compared to TIPS (see also Fleckenstein, Longstaff, and Lustig (2013)).[13] The synthetic TIPS yields are obtained as the difference between the nominal yields and the inflation swap rates

[10] Considering monthly end-of-month data avoids issues related to the CPI interpolation for the computation of TIPS payoffs.

[11] The one-month rate is available under the ticker <*GB1M Index*>.

[12] Swap rates are available under the ticker <*USSWITx Crncy*>, where x stands for the maturity. Swap interest rates are not available as continuously compounded rates and must be transformed. The continuously compounded yield is obtained as: $\kappa(t,h) = \log(1+\tilde{\kappa}(t,h))$ where $\tilde{\kappa}(t,h)$ is the quoted swap rate on Bloomberg terminal.

[13] For papers who focus on extracting the liquidity risk from TIPS data, see for instance Sack and Elasser (2004), Shen (2006), Gurkaynak, Sack, and Wright (2010), Grischenko and Huang (2013), Pflueger and Viceira (2013) or D'Amico, Kim, and Wei (2014). Fleckenstein, Longstaff, and Lustig (2014) note that the TIPS bonds were also subject to large mispricing during the crisis.

at the same maturities. Due to data limitations, the inflation-linked series start in July 2004. I also treat the months in the direct aftermath of Lehman failure – from September 2008 to February 2009 – as missing data since most movements on the TIPS interest rates during this period can likely be attributed to the large disruption of the inflation-indexed market (see for instance D'Amico, Kim, and Wei (2014)).

As observable macroeconomic variables M_t, I only consider the year-on-year inflation rate $\Pi_t := \pi_{t-12,t}$ at the monthly frequency, computed from the CPI-U series of the BLS database.[14] Last, I follow Kim and Orphanides (2012) and Chernov and Mueller (2012) adding two sets of survey forecasts in the observable variables. I obtain inflation forecasts from the Philadelphia Fed database, and take the series of expected average inflation over the next 1 and 10 years. These two series are respectively denoted by $S_{t,\Pi}^{(12)}$ and $S_{t,\Pi}^{(120)}$. From the same database, I obtain nominal yields forecasts for the 10-year maturity, respectively 3-months and 1-year ahead. These two series are respectively denoted by $S_{t,R}^{(3)}$ and $S_{t,R}^{(12)}$. All these surveys are quarterly data and contain missing values. In the end, I obtain 18 observable series with a maximum of 303 observation per series. Time series and standard descriptive statistics are respectively presented on Figure 1 and in Table 1.

[Insert Figure 1 Table 1 about here.]

The nominal interest rates are very persistent at all maturities with first-order autocorrelations around 0.98, and are upward sloping with maturity on average, from 2.9% to 5.1%. Nominal interest rate standard deviations are slightly decreasing with maturity. Their time-series show a globally decreasing behavior up to the recent zero lower bound period where the one-month interest rate is virtually zero from mid-2009 on (see Figure 1). From 1990 to 2015, periods of high and flat slope alternate regularly. Whereas these observations are well-known for the nominal yield curve, the real yield curve exhibits somehow similar specificities. Real interest rates are also very persistent, but slightly less than their nominal counterparts (from 0.94 to 0.97, see Table 1). They are also upward sloping on average, but starting with negative mean values at short maturities since they

[14] To be consistent with the reference price index of the inflation-indexed securities, the realized inflation series is lagged of 3 months. Mathematically, P_t =CPI-U$_{t-3}$ and $\Pi_t = \log(P_t/P_{t-12})$. I do not see this as a caveat for the following reason. The information available at date t is closer to the realized inflation rate of the reference index rather than to the real-time realized inflation rate due to the publication lag of the different price indices. Taking the lagged inflation rate is hence more consistent with the information set available by the representative agent. For long enough maturities, this difference is likely to be negligible.

are not constrained by the zero lower bound. Excluding the aftermath of the Lehman failure – where a huge peak can be observed at all maturities – the mean real yield curve becomes more negative, from -0.22% at the 1-year to 0.86% at the 10-year maturity. The real interest rate standard deviations are lower than the nominal ones, but are also decreasing with maturity. Also, Figure 1 shows that the slope of the real interest rates is on average lower than its nominal counterpart, and consistently non-negative through time except during the crisis period.

3.2 The state-space formulation

With the shared yield curve characteristics, enough flexibility and fit can be obtained considering only 3 latent yield factors Z_t ($n_Z = 3$). Hence $X_t = (\Pi_t, Z_t)'$ is a four-dimensional vector, and f_t is a vector of size 21. Since the transition equation of the model is given by Equation (15), I only need to express the different measurement equations.

I assume that both the yields and the surveys are measured with errors whereas the inflation rate is perfectly measured. Denote by $\widetilde{R}(t,h)$ and $\widetilde{R}_a^*(t,h)$ the measured nominal bonds and inflation-indexed bond yields respectively. Their measurement equations are given by (see Equations (17) and (22)):

$$\begin{aligned} \widetilde{R}(t,h) &= R(t,h) + \sigma_R \eta_{t,h}^{(R)} = a_h + \mathcal{B}_h' f_t + \sigma_R \eta_{t,h}^{(R)} && (26) \\ \widetilde{R}_a^*(t,h) &= R_a^*(t,h) + \sigma_R^* \eta_{t,h}^{(R_a^*)} = a_h^* + \mathcal{B}_h^{*'} f_t + \sigma_R^* \eta_{t,h}^{(R_a^*)} , && (27) \end{aligned}$$

where $\eta_{t,h}^{(R)}$ and $\eta_{t,h}^{(R_a^*)}$ are i.i.d. zero-mean unit-variance Gaussian shocks, and the standard deviations of the yields' measurement errors σ_R and σ_R^* are the same across maturities. Inflation is measured without errors, hence:

$$\Pi_t = e_1' X_t = \tilde{e}_1' f_t \, , \tag{28}$$

where e_1 is the first column of the identity matrix I_n and $\tilde{e}_1$ is the first column of the identity matrix I_{n+n^2+1}. All the survey series are assumed to be measured with errors. It is easy to obtain inflation forecasts with the physical VAR dynamics of Equation (2).

$$\begin{aligned} S_{t,\Pi}^{(12)} &= \mathbb{E}(\Pi_{t+12}|\underline{f_t}) + \sigma_{\Pi}^{(12)}\eta_{t,\Pi}^{(12)} \\ &= \widetilde{e}_1'(I_{n+n^2+1}-\Psi)^{-1}(I_{n+n^2+1}-\Psi^{12})\Psi_0 + \Psi^{12}f_t + \sigma_{\Pi}^{(12)}\eta_{t,\Pi}^{(12)} \qquad (29) \\ S_{t,\Pi}^{(120)} &= \frac{1}{10}\sum_{k=1}^{10}\mathbb{E}(\Pi_{t+12k}|\underline{f_t}) + \sigma_{\Pi}^{(120)}\eta_{t,\Pi}^{(120)} \\ &= \widetilde{e}_1'(I_{n+n^2+1}-\Psi)^{-1}\left(I_{n+n^2+1}-\frac{1}{10}\sum_{k=1}^{10}\Psi^{12k}\right)\Psi_0 + \frac{1}{10}\sum_{k=1}^{10}\Psi^{12k}f_t + \sigma_{\Pi}^{(120)}\eta_{t,\Pi}^{(120)} \qquad (30) \end{aligned}$$

where $\eta_{t,\Pi}^{(12)}$ and $\eta_{t,\Pi}^{(120)}$ are i.i.d. zero-mean unit-variance Gaussian shocks. Last, using Equation (19), surveys of 10-year maturity yield forecasts can be expressed easily:

$$\begin{aligned} S_{t,R}^{(3)} &= a_{120} + \mathcal{B}_{120}'\left[(I_{n+n^2+1}-\Psi)^{-1}(I_{n+n^2+1}-\Psi^{3})\Psi_0 + \Psi^{3}f_t\right] + \sigma_{S_R}^{(3)}\eta_{t,S_R}^{(3)} \qquad (31) \\ S_{t,R}^{(12)} &= a_{120} + \mathcal{B}_{120}'\left[(I_{n+n^2+1}-\Psi)^{-1}(I_{n+n^2+1}-\Psi^{12})\Psi_0 + \Psi^{12}f_t\right] + \sigma_{S_R}^{(12)}\eta_{t,S_R}^{(12)}, \qquad (32) \end{aligned}$$

where $\eta_{t,S_R}^{(3)}$ and $\eta_{t,S_R}^{(12)}$ are i.i.d. zero-mean unit-variance Gaussian shocks. Let us denote by Y_t the full set of observable variables at time t, that is:

$$Y_t = \left[\left(\widetilde{R}(t,h)\right)_{h=\{1,12,24,36,60,84,120\}}, \left(\widetilde{R}_a^*(t,h)\right)_{h=\{12,24,36,60,84,120\}}, \Pi_t, \left(S_{t,\Pi}^{(h)}\right)_{h=\{12,120\}}, \left(S_{t,R}^{(h)}\right)_{h=\{3,12\}}\right]'.$$

Putting together Equations (26) to (32), I can express the complete state-space model in an affine form with respect to f_t.

$$\begin{aligned} f_t &= \Psi_0 + \Psi f_{t-1} + \left[\text{Vec}^{-1}(\Omega_0 + \Omega f_{t-1}\right]^{1/2}\xi_t \\ Y_t &=: \mathcal{A} + \mathcal{B}'f_t + \mathcal{D}\eta_t, \qquad (33) \end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ stack the intercept and the loadings presented above, η_t stacks the measurement errors in a zero-mean unit-covariance matrix Gaussian vector, and $\mathcal{D}$ is the diagonal matrix containing the measurement errors standard deviations and a zero for the inflation measurement equation.

3.3 Estimation and identification constraints

I resort to filtering techniques to estimate the model and evaluate the factor values. Since the measurement equation (33) is an affine function of f_t, that is a linear-quadratic combination of X_t and a linear function of r_t, I use the Quadratic Kalman Filter (QKF) developed by Monfort, Renne, and Roussellet (2015). The original filtering algorithm is obtained by applying the linear Kalman filter to state-space models where the transition dynamics are given by a Gaussian VAR and the measurement equations are linear-quadratic. This algorithm is slightly modified to incorporate r_t (which is non-Gaussian) and is detailed in Appendix A.6. Assuming that the conditional distribution of Y_t given its past is Gaussian, I can estimate the model with the QKF-based quasi maximum likelihood.

For parsimony reasons, several constraints are imposed on the parameters. First, for the short-term interest rate physical dynamics, I impose that $\theta = 0$, and the bottom-right block of Θ, Θ_Z is diagonal. Hence $\theta_0 = 0$ and only the diagonal, the first row and the first column of Θ contain non-zero entries. Second, all the standard deviations of the survey measurement errors $\sigma_{\Pi}^{(12)}$, $\sigma_{\Pi}^{(120)}$, $\sigma_{S_R}^{(3)}$ and $\sigma_{S_R}^{(12)}$ are calibrated to the average forecaster disagreement during the estimation period.[15]

The identification of the physical dynamics is obtained imposing the sufficient condition that the bottom-right block of Φ, Φ_Z is diagonal (see Appendix A.7). In addition, to help the interpretation of the latent factors Z_t, I impose that the entries of the bottom-left block of Φ, $\Phi_{Z,M}$ are positive, that the entries of the upper-right block of Φ, $\Phi_{M,Z}$ are negative, and that μ_M is positive. Indeed, since the marginal mean of inflation is positive, the positivity constraint on $\Phi_{Z,M}$ imply that the marginal mean of Z_t is positive as well. Therefore, Z_t components are mostly located on the upward-sloping part of $Z_t'\Theta_Z Z_t$, and an increase in the short-term interest rate is mostly reflected by increases of Z_t components. This increase should be translated into a decrease of the inflation rate, which is obtained imposing the negativity constraint on $\Phi_{M,Z}$.

I end up estimating a set of 43 parameters in total, where 12 parameters describe X_t physical dynamics, 8 parameters describe r_t physical dynamics, 21 parameters describe the market prices of risk, and 2 are the yields measurement errors standard deviations.[16]

[15]In the data, only the interquartile range of forecasters answer is provided. Assuming the distribution among forecasters is Gaussian, in order to obtain a quantity comparable to a standard deviation, I divide the average interquartile range over the whole sample by $2 \times F^{-1}(0.75)$ where $F(\bullet)$ is the c.d.f of a normalized Gaussian distribution. Indeed, for any $\omega \sim \mathcal{N}(0, \sigma)$, if $F_\omega(\bullet)$ is the c.d.f of ω, I have $F_\omega^{-1}(0.75) - F_\omega^{-1}(0.25) = \sigma[F^{-1}(0.75) - F^{-1}(0.25)] = 2F^{-1}(0.75)\sigma$.

[16]More precisely, I have to estimate 1 parameter for μ_M, 10 for Φ, 1 for Σ_M, 7 for Θ since it is a symmetric positive matrix, 1 for ς, 4 for λ_0, 16 for λ, 1 for Λ_r, and 2 for σ_R and σ_R^*.

4 Estimation results

4.1 Empirical performance: factors and fit

The estimated parameters are presented on Tables 2 and 3, gathering respectively the parameters for the joint dynamics of the factors X_t on the one hand, and the parameters for the short-rate dynamics, the market prices of risk, and the measurement errors on the other hand.

[Insert Tables 2 and 3 about here.]

Inflation and all yield factors are highly persistent, and Φ possesses diagonal terms comprised between 0.89 and 0.99 under the physical measure. The inflation rate Π_t is significantly caused by fluctuations on the first yield factor Z_1 only. Conversely, Π_t only Granger-causes the third yield factor Z_3. Under the risk-neutral measure, Π_t is Granger-caused significantly by both Z_1 and Z_2, and causes only Z_2. For identification purposes, the conditional covariance matrix Σ has been imposed diagonal under the physical measure see Equation (1)). However, the $\mathbb{Q}$-dynamics authorizes non-zero conditional correlations between the variables present in X_t. Indeed, the risk-neutral conditional correlations between shocks on Π_t and shocks Z_2 and Z_3 are both significantly positive. This indicates positive covariance risk premia, and investors perceive the yield shocks and the inflation shocks going in the same direction as a bad outcome. The estimates for r_t dynamics show a positive discrepancy between risk-neutral and physical parameters, as indicated by the significantly positive price of risk Λ_r (see Table 3). The SDF therefore allows for richer risk-neutral dynamics than in most existing models and all risks in the model are priced significantly (all columns of λ are significantly different from 0, see Table 3).

Estimating the model with a filter presents the advantage of being able to reconstruct the model-implied time-series of the observables. I present the filtered factors extracted from the quadratic Kalman filter on Figure 2.

[Insert Figure 2 about here.]

The first factor is exactly the realized year-on-year inflation rate whereas the yield factors are evaluated by the filter so as to adjust to the nominal yield curve, the real yield curve and the survey data with the best possible fit. Looking at Figure 2, it can be seen that the yield factor Z_2 plays a role mainly at the long-end during the zero lower bound period as its fluctuations looks very much alike 10-year nominal yield variations after 2009. In comparison, the yield factor Z_3 seems to play more at the short-end of the nominal yield curve. This statement is confirmed looking at Figure 3 which plots the normalized factor loadings of nominal and real interest rates with respect to maturity. For the nominal yield curve, most of the fluctuations are explained by the first and last yield factors Z_1 and Z_3. The inflation rate has a slightly negative influence at short and medium maturities, and a slightly positive influence on maturities greater than 5 years. Focusing only on the linear loadings on the left and middle graphs of Figure 3, we could respectively interpret Z_2, Z_1 and Z_3 as level, slope and curvature factors for both the nominal and real yield curves (see Litterman and Scheinkman (1991)). Though the exact size of the loadings are different for nominal and real term structures, they vary in the same qualitative fashion with respect to maturity. However, the role of the factors is distorted by the quadratic loadings, which are dominant for Z_3. Such an interpretation of the factors is therefore difficult and inconsistent with their filtered values on Figure 2. In any case, all the factors are priced in both term structures of interest rates.

[Insert Figure 3 about here.]

Using the filtered factor series, it is easy to reconstruct the short-term interest rate r_t series along with its 95% confidence bounds (not presented here for the interest of space). This allows me to determine the starting date of the zero lower bound period as the first date when the lower confidence bound reaches 0, that is from **November 2008** on. Hence, every reference to the *zero lower bound period* is considered from November 2008 to the end of the estimation sample. I also reconstruct the rest of the nominal and the real yield curves and the obtained filtered survey data, and compute the associated fitting errors. The SPF data and model-implied series are presented on Figure 4 and the RMSEs for the yield curves are shown in Table 4.

[Insert Table 4 and Figure 4 about here.]

The model is able to provide both a reasonable fit on the survey data, consistently with the fairly large forecasters disagreement, and an impressively good fit on both the nominal and the real yield curve with only 3 unobservable factors. Indeed, RMSEs are around 5bps for nominal rates and around 12bps for real rates (see Table 4). This is partly linked to the rich linear-quadratic model formulated in the previous section, which – as stated by Leippold and Wu (2007) – fits the data better than a pure linear model with the same number of factors.

4.2 Inflation and real risk premia decomposition

A usual by-product of affine term structure models is that it is easy to obtain the decomposition of nominal yields in an expected component on the one hand, that is the component that would have been observed would the investors be risk-neutral, and a risk premia component on the other hand. Considering jointly inflation and real interest rate data further allows to decompose nominal interest rates at each maturity in four different parts: the expected compounded real interest rate, the expected inflation up to maturity, real risk premia and inflation risk premia.[17] Nominal bonds indeed contain inflation risk since the real return of nominal bonds decrease when inflation turns out to be higher than expected. The associated inflation premium is positive (resp. negative) if this situation is a bad (resp. good) outcome for the economic agents. I present the model-implied marginal decomposition and the time-series of the various components of the nominal interest rates on Figures 5 and 6 respectively.

[Insert Figures 5 and 6 about here.]

Consistently with the existing literature and the well-known violations of the expectation hypothesis, the model-implied nominal risk premium is time-varying and upward sloping with maturity (see Figure 5). In particular, most of the 10-year yield fluctuations during the zero lower bound are related to variations in risk premia whereas the expected component stays stable around 100bps. In contrast, the 1-year nominal yield is very close to zero during the ZLB, as well as its risk premia component (see first row of Figure 6). I turn now to the decomposition of nominal rates into real rates and inflation components. The second row of Figure 6 shows significantly time-varying real term premia components both at the 1-year and at the 10-year maturity are increasing with maturity on average. During the zero lower bound period, the expected real rates fall in negative territory with minimum values of -200bps and -100bps for the 1- and 10-year maturities, producing negative real interest rates at the short-end. The real term premia is nonetheless positive, reaching an all times high of more than 200bps during the crisis for both maturities. Again, as for the nominal term structure, the real term premium drives most fluctuations of the long-end of the 10-year real interest rate and the marginal term structure of the

[17] The methodology is as follows. First, I obtain the nominal expected component calculating the pricing formulas for nominal bonds and imposing all prices of risk λ_0, λ and Λ_r to be equal to zero. The pricing formulas are easily obtained using Equation (18) and replacing the risk-neutral parameters by the physical ones. The nominal risk premium is the spread between the observed yield and its expected component. The same method can be applied to get the real rates decomposition in expected real rates and real term premia. Last, expected inflation is given by the difference between nominal and real expected components, and inflation risk premia as the difference between nominal and real term premia.

real rates expected component is below 50bps at all maturities.

The inflation components are easily obtained as the difference between the nominal components and their real counterparts. Accordingly with the previous statements, short- and long-term inflation risk premia become largely negative when entering the crisis period reflecting fears of a deflation spiral, whereas it is whipsawing around zero during the rest of the sample. At the 1-year maturity, inflation risk premia stay in the negative territory during the ZLB, between -200bps and -10bps. This means that investors appreciate inflation shocks being higher than expected, potentially driving the economy out of the ZLB. Conversely, the 10-year inflation premia component comes back to close to zero values soon after Lehman's failure. In comparison, the 10-year expected inflation is roughly flat around 3% (the inflation marginal mean is 2.9%). Most of the fluctuations in the inflation component comes from inflation risk premia, whatever the maturity. However, for long-enough maturities, the inflation risk premia component is overall very small, fluctuating between -30bps and 30bps. This implies that long-term inflation expectation are well-anchored, and that there is low uncertainty around the level of the 10-year ahead compounded inflation rate. Economic agents are hence confident that the central bank will stabilize inflation in the long-run whereas short-term concerns produce sizable inflation risk premia at the one-year maturity.

The long-term inflation risk premia estimates are broadly in line with those of Abrahams, Adrian, Crump, and Moench (2013) and D'Amico, Kim, and Wei (2014), but differ sharply from those of Haubrich, Pennacchi, and Ritchken (2012) and to a lesser extent from those of Fleckenstein, Longstaff, and Lustig (2013). The model of Haubrich, Pennacchi, and Ritchken (2012) imposes the risk premia estimates to be functions of the conditional volatility of interest rates. This produces long-term real term premia and inflation risk premia that are roughly constant over time and consistently positive. In comparison Fleckenstein, Longstaff, and Lustig (2013) find that inflation risk premia changes through time but the 10-year inflation premia is slightly negative from 2010 on. Though they include more survey data, they do not include observed inflation series to estimate the model, which can lead to sizable differences. In addition, none of the aforementioned literature explicitly include the zero lower bound constraint in the estimation, which can bias the expected component estimates of nominal yields. The next section provide evidence of the reliability of the present model's decomposition.

4.3 Robustness check

There are probably as many interest rate decomposition as term structure models in the literature. It is therefore needed to provide supplementary evidence than just economic intuition that the obtained decomposition is meaningful. A conventional way of testing the reliability of model-implied risk premia is to compare the behavior of observed bond excess returns with the model-implied ones and use regression evidence. In this section, I generalize an existing procedure to the case of nominal and real term structure models and use it to assess the robustness of the present model.

The excess returns of any bond for k-holding periods can be defined as the return of a strategy consisting in buying the bond at time t and selling it at time $t+k$, minus the risk-less interest rate of maturity k. This k-period risk-less rate is equal to $R(t,k)$ in the nominal world and $R_a^*(t,k)$ in the real world. In the following, I focus on a 12-month holding period. In the nominal world, the one-year excess returns of holding a nominal bond of maturity h are given by:

$$\frac{1}{12}\log\left(\frac{B(t+12,h-12)}{B(t,h)}\right)-R(t,12)\,.$$

In the real world, the nominal return of this one-year holding-period strategy must be corrected from the realized inflation rate and compared to the real rates $R_a^*(t,12)$:

$$\begin{aligned}&\frac{1}{12}\log\left(\frac{B(t+12,h-12)}{B(t,h)}\right)-\frac{1}{12}\Pi_{t+12}-R_a^*(t,12)\\ =\;&\left[\underbrace{\frac{1}{12}\log\left(\frac{B(t+12,h-12)}{B(t,h)}\right)-R(t,12)}_{\textit{Nominal excess returns}}\right]+\left[\underbrace{R(t,12)-R_a^*(t,12)-\frac{1}{12}\Pi_{t+12}}_{\textit{Breakeven}\;-\;\textit{Inflation}}\right].\end{aligned}$$

The real excess returns of nominal bonds are the sum of the nominal excess returns and the spread between the so-called breakeven inflation rate $(R(t,12)-R_a^*(t,12))$ and the realized inflation during the holding period. This last term would be close to the inflation risk premium would the inflation forecasting errors be small. Therefore, real excess returns of nominal bonds include information about both the evolution of nominal term premia and inflation risk premia separately.

For the excess returns of TIPS, I denote by $B_t^*(t+12,h-12)$ the price at $t+12$ of the

TIPS issued at time t of maturity h.

$$B_t^*(t+12, h-12) = \mathbb{E}\left[m_{t+12,t+h}\frac{P_{t+h}}{P_t}|\underline{f_{t+12}}\right], \tag{34}$$

where the principal is adjusted by the reference price-index variation between the inception and the maturity date (t and $t+h$). Rearranging formula (34), this price can be expressed with the price of a newly issued TIPS at date $t+12$.

$$B_t^*(t+12, h-12) = \mathbb{E}\left[m_{t+12,t+h}\frac{P_{t+h}}{P_{t+12}}|\underline{f_{t+12}}\right]\frac{P_{t+12}}{P_t} = B^*(t+12, h-12)\exp(\Pi_{t+12}). \tag{35}$$

Therefore, the real and nominal excess returns of holding TIPS for k-holding periods are respectively given by:

$$\begin{aligned} & \frac{1}{12}\log\left(\frac{B^*(t+12,h-12)}{B^*(t,h)}\exp(\Pi_{t+12})\right) - \frac{1}{12}\Pi_{t+12} - R_a^*(t,12) \\ = & \frac{1}{12}\log\left(\frac{B^*(t+12,h-12)}{B^*(t,h)}\right) - R_a^*(t,12) \end{aligned}$$

and,

$$\begin{aligned} & \frac{1}{12}\log\left(\frac{B^*(t+12,h-12)}{B^*(t,h)}\exp(\Pi_{t+12})\right) - R(t,12) \\ = & \left[\underbrace{\frac{1}{12}\log\left(\frac{B^*(t+12,h-12)}{B^*(t,h)}\right) - R_a^*(t,12)}_{\textit{Real excess returns}}\right] - \left[\underbrace{R(t,12) - R_a^*(t,12) - \frac{1}{12}\Pi_{t+12}}_{\textit{Breakeven} \,-\, \textit{Inflation}}\right]. \end{aligned}$$

Similarly to nominal bonds, TIPS excess returns involve only real term premia in real terms, and both real term premia and inflation risk premia in nominal terms.

These excess returns computations can be used to test whether the model is able to reproduce the deviations from the expectation hypothesis consistently with the data, and whether the model-implied predictions of excess returns are reasonable. These two tests are respectively called LPY-I and LPY-II in the terminology of Dai and Singleton (2002). Both LPY-I and LPY-II reformulates the excess returns in the form of the well-known Campbell and Shiller regressions (see Campbell and Shiller (1991), CS henceforth). The two classical regressions exploit the above formulation of nominal excess returns of nominal bonds and the real excess returns of TIPS, and write:

$$R(t+12,h-12) - R(t,h) = \alpha_h + \beta_h \frac{12\,(R(t,h) - R(t,12))}{h-12} + \varepsilon_{t+12,h} \tag{CS.1}$$

$$R^*_a(t+12,h-12) - R^*_a(t,h) = \alpha^*_h + \beta^*_h \frac{12\,(R^*_a(t,h) - R^*_a(t,12))}{h-12} + \varepsilon^*_{t+12,h} \tag{CS.2}$$

Using the two other formulations of excess returns, I express two new Campbell and Shiller regressions as:[18]

$$R(t+12,h-12) - R(t,h) + \frac{\Pi_{t+12}}{h-12} = \alpha_h + \beta_h \frac{12\,(R(t,h) - R^*_a(t,12))}{h-12} + \varepsilon_{t+12,h} \tag{CS.3}$$

$$R^*_a(t+12,h-12) - R^*_a(t,h) - \frac{\Pi_{t+12}}{h-12} = \alpha^*_h + \beta^*_h \frac{12\,(R^*_a(t,h) - R(t,12))}{h-12} + \varepsilon^*_{t+12,h} \tag{CS.4}$$

These new regressions include the realized inflation on the left-hand side which is compensated by adding the one-year breakeven inflation rate on the right-hand side.[19] Whereas Regressions (CS.1) and (CS.2) focus respectively on nominal risk premia and real risk premia only, regressions (CS.3) and (CS.4) allow to isolate the behavior of inflation risk premia. All regressions have a predictive interpretation. If the expectation hypothesis was holding true, intercept and slopes would all be respectively equal to 0 and 1 and the corresponding excess return would average to zero. However, since the expectation is largely violated in practice, the current slope of nominal/real interest rates can predict the excess returns.

Testing LPY-I consists in estimating regressions (CS.1) to (CS.4) on the data for maturities ranging from 1 to 10 years, and comparing the estimated betas to the model-implied betas.[20] The latter are obtained easily in closed-form using the affine formulation of interest rates (see Equations (26) and (27)). Detailed formulas are provided in Appendix A.8. Testing LPY-II consists in performing the same regressions on the data adding the

[18] Note that Haubrich, Pennacchi, and Ritchken (2012) also formulate a similar exercise but they don't get a formulation with the realized inflation on the left-hand side. In essence, they obtain regressions (CS.1) and (CS.2). Evans (1998) formulates a slightly different regression with the Equation (20) of his paper. He expresses the expectation hypothesis equating the expected nominal excess returns of TIPS with the expected nominal excess returns of nominal bonds. As such, his formulation can be thought as a combination of Equations (CS.1) and (CS.4).

[19] Indeed, note that $R(t,h) - R^*_a(t,12) = R(t,h) - R(t,12) + R(t,12) - R^*_a(t,12)$ and that $R^*_a(t,h) - R(t,12) = R^*_a(t,h) - R^*_a(t,12) - (R(t,12) - R^*_a(t,12))$.

[20] To obtain the yields of nominal bonds and TIPS at all maturities for the whole sample period, I use the model-implied yield series reconstructed from the filtered factors and omit the measurement errors.

corresponding model-implied expected excess returns series in the independent variable on the right-hand side. Adding the expected excess return should in theory correct the deviations from the expectation hypothesis. A consistent model should hence be able to produce betas non significantly different from 1. Results of these regressions are provided on Figure 7 and 8.

[Insert Figures 7 and 8 about here.]

For all CS regressions testing LPY-I, the model-implied slopes lie inside the Newey-West 95% confidence bounds. In addition, except for the real excess returns of nominal bonds (bottom-left graph of Figure 7), model-implied regression slopes are very close to those obtained with the data. This provide evidence that the historical dynamics of both inflation, nominal and real interest rates are consistent with the LPY-I condition. However, the model still has difficulties in reproducing the slightly downward sloping regression slopes in nominal terms (see first row of Figure 7). Focusing on the LPY-II condition on Figure 8, we observe that theoretical unit values lie inside the 95% Newey-West confidence intervals of the CS regressions at all maturities, for all four regressions. Hence, I cannot reject the hypothesis that all the slopes are equal to one, indicating a strong capacity of the model to jointly reproduce the behavior of both inflation risk premia, real term premia, and nominal risk premia.[21]

5 Inflation shocks, liftoff, and the zero lower bound

5.1 Impulse-response analysis

With the presence of observable inflation, the estimated model allows to study the interactions between monetary policy, the nominal yield curve, and the inflation rate. I first focus on the Taylor-type rule interpretation of the short-term interest rate dynamics presented in Section 2.1. Using the linear-quadratic formulations of the short-rate expectation, I can isolate the monetary policy sensitivity to the inflation rate, that is the coefficients β_t and β_t^* of Equations (4) and (11). These time-series are represented on Figure 9.

[21] Since the nominal short-term interest rate has a conditional gamma distribution, this result might seem at odds with the incapacity of CIR processes (continuous-time equivalent of gamma processes) to verify both LPY-I and LPY-II at the same time (see Dai and Singleton (2002) or Backus, Foresi, and Telmer (2001) for example). It should be noted that this model incorporates richer dynamics compared to pure CIR processes, with a strong linear-quadratic component driving most of both yield curves.

[Insert Figure 9 about here.]

Figure 9 shows strongly time-varying loadings – both under the physical and the risk-neutral measure – to inflation in the monetary policy reaction function. At the beginning of the sample, the loading is close to 0.3 and falls to less than 0.1 after the beginning of the crisis. Interestingly, the loadings become slightly negative between 2010 and 2014, meaning that the central bank is willing to cut loose on fighting inflation increases. There is a small discrepancy between the risk-neutral and the physical loadings, indicating that investors fear a too high sensitivity to the inflation rate, implying a higher variance on the policy rate. These estimates are consistently lower than 1, violating the so-called Taylor principle. However, since the model is not forward-looking, it does not need an average loading on inflation greater than one to produce a stationary solution.[22] Second, though considerably lower, the monetary policy sensitivity movements are consistent with estimates of Callot and Kristensen (2014) or Ang, Boivin, Dong, and Loo-Kung (2011). Since the latter only have two latent factors, their model is likely to overestimate the role of inflation in the yield curve, leading to higher loadings than in the present estimation. In addition, though they obtain a quadratic model, their formulation is not consistent with the zero lower bound and movements in macroeconomic loadings in the monetary policy rule are less constrained.

Second, I use the estimated model to study the impact of an inflation shock on the yield curve, and of a monetary policy shock on the inflation rate. Since the model is non-linear in Z_t and Π_t however, the derivation of the impulse-response functions require a few computations. The problem can be generalized in the following fashion. All the variables used in this section can be expressed as linear combinations of f_t components. I want to express the impact of a shock of size s of variable v_2 on variable v_1:

$$\mathcal{I}_{t,k}^{v_2 \to v_1} = \mathbb{E}\left(e'_{v_1} f_{t+k} \middle| \underline{f_{t-1}},\, e'_{v_2}[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = s\right) - \mathbb{E}\left(e'_{v_1} f_{t+k} \middle| \underline{f_{t-1}}\right), \tag{36}$$

where e_{v_1} and e_{v_2} are vectors weighting and selecting the right entries of f_t depending on the variables of interest. Using the semi-strong VAR form of Equation (15), I obtain:

$$\mathcal{I}_{t,k}^{v_2 \to v_1} = e'_{v_1} \Psi^k \left[\mathbb{E}\left(f_t \middle| \underline{f_{t-1}},\, e'_{v_2}[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = s\right) - \Psi_0 - \Psi f_{t-1}\right]. \tag{37}$$

[22]For example, Ang, Boivin, Dong, and Loo-Kung (2011) get an estimate of the average reaction to inflation of 0.609 on the 1945-2005 period while preserving the stationarity of the model.

The conditional expectation of Equation (37) can be easily computed using the quadratic Kalman filter. Indeed, it is the expectation of unobservable factors f_t given initial conditions $\underline{f_{t-1}}$ and that the observable variable $e'_{v_2} f_t = e'_{v_2}(\Psi_0 + \Psi f_{t-1}) + s$. I use the filter for one iteration and the update step provides the needed filtered factors.

Since the variance-covariance matrix of the reduced-form residuals Σ is diagonal, I designate by *structural inflation shock* a shock on Π_t which has no contemporaneous impact on Z_t, and by *structural monetary policy shock* a shock on Z_t which has no contemporaneous impact on Π_t. More precisely, the effect of a structural inflation shock on any variable v_1 is trivially expressed as:

$$
\begin{aligned}
X_t^{(s)} &= \begin{pmatrix} \mu_M + \Phi_M \Pi_{t-1} + \Phi_{M,Z} Z_{t-1} + s \\ \\ \Phi_{Z,M} \Pi_{t-1} + \Phi_Z Z_{t-1} \end{pmatrix}, \\
\mathcal{I}_{t,k}^{\Pi \to v_1} &= e'_{v_1} \Psi^k \left[\mathbb{E}\left(f_t | \underline{f_{t-1}},\, X_t = X_t^{(s)} \right) - \Psi_0 - \Psi f_{t-1} \right] \\
&= e'_{v_1} \Psi^k \left[f_t^{(s)} - \Psi_0 - \Psi f_{t-1} \right], \qquad (38)
\end{aligned}
$$

where the conditioning $X_t = X_t^{(s)}$ represents the structural shock on the inflation rate, and $f_t^{(s)} = \left[X_t^{(s)\prime},\, \text{Vec}\left(X_t^{(s)} X_t^{(s)\prime} \right)',\, \varsigma\left(\theta_0 + \theta' X_t^{(s)} + X_t^{(s)\prime} \Theta X_t^{(s)} \right) \right]'$. For example, if the variable of interest v_1 is the nominal interest rate $R(t,h)$, $e_{v_1} = \mathcal{B}_h$ and the previous formula can be applied simply. Similarly, the impulse-response of a monetary policy shock on any variable v_1 is given by:

$$
\mathcal{I}_{t,k}^{r_t \to v_1} = e'_{v_1} \Psi^k \left[\mathbb{E}\left(f_t | \underline{f_{t-1}},\, \Pi_t = \mathbb{E}(\Pi_t | \underline{f_{t-1}}),\, r_t = \mathbb{E}(r_t | \underline{f_{t-1}}) + s \right) - \Psi_0 - \Psi f_{t-1} \right] \qquad (39)
$$

For a structural monetary policy shock, the quadratic Kalman filter naturally divides the shock s between the different components of Z_t while not instantaneously affecting the value of Π_t. Depending on the initial conditions f_{t-1}, a monetary policy shock can thus have different persistence and a different impact on the variables of interest v_1. For this shock, I draw a 1% positive shock on the one-month nominal rate and only keep the persistent part, that is the one that does not die instantly and which has an impact on inflation.

Since the model is non-linear, the shape and amplitude of the IRFs depend on the initial condition f_{t-1}. I choose to consider two specific cases. I first compute the impulse-responses of structural inflation and monetary policy shocks when the economy is at the steady-state, that is when $X_{t-1} = \mathbb{E}(X_{t-1}) =: \overline{X}$. Hence, the initial condition is given by $f_{t-1} = \left[\overline{X}', \text{Vec}\left(\overline{X}\overline{X}'\right)', \varsigma(\theta_0 + \theta'\overline{X} + \overline{X}'\Theta\overline{X})\right]'$.[23] Second, I focus on the impulse-responses when the economy is at the ZLB. The initial conditional $X_{t-1} = \overline{X}_{zlb}$ is now equal to the sample mean of the filtered X_t's when the short-term interest rate r_t is not significantly different from zero. I obtain the initial condition $f_{t-1} = \left[\overline{X}'_{zlb}, \text{Vec}\left(\overline{X}_{zlb}\overline{X}'_{zlb}\right)', \varsigma(\theta_0 + \theta'\overline{X}_{zlb} + \overline{X}'_{zlb}\Theta\overline{X}_{zlb})\right]'$ accordingly. The results are respectively presented on Figures 10 and 11.

[Insert Figures 10 and 11 about here.]

At the steady-state, the monetary policy shock translates into a 30bps persistent shock. This results in a persistent increase of the 10-year nominal rate of around 10bps and a decrease of the 10-year nominal risk premia of about 10bps. The shock is very persistent and the one-month interest rate is still 15bps above its steady-state value after 10 years. Looking at the decomposition of nominal rates, nearly all this shock can be attributed to a shock in real interest rates, and the IRFs on the two term structures are different by only a few basis points. The effect of this shock on inflation is negative, if anything, but has a very small impact. At the steady state, a monetary policy shock therefore possesses only a small contracting effect on the inflation rate. This result is consistent with the IRFs obtained with FAVAR approaches of Bernanke, Boivin, and Eliasz (2005) or Wu and Xia (2013), where the authors find that the effect of a monetary policy shock has a non-significant impact on the CPI index. In contrast, a one-standard deviation structural inflation shock translates into approximately a 40bps positive shock at period 0, which dies completely only after 2 years. Consistently with the monetary policy reaction function of Figure 9, this shock instantaneously shifts the nominal yield curve upward by 7bps for the 1-month interest rate to 2bps for the 10-year expected component. Because the effect on the 10-year yield is virtually zero, the inflation shock has a -2bps instantaneous effect on the 10-year nominal risk premium, mostly driven by a fall in inflation risk premia, making it closer to zero (see Figure 5). In contrast, the effect on the long-term expected inflation is close to zero, meaning that economic agents are still confident

[23] It is worth mentioning first that the initial condition $f_{t-1} = [\overline{X}', \text{Vec}(\overline{X}\overline{X}')', \varsigma(\theta_0 + \theta'\overline{X} + \overline{X}'\Theta\overline{X})]'$ is different from $f_{t-1} = \mathbb{E}(f_{t-1})$ since $\mathbb{E}[XX'] \neq \mathbb{E}(X)\mathbb{E}(X')$. However, once you condition by $X_{t-1} = \overline{X}$, it follows directly that $\text{Vec}(X_{t-1}X'_{t-1}) = \text{Vec}(\overline{X}\overline{X}')$ with probability one. Second, it is not the only possible way of computing non-linear impulse-response functions. An alternative would consist in simulating many initial conditions f_{t-1} using its known marginal distribution, compute the IRFs for each initial condition, and average over the responses. The two approaches are not equivalent since they flip the order of integration (see for example Gallant, Rossi, and Tauchen (1993) or Koop, Pesaran, and Potter (1996)).

that monetary policy will preserve long-term price stability. Though moderately high, the nominal yield curve is still above its steady-state after 10 years, emphasizing interest rates high persistence.

The effects of the structural shocks are completely different during the ZLB period and the monetary policy shock translates into a 15bps persistent shock on the one-month interest rate (see top-left graph of Figure 11). Even though this shock has a positive instantaneous impact on the 10-year yield, All the nominal yield curve stays below its normal path from only 20 months on after the shock. Combined with the fact that the one-period probability of staying at 0 for the short-rate is 40% (see Table 4), these two elements emphasize the persistence of the ZLB state. The persistent 10bps shock however has a stronger negative effect on realized inflation peaking at -7bps after a year. The impact of lifting-off can hence be very detrimental with respect to stabilizing inflation. This is partly due to the large increases in the real yield curve: from 80bps at the one-year maturity to 40bps at the ten year. Thus, the inflation risk premia falls dramatically of about -20bps, reflecting high fears of long-term deflation. Looking at the second row of Figure 11, we see that a 40 bps structural inflation shock has virtually no effect on the nominal yield curve, consistently with the fact that the central bank is trying to restore stable long-term inflation and is stuck at the ZLB.

5.2 The price of lifting-off

As emphasized in Section 2.6, it is easy to derive the so called liftoff probabilities within this framework. Deriving these probabilities allows to investigate how economic agents perceive the monetary policy while the economy is at the ZLB. I first focus on the time series of physical and risk-neutral probabilities for horizons of 6 months and 1 year. The results of these computations are respectively presented on the left and right columns of Figure 12.

[Insert Figure 12 about here.]

When the economy enters the ZLB, the liftoff probabilities begin to show high time-variation. At the 6-month horizon, they mostly fluctuate between 0.6 and 0.9. At the 1-year horizon, the series exhibit a smaller variance and are slightly higher on average. For both horizons, the physical liftoff probability series are very stable during the period 2011-2013, around 0.9 (see black solid lines on Figure 12). The risk-neutral probabilities also stabilizes, but nearly 0.2 units below. This is the only period when the difference between risk-neutral and physical probabilities becomes negative, reflecting the time-changing perception of the investors on the liftoff event. Between 2011 and 2013, investors seem to perceive the liftoff as a desirable event, with a negative associated risk premium. At the end of the sample, both physical and risk-neutral probabilities increase greatly consistently with a more optimistic macroeconomic perspective. However, the liftoff risk premium becomes positive again from 2014 on, and investors still perceive liftoff as a bad outcome as of March 2015.

I now investigate whether the fluctuations of both the liftoff probabilities and liftoff risk premia are linked to inflation shocks during the sample. Figure 13 presents the series of partial derivatives of the liftoff probabilities with respect to inflation, under both the physical and risk-neutral measures. On both the 6-months and 1-year probability sensitivities to inflation, three cycles can be observed. An increasing phase up to 2011, decreasing up to 2013 where the sensitivities both reached negative values, and increasing again up to mid 2014. At its highest in 2009, a 1% positive inflation shock would have increased the 6-months probability of lifting off by 6 percentage points, and reduced the liftoff premium by nearly the same amount. In particular, during the 2011-2013 period, risk-neutral sensitivities tend to become more negative than the physical ones, indicating that positive inflation shocks decrease the fear of lifting off. It is consistent with negative short-term inflation risk-premia, and restoring a stable short-term inflation is a primary concern. Interestingly, looking after 2014 at both the 6-months and the 1-year horizon, risk-neutral sensitivities become negative whereas physical sensitivities stay in the positive territory. Hence any 1% positive inflation shock will result in a higher probability of lifting off of 3 percentage points for the 6-months horizon and 2 percentage points for the 1-year horizon while reducing liftoff risk premia by resp. 4 and 3 percentage points.

[Insert Figure 13 about here.]

Last, I can directly interpret the risk-neutral liftoff probabilities as the price of lifting off. As such, I compute the *ex-ante* interest rate on a ZLB insurance bond, as presented in Section 2.6. Note that when the interest rate of this bond is above its expected component, it indicates positive liftoff risk premia. Lifting off can be seen as a default event on this insurance from the economic agents' point of view, with a zero recovery rate. When they fear the liftoff event, they charge a positive premium on this bond. Both the interest rate and the expected component can be further decomposed to shed light on the inflation contribution to their fluctuations. Decomposition for the 6-month and the 1-year rate are presented on Figure 14.

[Insert Figure 14 about here.]

Consistently with the results on Figure 12, the interest rate on the ZLB insurance bond is always above its expected component except during the 2011-2013 period. Since the probabilities can reach 0.7 at the short-term, the associated interest rate can reach very high annualized levels, up to 200% at the 6-month maturity. Focusing on the inflation contribution, we see that most fluctuations of the interest rate and its expected component can indeed be explained by the inflation component during the ZLB period. Linear regressions of the interest rate or its expected component on the corresponding inflation-related component (not presented here) yield R-squared values between 0.5 and 0.8 for both maturities and regression slopes are not statistically different from 1. While the expected component goes above the interest rate during the 2011-2013 period forming the negative liftoff risk premia, the inflation components show an inverted spread at the 6-months maturity and nearly no spread at the 1-year. If anything, inflation concerns increases the liftoff risk premia during this period, consistently with the intuition. Therefore, movements in the price of lifting off is mostly related to inflation, it seems that the fear of lifting off disappears between 2011 and 2013 for reasons that are less related to inflation.

6 Conclusion

In this paper, I provide a new way of modeling both nominal and real yield curves in an affine framework, which allows for the presence of observable macroeconomic variables and is consistent with the zero lower bound. Relying on a combination of quadratic term structure models and the gamma-zero distribution, the model is able to generate a short-term nominal rate stuck at the zero lower bound for several periods, where the liftoff probability depends explicitly on the observable macroeconomic state. I show that the short-term interest rate specification can have a convenient economic interpretation in terms of a time-varying Taylor-type rule. Most importantly, I show that the model is an ATSM such that it provides closed-form formulas for nominal and real interest rates, interest rate forecasts, macroeconomic forecasts, impulse-response functions, and liftoff probabilities under both physical and risk-neutral measure. The latter can be easily interpreted as the price of a ZLB insurance bond, which is also obtained as a closed-form combination of observable macroeconomic variables and latent factors.

The relevance of this new framework is explored with an empirical application on U.S. data to study the interactions between inflation and the monetary policy in and out of the zero lower bound. During the ZLB period, inflation risk premia become negative at the short-end of the yield curve, while staying close to zero at longer horizons, emphasizing the horizon-dependent deflation risk aversion. I provide evidence that monetary policy becomes more accommodating during the ZLB to try to restore short-term positive and long-term stable inflation around target. Last, I study the effect and the cost of lifting-off. While an increase of the short-term interest rate has little impact on inflation during normal times, it severely and negatively impacts inflation during the ZLB, and a 15bps persistent increase is reflected by a 7bps decrease in inflation. Second, I provide evidence that the fear of lifting-off changes over time, and that the liftoff risk premium was negative between 2011 and 2013. As such, the model provides a convenient tool for policy-makers to monitor not only market views on the timing of interest rate increases, but also on the harm done to economic agents when the liftoff occurs. As advocated by Feldman et al. (2015), an optimal liftoff time would therefore be a period when the objective liftoff probabilities are high but also when the market-based probabilities are lower, indicating that the liftoff event is well-anticipated and is associated with a positive marginal utility from the agents' point of view.

A Appendix

A.1 The Gamma-zero (γ_0) distribution

The gamma-zero autoregressive process was introduced by Monfort, Pegoraro, Renne, and Roussellet (2014) as a generalization of the autoregressive gamma process of Gouriéroux and Jasiak (2006). Its construction is summarized hereafter.

Let $\mathcal{I}_t = \mathcal{I}(X_t)$ be a non-negative process which is a function of the risk factors X_t, and P_t be a Poisson variable with intensity $\mathcal{I}_t$. r_t is conditionally gamma-zero distributed if:

$$P_t|X_t \sim \mathcal{P}(\mathcal{I}(X_t)) \quad \text{and} \quad r_t|P_t \sim \gamma_{P_t}(\varsigma) \ , \tag{40}$$

that is, conditionally on the Poisson mixing variable, r_t has a gamma distribution of shape (or degree of freedom) parameter P_t and a scale parameter ς. When $P_t = 0$, the conditional distribution of r_t converges to a Dirac point mass at zero. Integrating with respect to P_t, we obtain the conditional distribution of r_t given X_t that we call gamma-zero, encompassing a zero point mass. In this paper, the intensity $\mathcal{I}_t$ is given by a quadratic combination of the Gaussian vector X_t, that is:

$$\mathcal{I}_t = \theta_0 + \theta' X_t + X_t' \Theta X_t \,.$$

The conditional distribution of r_t given X_t can be expressed with its conditional Laplace transform:

$$\mathbb{E}\left[\exp(u_r r_t)\big|\underline{X_t}\right] = \exp\left(\frac{u_r\varsigma}{1-u_r\varsigma}(\theta_0 + \theta' X_t + X_t'\Theta X_t)\right) \ , \tag{41}$$

which is an exponential-quadratic function of X_t. From the conditional Laplace transform expression it is easy to derive the first-two conditional moments of r_t:

$$\begin{aligned} \mathbb{E}(r_t|\underline{X_t}) &= \varsigma\,(\theta_0 + \theta' X_t + X_t'\Theta X_t) \\ \mathbb{V}(r_t|\underline{X_t}) &= 2\varsigma^2\,(\theta_0 + \theta' X_t + X_t'\Theta X_t) \ . \end{aligned}$$

A.2 Affine $\mathbb{P}$-property and risk neutral dynamics of f_t

Define $u = [u_x', \,\mathrm{Vec}(U_x)', \,u_r']'$, where the blocks have respective size n, n^2 and 1. I first introduce the following Lemma.

Lemma A.1 *The conditional Laplace transform of* $[X_t', \text{Vec}(X_tX_t')']'$ *given its past is given by:*

$$\begin{aligned} & \mathbb{E}\left[\exp\left(u_x'X_t + X_t'U_xX_t\right)|\underline{X_{t-1}}\right] \\ = \;& \exp\left\{u_x'(I_n - 2\Sigma U_x)^{-1}\left(\mu + \frac{1}{2}\Sigma u_x\right) + \mu' U_x(I_n - 2\Sigma U_x)^{-1}\mu - \frac{1}{2}\log\left|I_n - 2\Sigma U_x\right|\right. \\ + \;& \left.(u_x + 2U_x\mu)'(I_n - 2\Sigma U_x)^{-1}\Phi X_{t-1} + X_{t-1}\Phi' U_x(I_n - 2\Sigma U_x)^{-1}\Phi X_{t-1}\right\} \end{aligned}$$

Proof See Cheng and Scaillet (2007). ■

Let us now calculate the conditional Laplace transform of f_t given $\underline{f_{t-1}}$.

$$\begin{aligned} & \mathbb{E}\left[\exp\left(u'f_t\right)|\underline{f_{t-1}}\right] \\ = \;& \mathbb{E}\left[\exp\left(u_x'X_t + X_t'U_xX_t + u_r r_t\right)|\underline{f_{t-1}}\right] \\ = \;& \mathbb{E}\left\{\mathbb{E}\left[\exp\left(u_x'X_t + X_t'U_xX_t + u_r r_t\right)|\underline{f_{t-1}}, X_t\right]|\underline{f_{t-1}}\right\} \\ = \;& \mathbb{E}\left[\exp\left\{u_x'X_t + X_t'U_xX_t + \frac{u_r\varsigma}{1-u_r\varsigma}\left[\theta_0 + \theta'X_t + X_t'\Theta X_t\right]\right\}|\underline{f_{t-1}}\right], \\ = \;& \exp\left(\frac{u_r\varsigma\theta_0}{1-u_r\varsigma}\right)\mathbb{E}\left[\exp\left\{\left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)'X_t + X_t'\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)X_t\right\}|\underline{f_{t-1}}\right], \end{aligned}$$

which is obtained using the law of iterated expectations and the conditional Laplace transform of r_t given X_t (see Equation (41)). I hence obtain the conditional Laplace transform of $[X_t', \text{Vec}(X_tX_t')']'$ applied in the two arguments $\left[\left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)'; \text{Vec}\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)'\right]'$. Using Lemma A.1, I have:

$$\begin{aligned} & \mathbb{E}\left[\exp\left(u'f_t\right)|\underline{f_{t-1}}\right] \\ = \;& \exp\left\{\frac{u_r\varsigma\theta_0}{1-u_r\varsigma} + \left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)'\left[I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]^{-1}\left[\mu + \frac{1}{2}\Sigma\left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)\right]\right. \\ + \;& \mu'\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\left[I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]^{-1}\mu - \frac{1}{2}\log\left|I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right| \\ + \;& \left[\left(u_x + \frac{u_r\varsigma}{1-u_r\varsigma}\theta\right)' + 2\mu'\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]\left[I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]^{-1}\Phi X_{t-1} \\ + \;& \left.X_{t-1}'\Phi'\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\left[I_n - 2\Sigma\left(U_x + \frac{u_r\varsigma}{1-u_r\varsigma}\Theta\right)\right]^{-1}\Phi X_{t-1}\right\}. \end{aligned} \tag{42}$$

This conditional Laplace transform is obviously an exponential-quadratic function of X_{t-1}, that is by extension an exponential-affine function of f_{t-1}. (f_t) is therefore an affine process under the physical measure.

To derive the risk-neutral conditional Laplace transform of f_t given $\underline{f_{t-1}}$, I use the transition formulas provided in Roussellet (2015), Chapter 4. Using the block recursive affine structure of f_t, the risk-neutral conditional Laplace transform of r_t given X_t and $\underline{f_{t-1}}$ is given by:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(\exp\{u_r' r_t\}\middle|X_t,\,\underline{f_{t-1}}\right) &= \frac{\mathbb{E}\left(\exp\left\{[u_r+\Lambda_r]' r_t\right\}\middle|X_t,\,\underline{f_{t-1}}\right)}{\mathbb{E}\left(\exp\left\{\Lambda_r' r_t\right\}\middle|X_t,\,\underline{f_{t-1}}\right)} \\
&= \exp\left\{\left(\frac{(u_r+\Lambda_r)\varsigma}{1-(u_r+\Lambda_r)\varsigma}-\frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma}\right)(\theta_0+\theta' X_t+X_t'\Theta X_t)\right\},
\end{aligned} \tag{43}$$

where $\mathbb{E}^{\mathbb{Q}}(\cdot)$ is the expectation operator under the risk-neutral measure. The difference of ratios can be simplified as follows.

$$\begin{aligned}
\frac{(u_r+\Lambda_r)\varsigma}{1-(u_r+\Lambda_r)\varsigma}-\frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma} &= \frac{(1-\Lambda_r\varsigma)\,(u_r+\Lambda_r)\varsigma-[1-(u_r+\Lambda_r)\varsigma]\,\Lambda_r\varsigma}{[1-\Lambda_r\varsigma]\,[1-(u_r+\Lambda_r)\varsigma]} \\
&= \varsigma\frac{u_r-\Lambda_r u_r\varsigma+u_r\Lambda_r\varsigma}{[1-\Lambda_r\varsigma]\,[1-(u_r+\Lambda_r)\varsigma]} \\
&= \frac{u_r\varsigma}{[1-\Lambda_r\varsigma]\,[1-(u_r+\Lambda_r)\varsigma]}\,.
\end{aligned}$$

Define now $\varsigma^{\mathbb{Q}}=\dfrac{\varsigma}{1-\Lambda_r\varsigma}$, that is $\varsigma=\dfrac{\varsigma^{\mathbb{Q}}}{1+\Lambda_r\varsigma^{\mathbb{Q}}}$. I obtain:

$$\begin{aligned}
\frac{u_r\varsigma}{1-(u_r+\Lambda_r)\varsigma} &= \frac{u_r\frac{\varsigma^{\mathbb{Q}}}{1+\Lambda_r\varsigma^{\mathbb{Q}}}}{1-(u_r+\Lambda_r)\frac{\varsigma^{\mathbb{Q}}}{1+\Lambda_r\varsigma^{\mathbb{Q}}}} \\
&= \frac{1+\Lambda_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\times\frac{u_r\varsigma^{\mathbb{Q}}}{1+\Lambda_r\varsigma^{\mathbb{Q}}} \\
&= \frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\,.
\end{aligned}$$

Hence the conditional Laplace transform of Equation (43) is given by:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(\exp\{u_r' r_t\}\middle|X_t,\,\underline{f_{t-1}}\right) &= \exp\left\{\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\times\frac{\theta_0+\theta' X_t+X_t'\Theta X_t}{1-\Lambda_r\varsigma}\right\} \\
&=: \exp\left\{\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\left(\theta_0^{\mathbb{Q}}+\theta^{\mathbb{Q}\prime}X_t+X_t'\Theta^{\mathbb{Q}}X_t\right)\right\}.
\end{aligned}$$

r_t is therefore conditionally gamma-zero distributed given X_t, where the risk-neutral parameters are given by:

$$\theta_0^{\mathbb{Q}} = \frac{\theta_0}{1-\Lambda_r\varsigma}\,,\quad \theta^{\mathbb{Q}}=\frac{\theta}{1-\Lambda_r\varsigma}\,,\quad \Theta^{\mathbb{Q}}=\frac{\Theta}{1-\Lambda_r\varsigma}\,,\quad \varsigma^{\mathbb{Q}}=\frac{\varsigma}{1-\Lambda_r\varsigma}$$

I turn now to the computation of the risk-neutral conditional Laplace transform of $(X_t', \text{Vec}(X_tX_t')')'$ given $\underline{f_{t-1}}$. Again, using the property in Roussellet (2015) Chapter 4, I have:

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\left\{u_x'X_t + X_t'U_xX_t\right\}|\underline{f_{t-1}}\right) = \frac{\mathbb{E}\left[\exp\left\{(u_x+\widetilde{\Lambda}_{t-1})'X_t + X_t'(U_x+\widetilde{\Lambda}_r)X_t\right\}|\underline{f_{t-1}}\right)}{\mathbb{E}\left[\exp\left\{\widetilde{\Lambda}_{t-1}'X_t + X_t'(U_x+\widetilde{\Lambda}_r)X_t\right\}|\underline{f_{t-1}}\right)},$$

where $\widetilde{\Lambda}_{t-1}$ and $\widetilde{\Lambda}_r$ are given by:

$$\widetilde{\Lambda}_{t-1} = \lambda_0 + \theta\frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma} + \lambda X_{t-1}\,, \quad \widetilde{\Lambda}_r = \frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma}\Theta\,.$$

The transition between the physical and risk-neutral dynamics of X_t are as if the SDF was exponential-quadratic, with adjusted prices of risk $\widetilde{\Lambda}_{t-1}$ and $\widetilde{\Lambda}_r$. Since $\widetilde{\Lambda}_r$ the price associated to $\text{Vec}(X_tX_t')$ is constant through time, I can rely on the results of Monfort and Pegoraro (2012). I obtain that X_t follows a Gaussian VAR(1) under the risk-neutral measure and:

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}}X_{t-1} + \Sigma^{\mathbb{Q}^{1/2}}\varepsilon_t^{\mathbb{Q}}\,,$$

where $\varepsilon_t^{\mathbb{Q}}$ is a zero-mean unit-variance Gaussian white noise, and $\mu^{\mathbb{Q}}$, $\Phi^{\mathbb{Q}}$ and $\Sigma^{\mathbb{Q}}$ are given by:

$$\begin{cases} \mu^{\mathbb{Q}} &= \Sigma^{\mathbb{Q}}\left(\lambda_0 + \theta\frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma} + \Sigma^{-1}\mu\right)\,, \\ \Phi^{\mathbb{Q}} &= \Sigma^{\mathbb{Q}}\left(\lambda + \Sigma^{-1}\Phi\right)\,, \\ \Sigma^{\mathbb{Q}} &= \left(\Sigma^{-1} - 2\frac{\Lambda_r\varsigma}{1-\Lambda_r\varsigma}\Theta\right)^{-1}\,. \end{cases}$$

Since r_t is conditionally gamma-zero given X_t and that X_t follows a VAR(1) under the risk-neutral measure, the class of distributions are the same under the physical and the risk-neutral measure. Trivially transforming Formula 42, the risk-neutral Laplace transform of f_t given $\underline{f_{t-1}}$ is given by:

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\exp\left(u'f_t\right)|\underline{f_{t-1}}\right] \\
= & \exp\left\{\frac{u_r\varsigma^{\mathbb{Q}}\theta_0^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}+\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\left[\mu^{\mathbb{Q}}+\frac{1}{2}\Sigma^{\mathbb{Q}}\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)\right]\right. \\
+ & \mu^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\mu^{\mathbb{Q}}-\frac{1}{2}\log\left|I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right| \\
+ & \left[\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'+2\mu^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}}X_{t-1} \\
+ & \left.X_{t-1}'\Phi^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}}X_{t-1}\right\}. \qquad (44)
\end{aligned}
$$

This conditional Laplace transform is also exponential-quadratic in X_{t-1}, that is an exponential-affine function of f_{t-1}. (f_t) is therefore an affine process under the risk-neutral measure.

A.3 Multi-horizon Laplace transform of f_t

Since the one-period ahead conditional risk-neutral Laplace transform of f_t given $\underline{f_{t-1}}$ is exponential-affine in f_{t-1}, it is well-known that the conditional multi-horizon risk-neutral Laplace transform of $(f_t, \dots, f_{t+k})$ is also exponential-affine in f_{t-1} (see e.g. Darolles, Gourieroux, and Jasiak (2006)). Using the notation:

$$
\mathbb{E}^{\mathbb{Q}}\left[\exp\left(u'f_t\right)|\underline{f_{t-1}}\right] =: \exp\left\{\mathbb{A}^{\mathbb{Q}}(u)+\mathbb{B}^{\mathbb{Q}'}(u)X_{t-1}+X_{t-1}'\mathbb{C}^{\mathbb{Q}}(u)X_{t-1}\right\},
$$

with:

$$
\begin{aligned}
\mathbb{A}^{\mathbb{Q}}(u) &= \frac{u_r\varsigma^{\mathbb{Q}}\theta_0^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}-\frac{1}{2}\log\left|I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right| \\
&+ \left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\left[\mu^{\mathbb{Q}}+\frac{1}{2}\Sigma^{\mathbb{Q}}\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)\right] \\
&+ \mu^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\mu^{\mathbb{Q}} \\
\mathbb{B}^{\mathbb{Q}'}(u) &= \left[\left(u_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\theta^{\mathbb{Q}}\right)'+2\mu^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}} \\
\mathbb{C}^{\mathbb{Q}}(u) &= \Phi^{\mathbb{Q}'}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\left[I_n-2\Sigma^{\mathbb{Q}}\left(U_x+\frac{u_r\varsigma^{\mathbb{Q}}}{1-u_r\varsigma^{\mathbb{Q}}}\Theta^{\mathbb{Q}}\right)\right]^{-1}\Phi^{\mathbb{Q}},
\end{aligned}
$$

I obtain:

$$\begin{aligned}\phi^{\mathbb{Q}}_{t-1}(u_0,\ldots,u_k) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sum_{i=0}^{k} u_i' f_{t+i}\right)|\underline{f_{t-1}}\right] \\ &= \exp\left(\mathbb{A}^{\mathbb{Q}}_k(u_0,\ldots,u_k) + \mathbb{B}^{\mathbb{Q}'}_k(u_0,\ldots,u_k)X_{t-1} + X'_{t-1}\mathbb{C}^{\mathbb{Q}}_k(u_0,\ldots,u_k)\,X_{t-1}\right),\end{aligned}$$

where:

$$\begin{aligned}\mathbb{A}^{\mathbb{Q}}_k(u_0,\ldots,u_k) &:= \mathbb{A}^{\mathbb{Q}}_{k,k}(u_0,\ldots,u_k) \\ \mathbb{B}^{\mathbb{Q}}_k(u_0,\ldots,u_k) &:= \mathbb{B}^{\mathbb{Q}}_{k,k}(u_0,\ldots,u_k) \\ \mathbb{C}^{\mathbb{Q}}_k(u_0,\ldots,u_k) &:= \mathbb{C}^{\mathbb{Q}}_{k,k}(u_0,\ldots,u_k)\,,\end{aligned}$$

with initial conditions $\mathbb{A}^{\mathbb{Q}}_{k,1}(u_0,\ldots,u_k) = \mathbb{A}^{\mathbb{Q}}(u_k)$, $\mathbb{B}^{\mathbb{Q}}_{k,1}(u_0,\ldots,u_k) = \mathbb{B}^{\mathbb{Q}}(u_k)$, and $\mathbb{C}^{\mathbb{Q}}_{k,1}(u_0,\ldots,u_k) = \mathbb{C}^{\mathbb{Q}}(u_k)$, and $\forall i \in \{2,\ldots,k\}$,

$$\begin{aligned}\mathbb{A}^{\mathbb{Q}}_{k,i}(u_0,\ldots,u_k) &= \mathbb{A}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k) \\ &+ \mathbb{A}^{\mathbb{Q}}\left(u_{k-i+1} + \left[\mathbb{B}^{\mathbb{Q}'}_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,0\right]'\right) \\ \mathbb{B}^{\mathbb{Q}}_{k,i}(u_0,\ldots,u_k) &= \mathbb{B}^{\mathbb{Q}}\left(u_{k-i+1} + \left[\mathbb{B}^{\mathbb{Q}'}_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,0\right]'\right) \\ \mathbb{C}^{\mathbb{Q}}_{k,i}(u_0,\ldots,u_k) &= \mathbb{C}^{\mathbb{Q}}\left(u_{k-i+1} + \left[\mathbb{B}^{\mathbb{Q}'}_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,0\right]'\right).\end{aligned}$$

A.4 Conditional moments of f_t

From Cheng and Scaillet (2007) and using the same notations as in Monfort, Renne, and Roussellet (2015), the conditional first two moments of $(X_t', \mathrm{Vec}(X_t X_t')')'$ given the past can be expressed as:

$$\mathbb{E}\left[\left(\begin{array}{c} X_t \\ \mathrm{Vec}(X_t X_t') \end{array}\right)|\underline{f_{t-1}}\right] = \left(\begin{array}{c} \mu \\ \mathrm{Vec}(\mu\mu' + \Sigma) \end{array}\right) + \left(\begin{array}{c|c} \Phi & 0 \\ \hline \mu\otimes\Phi + \Phi\otimes\mu & \Phi\otimes\Phi \end{array}\right)\left(\begin{array}{c} X_{t-1} \\ \mathrm{Vec}(X_{t-1}X'_{t-1}) \end{array}\right)$$

$$\mathbb{V}\left[\left(\begin{array}{c} X_t \\ \mathrm{Vec}(X_t X_t') \end{array}\right)|\underline{f_{t-1}}\right] = \left(\begin{array}{c|c} \Sigma & \Sigma\Gamma'_{t-1} \\ \hline \Gamma_{t-1}\Sigma & \Gamma_{t-1}\Sigma\Gamma'_{t-1} + (I_{n^2} + K_n)(\Sigma\otimes\Sigma) \end{array}\right).$$

where $\otimes$ is the standard Kronecker product, $\Gamma_{t-1} = [I_n\otimes(\mu+\Phi X_{t-1}) + (\mu+\Phi X_{t-1})\otimes I_n]$, and K_n is the $(n^2\times n^2)$ commutation matrix.

Using the law of iterated expectations and the conditional first two moments of r_t given

X_t, I have:

$$
\begin{aligned}
\mathbb{E}\left[r_t|\underline{f_{t-1}}\right] &= \varsigma\left(\theta_0 + \theta'\mathbb{E}(X_t|\underline{f_{t-1}}) + \text{Vec}(\Theta)'\mathbb{E}[\text{Vec}(X_tX_t')|\underline{f_{t-1}}]\right) \\
&= \varsigma\Big(\theta_0 + \theta'\left(\mu + \Phi X_{t-1}\right) \\
&+ \text{Vec}(\Theta)'\left[\text{Vec}(\mu\mu' + \Sigma) + \left(\mu\otimes\Phi + \Phi\otimes\mu\right)X_{t-1} + (\Phi\otimes\Phi)\text{Vec}(X_{t-1}X_{t-1}')\right]\Big) \\
\mathbb{V}\left[r_t|\underline{f_{t-1}}\right] &= \mathbb{E}\left(\mathbb{V}\left[r_t|\underline{f_{t-1}}, X_t\right]|\underline{f_{t-1}}\right) + \mathbb{V}\left(\mathbb{E}\left[r_t|\underline{f_{t-1}}, X_t\right]|\underline{f_{t-1}}\right) \\
&= 2\varsigma\mathbb{E}\left[r_t|\underline{f_{t-1}}\right] + \mathbb{V}\left(\varsigma\left[\theta'X_t + X_t'\Theta X_t\right]|\underline{f_{t-1}}\right) \\
&= 2\varsigma^2\Big(\theta_0 + \theta'\left(\mu + \Phi X_{t-1}\right) \\
&+ \text{Vec}(\Theta)'\left[\text{Vec}(\mu\mu' + \Sigma + \left(\mu\otimes\Phi + \Phi\otimes\mu\right)X_{t-1} + (\Phi\otimes\Phi)\text{Vec}(X_{t-1}X_{t-1}')\right]\Big) \\
&+ \varsigma^2\Big(\theta'\Sigma\theta + 2\text{Vec}(\Theta)'\Gamma_{t-1}\Sigma\theta + \text{Vec}(\Theta)'\left[\Gamma_{t-1}\Sigma\Gamma_{t-1}' + (I_{n^2} + K_n)(\Sigma\otimes\Sigma)\right]\text{Vec}(\Theta)\Big).
\end{aligned}
$$

The conditional covariance is given by:

$$
\begin{aligned}
\mathbb{C}ov\left[\begin{pmatrix} X_t \\ \text{Vec}(X_tX_t') \end{pmatrix}, r_t|\underline{f_{t-1}}\right] &= \mathbb{E}\left[\begin{pmatrix} X_t \\ \text{Vec}(X_tX_t') \end{pmatrix} r_t|\underline{f_{t-1}}\right] - \mathbb{E}\left[r_t|\underline{f_{t-1}}\right]\mathbb{E}\left[\begin{pmatrix} X_t \\ \text{Vec}(X_tX_t') \end{pmatrix}|\underline{f_{t-1}}\right] \\
&= \mathbb{E}\left[\begin{pmatrix} X_t \\ \text{Vec}(X_tX_t') \end{pmatrix}\varsigma\left(\theta_0 + \theta'X_t + X_t'\Theta X_t\right)|\underline{f_{t-1}}\right] \\
&- \mathbb{E}\left[\begin{pmatrix} X_t \\ \text{Vec}(X_tX_t') \end{pmatrix}|\underline{f_{t-1}}\right]\mathbb{E}\left[\varsigma\left(\theta_0 + \theta'X_t + X_t'\Theta X_t\right)|\underline{f_{t-1}}\right] \\
&= \varsigma\begin{pmatrix} \Sigma\left[\theta + \Gamma_{t-1}'\text{Vec}(\Theta)\right] \\ \Gamma_{t-1}\Sigma\left[\theta + \Gamma_{t-1}'\text{Vec}(\Theta)\right] + (I_{n^2} + K_n)(\Sigma\otimes\Sigma)\text{Vec}(\Theta) \end{pmatrix}.
\end{aligned}
$$

In the end, putting the previous results together, I obtain the transition equation in the form of Equation (15) with parameters given by:

$$
\Psi_0 = \begin{pmatrix} \mu \\ \text{Vec}(\mu\mu' + \Sigma) \\ \varsigma\left(\theta_0 + \theta'\mu + \text{Vec}(\Theta)'\text{Vec}(\mu\mu' + \Sigma)\right) \end{pmatrix},
$$

$$
\Psi = \left(\begin{array}{c|c|c} \Phi & 0 & 0 \\ \hline \mu \otimes \Phi + \Phi \otimes \mu & \Phi \otimes \Phi & 0 \\ \hline \varsigma \left(\theta' \Phi + \mathrm{Vec}(\Theta)' [\mu \otimes \Phi + \Phi \otimes \mu]\right) & \varsigma \left(\Phi \otimes \Phi\right) & 0 \end{array} \right),
$$

and,

$$
\mathrm{Vec}^{-1}(\Omega_0 + \Omega f_{t-1}) =
$$

$$
\left(\begin{array}{c|l|l} \Sigma & \Sigma \Gamma'_{t-1} & \varsigma \Sigma \left(\theta + \Gamma'_{t-1} \mathrm{Vec}(\Theta)\right) \\ \hline & \Gamma_{t-1} \Sigma \Gamma'_{t-1} & \varsigma \Gamma_{t-1} \Sigma \left[\theta + \Gamma'_{t-1} \mathrm{Vec}(\Theta)\right] \\ & +(I_{n^2} + K_n)(\Sigma \otimes \Sigma) & \quad + \quad \varsigma (I_{n^2} + K_n)(\Sigma \otimes \Sigma) \mathrm{Vec}(\Theta) \\ \hline & & 2\varsigma^2 \Big(\theta_0 + \theta' \left(\mu + \Phi X_{t-1}\right) \\ & & + \mathrm{Vec}(\Theta)' \left[\mathrm{Vec}(\mu\mu' + \Sigma + (\mu \otimes \Phi + \Phi \otimes \mu) X_{t-1} + (\Phi \otimes \Phi) \mathrm{Vec}(X_{t-1} X'_{t-1})\right] \Big) \\ & & + \varsigma^2 \Big(\theta' \Sigma \theta + 2 \mathrm{Vec}(\Theta)' \Gamma_{t-1} \Sigma \theta + \mathrm{Vec}(\Theta)' \left[\Gamma_{t-1} \Sigma \Gamma'_{t-1} + (I_{n^2} + K_n)(\Sigma \otimes \Sigma)\right] \mathrm{Vec}(\Theta) \Big) \end{array} \right).
$$

A.5 Including longer period price variations in M_t

Remember that the class of models we consider is models where the inflation rate between $t-k$ and t, denoted by $\pi_{t-k,t}$, is directly included as the first macroeconomic variable, that is the first component of M_t. For notation simplicity let us also assume that there is no other macroeconomic variable, that is $M_t = \pi_{t-k,t}$. By definition I have:

$$
\pi_{t-k,t} = \sum_{i=1}^{k} \pi_{t-k+i-1,t-k+i} \Longleftrightarrow \pi_{t-1,t} = \pi_{t-k,t} - \pi_{t-k-1,t-1} + \pi_{t-k-1,t-k} \,.
$$

Hence, using the VAR(1) dynamics of X_t (see Equation (1)):

$$
\pi_{t-1,t} = \mu_\pi + (\Phi_\pi - 1)\pi_{t-k-1,t-1} + \Phi_{\pi Z} Z_{t-1} + \Sigma_\pi^{1/2} \varepsilon_{\pi,t} + \pi_{t-k-1,t-k} \,.
$$

Denoting by:

$$
\widetilde{X}_t = \left[\pi_{t-k,t},\, X'_t,\, \pi_{t-1,t},\, \ldots,\, \pi_{t-k,t-k+1}\right]' ,
$$

the vector of size $n+k$, I can form a new Gaussian VAR(1) dynamic system with $\widetilde{X}_t$ as:

$$\widetilde{X}_t = \left(\begin{array}{c} \mu_\pi \\ 0 \\ \hline \mu_\pi \\ \hline 0 \\ \vdots \\ 0 \end{array}\right) + \left(\begin{array}{cc|cccc} \Phi_\pi & \Phi_{\pi,Z} & 0 & & \cdots & 0 \\ \Phi_{Z,\pi} & \Phi_Z & 0 & & \cdots & 0 \\ \hline \Phi_\pi - 1 & \Phi_{\pi,Z} & 0 & \cdots & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \widetilde{X}_{t-1} + \left(\begin{array}{cc} \Sigma_\pi^{1/2} & 0 \\ 0 & I_{n_Z} \\ \hline \Sigma_\pi^{1/2} & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array}\right) \varepsilon_t \,.$$

It is immediate to see that the short-rate dynamics can also be transformed in terms of $\widetilde{X}_t$, that is:

$$r_t|(\underline{r_{t-1}}, \underline{\widetilde{X}_t}) \sim \gamma_0 \left(\theta_0 + \widetilde{\theta}' \widetilde{X}_t + \widetilde{X}_t' \widetilde{\Theta} \widetilde{X}_t, \varsigma\right) ,$$

where

$$\widetilde{\theta} = [\theta', 0, \cdots 0]' \quad \text{and} \quad \widetilde{\Theta} = \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix} .$$

Risk-neutral dynamics and conditional Laplace transforms under any measure can be easily derived for the vector $\widetilde{f}_t = [\widetilde{X}_t', \mathrm{Vec}(\widetilde{X}_t \widetilde{X}_t')', r_t]'$. The pricing of nominal bonds and inflation-indexed bonds hence follow exactly the same pattern as presented in the main text.

A.6 The quadratic Kalman filter

The original version of the QKF is developed in Monfort, Renne, and Roussellet (2015) for state-space models where the transition equations are given by a standard Gaussian VAR(1) and the measurement equations are linear-quadratic in the state variables. In the present paper, I also include a conditionally Gamma-distributed variable r_t in the state vector, which imposes to slightly modify the filtering algorithm. I hereby present the version of the filter that is used to estimate the model. Remember that the state-space model is written as:

$$
\begin{aligned}
f_t &= \Psi_0 + \Psi f_{t-1} + \left[\text{Vec}^{-1}(\Omega_0 + \Omega f_{t-1}\right]^{1/2} \xi_t \\
Y_t &= \mathcal{A} + \mathcal{B}' f_t + \mathcal{D}\eta_t \,,
\end{aligned}
$$

where ξ_t is a martingale difference with zero mean and unit variance. Assuming ξ_t is a Gaussian factor process, I obtain a fully linear state-space model where the conditional variance-covariance matrix of f_t given $\underline{f_{t-1}}$ is time-varying and affine in f_{t-1}. I can apply the linear Kalman filter on the previous state-space model. I use the following notations:

$$
\begin{aligned}
f_{t|t-1} &= \mathbb{E}(f_t|\underline{Y_{t-1}}) & \qquad Y_{t|t-1} &= \mathbb{E}(Y_t|\underline{Y_{t-1}}) \\
P_{t|t-1} &= \mathbb{V}(f_t|\underline{Y_{t-1}}) & \qquad M_{t|t-1} &= \mathbb{V}(Y_t|\underline{Y_{t-1}}) \\
f_{t|t} &= \mathbb{E}(f_t|\underline{Y_t}) & \qquad P_{t|t} &= \mathbb{V}(f_t|\underline{Y_t}) \,.
\end{aligned}
$$

The steps in the algorithm are:

- Initialize the filter at $f_{0|0} = \mathbb{E}(f_t)$ and $P_{0|0} = \mathbb{V}(f_t)$. These marginal moments are known in closed-form:

  $$f_{0|0} = (I_{n+n^2+1} - \Psi)^{-1}\Psi_0 \quad \text{and} \quad \text{Vec}(P_{0|0}) = \left(I_{(n+n^2+1)^2} - (\Psi \otimes \Psi)\right)^{-1}\left(\Omega_0 + \Omega f_{0|0}\right)$$

  Then, for each period t,

- Prediction of the latent:

$$
\begin{aligned}
f_{t|t-1} &= \Psi_0 + \Psi f_{t-1|t-1} \\
P_{t|t-1} &= \Psi P_{t-1|t-1}\Psi' + \text{Vec}^{-1}(\Omega_0 + \Omega f_{t-1|t-1}) \,.
\end{aligned}
$$

- Prediction of the observable:

$$
\begin{aligned}
Y_{t|t-1} &= \mathcal{A} + \mathcal{B}' f_{t|t-1} \\
M_{t|t-1} &= \mathcal{B}' P_{t|t-1}\mathcal{B} + \mathcal{D}\mathcal{D}'
\end{aligned}
$$

- Updating step:

$$
\begin{aligned}
f_{t|t} &= f_{t|t-1} + P_{t|t-1}\mathcal{B}M_{t|t-1}^{-1}\left(Y_t - Y_{t|t-1}\right)\\
P_{t|t} &= P_{t|t-1} - P_{t|t-1}\mathcal{B}M_{t|t-1}^{-1}\mathcal{B}'P_{t|t-1}
\end{aligned}
$$

- Computation of the log-likelihood: As for quasi-maximum likelihood, the conditional distribution of Y_t given $\underline{Y_{t-1}}$ is assumed Gaussian with mean $Y_{t|t-1}$ and variance $M_{t|t-1}$. Let m be the dimension of the observable vector Y_t. The log-likelihood therefore writes:

$$
\mathcal{L}_t = -\frac{1}{2}\left[m\log(2\pi) + \log|M_{t|t-1}| + (Y_t - Y_{t|t-1})'M_{t|t-1}^{-1}(Y_t - Y_{t|t-1})\right].
$$

In order to be consistent with the theoretical properties of the processes, two corrections are applied to the filtered values after storing the results. First, if the components of $r_{t|t}$ are negative, they are set to zero. Second, the filtered values of $\mathrm{Vec}(XX')_{t|t}$ are imposed to be exactly equal to $\mathrm{Vec}(X_{t|t}X'_{t|t})$.

A.7 Identification constraints

In this section, I prove that the constraints imposed for the estimation are sufficient to identify the physical parameters and that the latent factors cannot be rotated. Let us consider an alternative vector of factors W_t such that:

$$
W_t = \widetilde{\rho}_0 + \widetilde{\rho}X_t \quad \text{with} \quad \widetilde{\rho}_0 = \begin{pmatrix} 0 \\ \rho_0 \end{pmatrix}, \quad \widetilde{\rho} = \begin{pmatrix} I_{n_M} & 0 \\ 0 & \rho \end{pmatrix},
$$

and ρ_0 and ρ are respectively a vector of size n_Z and a $(n_Z \times n_Z)$ matrix. The dynamics given by Equation (1) trivially impose that $\rho_0 = 0$. Then, I have:

$$
\rho\Phi_Z\rho^{-1} = \Phi_Z \quad \text{and} \quad \rho\rho' = I_{n_Z}
$$

The second condition, corresponding to the equality of the conditional covariance matrices impose that ρ is orthogonal, that is $\rho^{-1} = \rho'$. Since Φ_Z is imposed to be diagonal, $\rho\Phi_Z\rho'$ should be diagonal as well. This obtained if and only if $\rho = I_{n_Z}$.

A.8 Campbell-Shiller regression coefficients

In this section I present the model-implied betas for Campbell-Shiller regressions (CS.1) to (CS.4). Due to the similarities of the different specifications, I only present the computations for (CS.1).

Let us consider the general linear regression:

$$Y = \alpha + \beta X + \epsilon\,,$$

where the optimal β is given by:

$$\beta = \frac{\mathbb{C}ov\,(Y, X)}{\mathbb{V}(X)}\,.$$

Replacing Y and X by the Campbell-Shiller variables, I obtain:

$$\beta_h = \frac{h-12}{12} \times \frac{\mathbb{C}ov\,[R(t+12, h-12) - R(t,h), R(t,h) - R(t,12)]}{\mathbb{V}\,[R(t,h) - R(t,12)]}\,.$$

Using the affine interest rate formulas, I obtain:

$$\begin{aligned}
\beta_h &= \frac{h-12}{12} \times \frac{\mathbb{C}ov\,\left[\mathcal{B}'_{h-12} f_{t+12} - \mathcal{B}'_h f_t, \mathcal{B}'_h f_t - \mathcal{B}'_{12} f_t\right]}{\mathbb{V}\,[\mathcal{B}'_h f_t - \mathcal{B}'_{12} f_t]} \\
&= \frac{h-12}{12} \times \frac{\mathbb{C}ov\,\left[\left(\mathcal{B}'_{h-12}\Psi^{12} - \mathcal{B}'_h\right) f_t, \left(\mathcal{B}'_h - \mathcal{B}'_{12}\right) f_t\right]}{\mathbb{V}\,[\left(\mathcal{B}'_h - \mathcal{B}'_{12}\right) f_t]} \\
&= \frac{h-12}{12} \times \frac{\left(\mathcal{B}'_{h-12}\Psi^{12} - \mathcal{B}'_h\right) \mathrm{Vec}^{-1}\left(\left(I_{(n+n^2+1)^2} - (\Psi \otimes \Psi)\right)^{-1} \left(\Omega_0 + \Omega\mathbb{E}(f_t)\right)\right) \left(\mathcal{B}_h - \mathcal{B}_{12}\right)}{\left(\mathcal{B}'_h - \mathcal{B}'_{12}\right) \mathrm{Vec}^{-1}\left(\left(I_{(n+n^2+1)^2} - (\Psi \otimes \Psi)\right)^{-1} \left(\Omega_0 + \Omega\mathbb{E}(f_t)\right)\right) \left(\mathcal{B}'_h - \mathcal{B}'_{12}\right)}\,,
\end{aligned}$$

where $\mathbb{E}(f_t) = (I_{n+n^2+1} - \Psi)^{-1}\Psi_0$. The proofs for the other regressions are of similar fashion, since all dependent and independent variables of all regressions can be expressed as affine functions of the process f_t.

A.9 Tables and figures

Table 1: Descriptive statistics

	Nominal rates (1990-2015)						
	1-month	1-year	2-year	3-year	5-year	7-year	10-year
mean	2.883	3.375	3.642	3.893	4.336	4.699	5.103
sd	2.228	2.382	2.351	2.272	2.098	1.952	1.800
$\rho(1)$	0.981	0.986	0.985	0.984	0.982	0.980	0.979
	Inflation	Real rates (2004-2015)					
	y-o-y	1-year	2-year	3-year	5-year	7-year	10-year
mean	2.607	-0.071	-0.100	-0.034	0.234	0.532	0.909
mean (excl. crisis)		-0.221	-0.206	-0.111	0.174	0.473	0.855
sd	1.237	1.592	1.423	1.312	1.153	1.069	0.954
sd (excl. crisis)		1.431	1.362	1.286	1.146	1.055	0.936
$\rho(1)$	0.942	0.938	0.963	0.964	0.969	0.962	0.956

Notes: All units are annualized percentage points. 'mean' are sample averages, 'sd' are sample standard deviations, and '$\rho(1)$' are autocorrelation of order 1. The 'excl. crisis' rows present descriptive statistics calculated on the TIPS data excluding the period from September 2008 to February 2009.

Table 2: Parameter estimates: X_t dynamics

	estimates	std.		estimates	std.
μ_Π	0.4132**	(0.1746)	$\mu^{\mathbb{Q}}_\Pi$	0.8792	(0.6619)
μ_{Z_1}	0	–	$\mu^{\mathbb{Q}}_{Z_1}$	0.0717	(0.1526)
μ_{Z_2}	0	–	$\mu^{\mathbb{Q}}_{Z_2}$	-0.4226	(0.3277)
μ_{Z_3}	0	–	$\mu^{\mathbb{Q}}_{Z_3}$	1.0737*	(0.5936)
Φ_Π	0.887***	(0.0174)	$\Phi^{\mathbb{Q}}_\Pi$	0.8138***	(0.0251)
$\Phi_{Z_1,\Pi}$	0.0188	(0.0366)	$\Phi^{\mathbb{Q}}_{Z_1,\Pi}$	0.0078	(0.0385)
$\Phi_{Z_2,\Pi}$	0.0472	(0.0527)	$\Phi^{\mathbb{Q}}_{Z_2,\Pi}$	0.1029**	(0.0498)
$\Phi_{Z_3,\Pi}$	0.0294*	(0.0162)	$\Phi^{\mathbb{Q}}_{Z_3,\Pi}$	-0.0469	(0.0325)
Φ_{Π,Z_1}	-0.0196***	(0.0046)	$\Phi^{\mathbb{Q}}_{\Pi,Z_1}$	-0.0633***	(0.0105)
Φ_{Z_1}	0.9862***	(0.0072)	$\Phi^{\mathbb{Q}}_{Z_1}$	0.9853***	(0.0134)
Φ_{Z_2,Z_1}	0	–	$\Phi^{\mathbb{Q}}_{Z_2,Z_1}$	0.021	(0.0174)
Φ_{Z_3,Z_1}	0	–	$\Phi^{\mathbb{Q}}_{Z_3,Z_1}$	-0.0557***	(0.0126)
Φ_{Π,Z_2}	-0.0073	(0.0085)	$\Phi^{\mathbb{Q}}_{\Pi,Z_2}$	-0.0646***	(0.0117)
Φ_{Z_1,Z_2}	0	–	$\Phi^{\mathbb{Q}}_{Z_1,Z_2}$	-0.0118	(0.0144)
Φ_{Z_2}	0.9028***	(0.0274)	$\Phi^{\mathbb{Q}}_{Z_2}$	1.016***	(0.0152)
Φ_{Z_3,Z_2}	0	–	$\Phi^{\mathbb{Q}}_{Z_3,Z_2}$	-0.0209	(0.0166)
Φ_{Π,Z_3}	$0^\dagger$	–	$\Phi^{\mathbb{Q}}_{\Pi,Z_3}$	0.0002	(0.0049)
Φ_{Z_1,Z_3}	0	–	$\Phi^{\mathbb{Q}}_{Z_1,Z_3}$	-0.0023*	(0.0013)
Φ_{Z_2,Z_3}	0	–	$\Phi^{\mathbb{Q}}_{Z_2,Z_3}$	0.0015	(0.0034)
Φ_{Z_3}	0.9939***	(0.0034)	$\Phi^{\mathbb{Q}}_{Z_3}$	0.9667***	(0.0029)
Σ_Π	0.1507***	(0.0099)	$\Sigma^{\mathbb{Q}}_\Pi$	0.1511***	(0.01)
$\Sigma_{Z_1,\Pi}$	0	–	$\Sigma^{\mathbb{Q}}_{Z_1,\Pi}$	$0^\dagger$	–
$\Sigma_{Z_2,\Pi}$	0	–	$\Sigma^{\mathbb{Q}}_{Z_2,\Pi}$	0.001*	(5e-04)
$\Sigma_{Z_3,\Pi}$	0	–	$\Sigma^{\mathbb{Q}}_{Z_3,\Pi}$	0.002**	(0.001)
Σ_{Z_1}	1	–	$\Sigma^{\mathbb{Q}}_{Z_1}$	$1^\dagger$	–
Σ_{Z_2,Z_1}	0	–	$\Sigma^{\mathbb{Q}}_{Z_2,Z_1}$	$0^\dagger$	–
Σ_{Z_3,Z_1}	0	–	$\Sigma^{\mathbb{Q}}_{Z_3,Z_1}$	$0^\dagger$	–
Σ_{Z_2}	1	–	$\Sigma^{\mathbb{Q}}_{Z_2}$	1.0037***	(0.0021)
Σ_{Z_3,Z_2}	0	–	$\Sigma^{\mathbb{Q}}_{Z_3,Z_2}$	0	(0)
Σ_{Z_3}	1	–	$\Sigma^{\mathbb{Q}}_{Z_3}$	1.0266***	(0.0057)

Notes: Standard deviations are in parentheses and are calculated using the outer-product Hessian approximation. The '–' sign indicates that the parameter has been calibrated hence does not possess any standard deviation. Whereas some parameters are calibrated for parsimony purposes (see main), some other are imposed to 0 due to a prior estimation and are indicated with the † sign. Significance level: * <0.1, ** <0.05, *** <0.01.

Table 3: Parameter estimates: short-rate and the prices of risk

	r_t dynamics				
	estimates	std.		estimates	std.
θ_0	0	–	$\theta_0^{\mathbb{Q}}$	0	–
θ_Π	0	–	$\theta_\Pi^{\mathbb{Q}}$	0	–
θ_{Z_1}	0	–	$\theta_{Z_1}^{\mathbb{Q}}$	0	–
θ_{Z_2}	0	–	$\theta_{Z_2}^{\mathbb{Q}}$	0	–
θ_{Z_3}	0	–	$\theta_{Z_3}^{\mathbb{Q}}$	0	–
Θ_Π	0.1469*	(0.0753)	$\Theta_\Pi^{\mathbb{Q}}$	0.1554*	(0.0795)
$\Theta_{Z_1,\Pi}$	$0^\dagger$	–	$\Theta_{Z_1,\Pi}^{\mathbb{Q}}$	$0^\dagger$	–
$\Theta_{Z_2,\Pi}$	0.0547*	(0.0277)	$\Theta_{Z_2,\Pi}^{\mathbb{Q}}$	0.0579*	(0.0293)
$\Theta_{Z_3,\Pi}$	0.1086**	(0.0505)	$\Theta_{Z_3,\Pi}^{\mathbb{Q}}$	0.1149**	(0.0533)
Θ_{Z_1}	$0^\dagger$	–	$\Theta_{Z_1}^{\mathbb{Q}}$	$0^\dagger$	–
Θ_{Z_2}	0.0318*	(0.0169)	$\Theta_{Z_2}^{\mathbb{Q}}$	0.0336*	(0.0179)
Θ_{Z_3}	0.2236***	(0.033)	$\Theta_{Z_3}^{\mathbb{Q}}$	0.2365***	(0.035)
$\varsigma \cdot 10^{-5}$	3.7621***	(0.4125)	$\varsigma^{\mathbb{Q}} \cdot 10^{-5}$	3.98***	(0.4335)
	Prices of risk and measurement errors standard deviations				
	estimates	std.		estimates	std.
$\lambda_{0,r}$	1455.5***	(276.29)			
$\lambda_{0,\Pi}$	3.0672	(3.3402)	λ_{0,Z_2}	-0.4266	(0.3293)
λ_{0,Z_1}	0.0717	(0.1526)	λ_{0,Z_3}	1.0348*	(0.5722)
λ_Π	-0.4992***	(0.1707)	λ_{Π,Z_2}	-0.3857***	(0.0656)
$\lambda_{Z_1,\Pi}$	-0.011	(0.0336)	λ_{Z_1,Z_2}	-0.0118	(0.0144)
$\lambda_{Z_2,\Pi}$	0.0501	(0.0811)	λ_{Z_2}	0.1099***	(0.0296)
$\lambda_{Z_3,\Pi}$	-0.0853**	(0.0322)	λ_{Z_3,Z_2}	-0.0196	(0.0164)
λ_{Π,Z_1}	-0.2884***	(0.0742)	λ_{Π,Z_3}	-0.011	(0.0343)
λ_{Z_1}	-0.0009	(0.0115)	λ_{Z_1,Z_3}	-0.0023*	(0.0013)
λ_{Z_2,Z_1}	0.0214	(0.0174)	λ_{Z_2,Z_3}	0.0015	(0.0034)
λ_{Z_3,Z_1}	-0.0535***	(0.0126)	λ_{Z_3}	-0.0522***	(0.0067)
$\sigma_\Pi^{(12)}$	0.509	–	$\sigma_\Pi^{(120)}$	0.389	–
$\sigma_{S_R}^{(3)}$	0.231	–	$\sigma_{S_R}^{(12)}$	0.422	–
σ_R	0.0516***	(0.0005)	σ_R^*	0.1313***	(0.003)

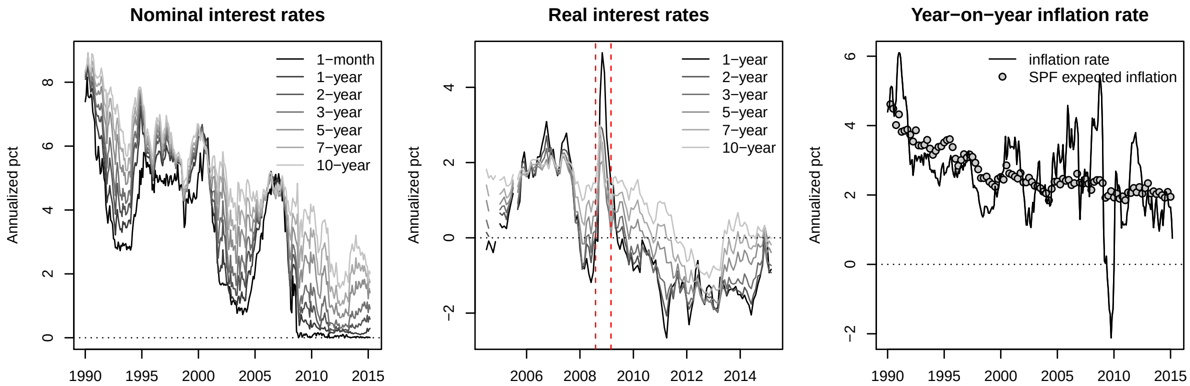
Notes: Standard deviations are in parentheses and are calculated using the outer-product Hessian approximation. The '–' sign indicates that the parameter has been calibrated hence does not possess any standard deviation. Whereas some parameters are calibrated for parsimony purposes (see main), some other are imposed to 0 due to a prior estimation and are indicated with the † sign. Significance level: * <0.1, ** <0.05, *** <0.01.

Table 4: Model fit and characteristics

Maturities (months)	1	12	24	36	60	84	120
Nominal rates RMSE (in bps)	1	7	5	5	4	2	6
Real rates RMSE (in bps)	–	16	8	11	14	12	9
Probabilities (in %)	$\mathbb{P}(r_t = 0) = 2.14$			$\mathbb{P}(r_t = 0 \vert r_{t-1} = 0) = 40.82$			

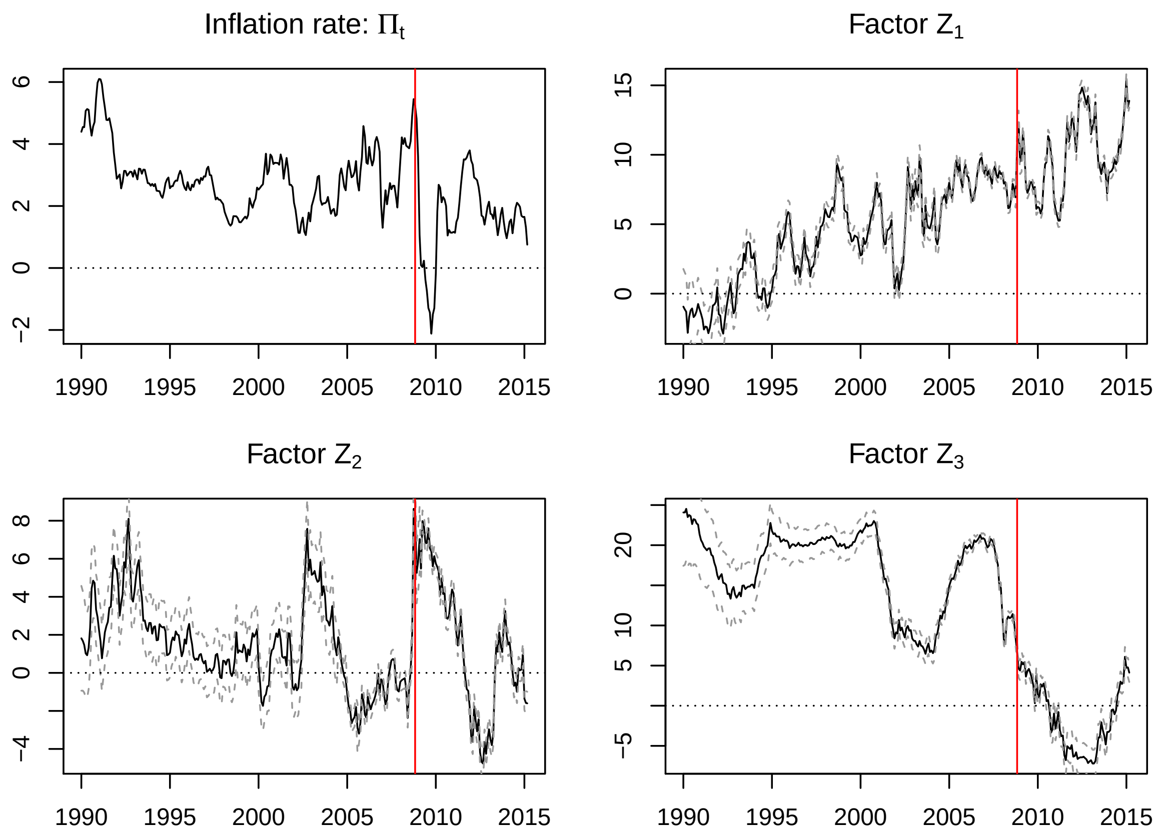
Note: Probabilities are calculated with simulated paths of length 2,000,000.

Figure 1: Nominal and real term structures and inflation data



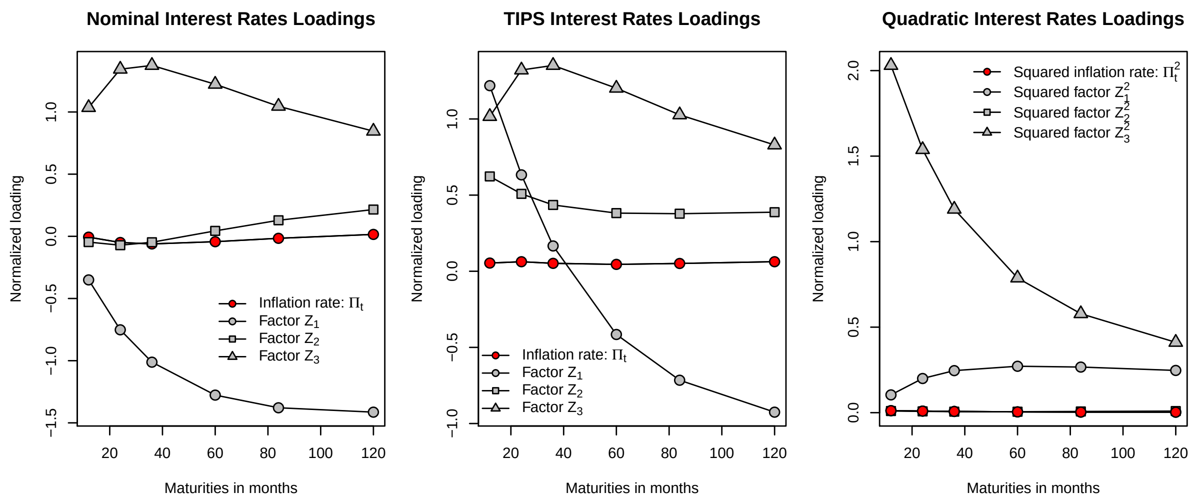
Notes: The left plot presents the time-series of the nominal term structure of interest rates from January 1990 to March 2015. Maturities range from 1 month to 10 years. The middle plot presents the term structure of real rates built as the difference between the nominal zero-coupon interest rates and the inflation swap rates of the same maturity. Observations start in July 2004 and run to March 2015. The vertical red dashed lines indicate the beginning and end of a reduced market liquidity period, that I treat as missing data in the estimation. The right plot presents the realized year-on-year inflation lagged of 3 months (black solid line). The dots superimpose the expected average inflation rate over the next year as measure by the survey of professional forecasters data.

Figure 2: Filtered factors



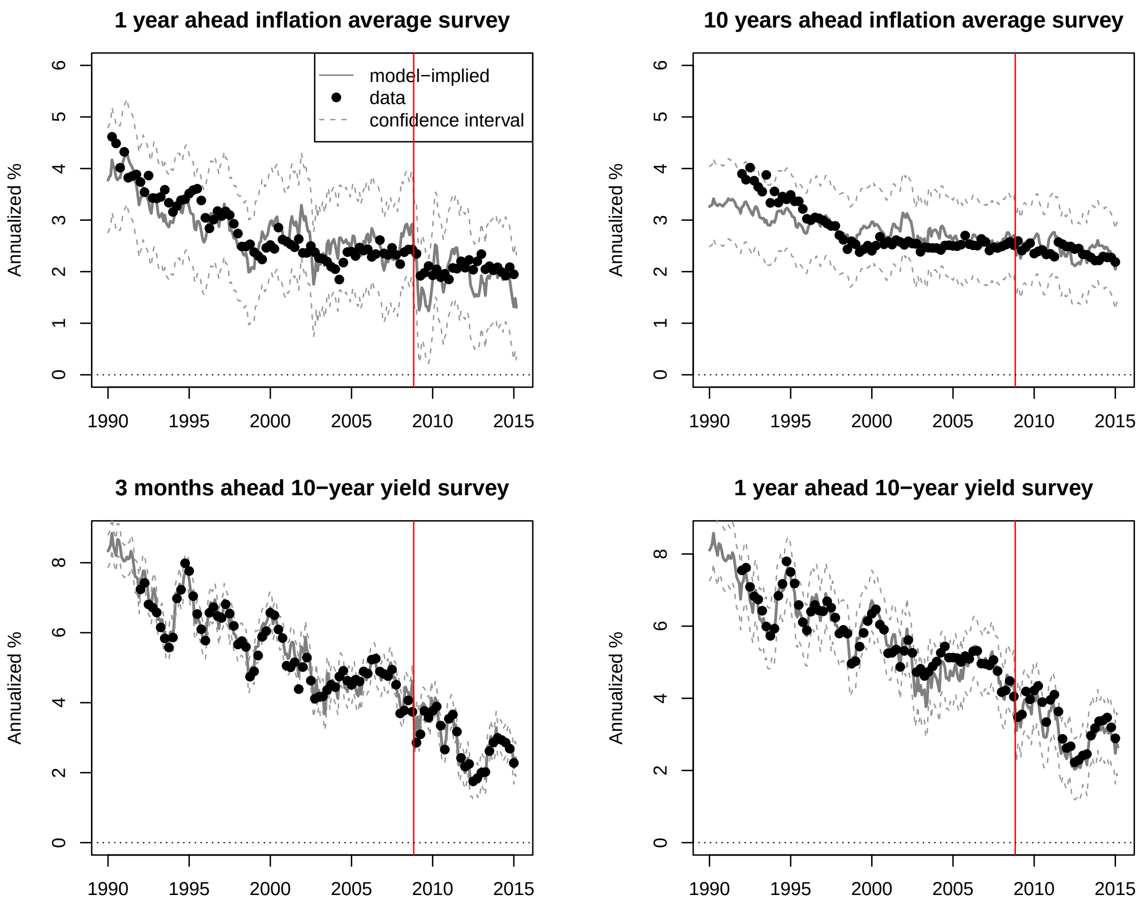
Notes: The first factor is the observed year-on-year inflation rate hence it possesses no filtering standard deviations. The other 3 factors are estimated using the quadratic Kalman filter and 95% confidence bounds are plotted with dashed grey lines. Red vertical lines delimit the beginning of the zero lower bound period.

Figure 3: Factors loadings



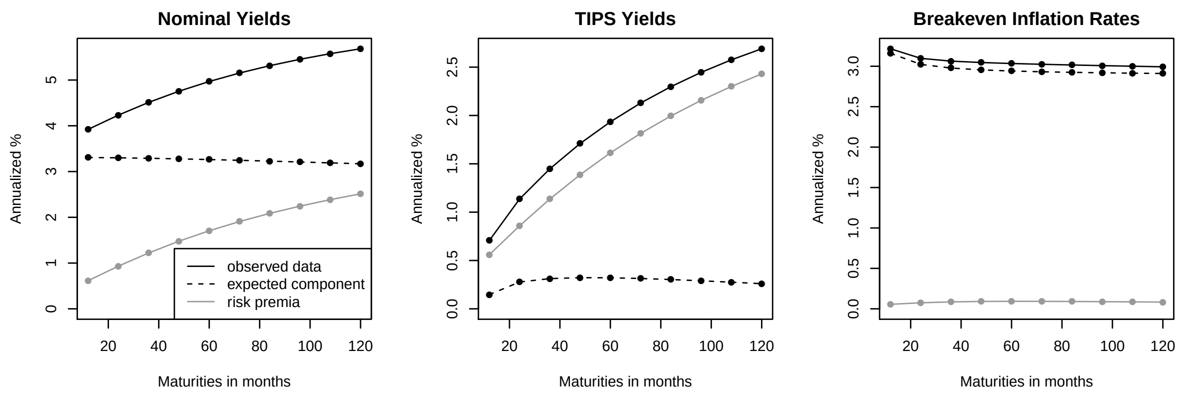
Notes: From left to right, this plot gathers the linear loadings of the nominal interest rates, of the real rates, and the quadratic loadings (which are the same for both yield curves) with respect to maturity. These loadings are normalized by the in-sample standard deviation of the corresponding filtered factor to be comparable with each other.

Figure 4: Fitted series of survey data



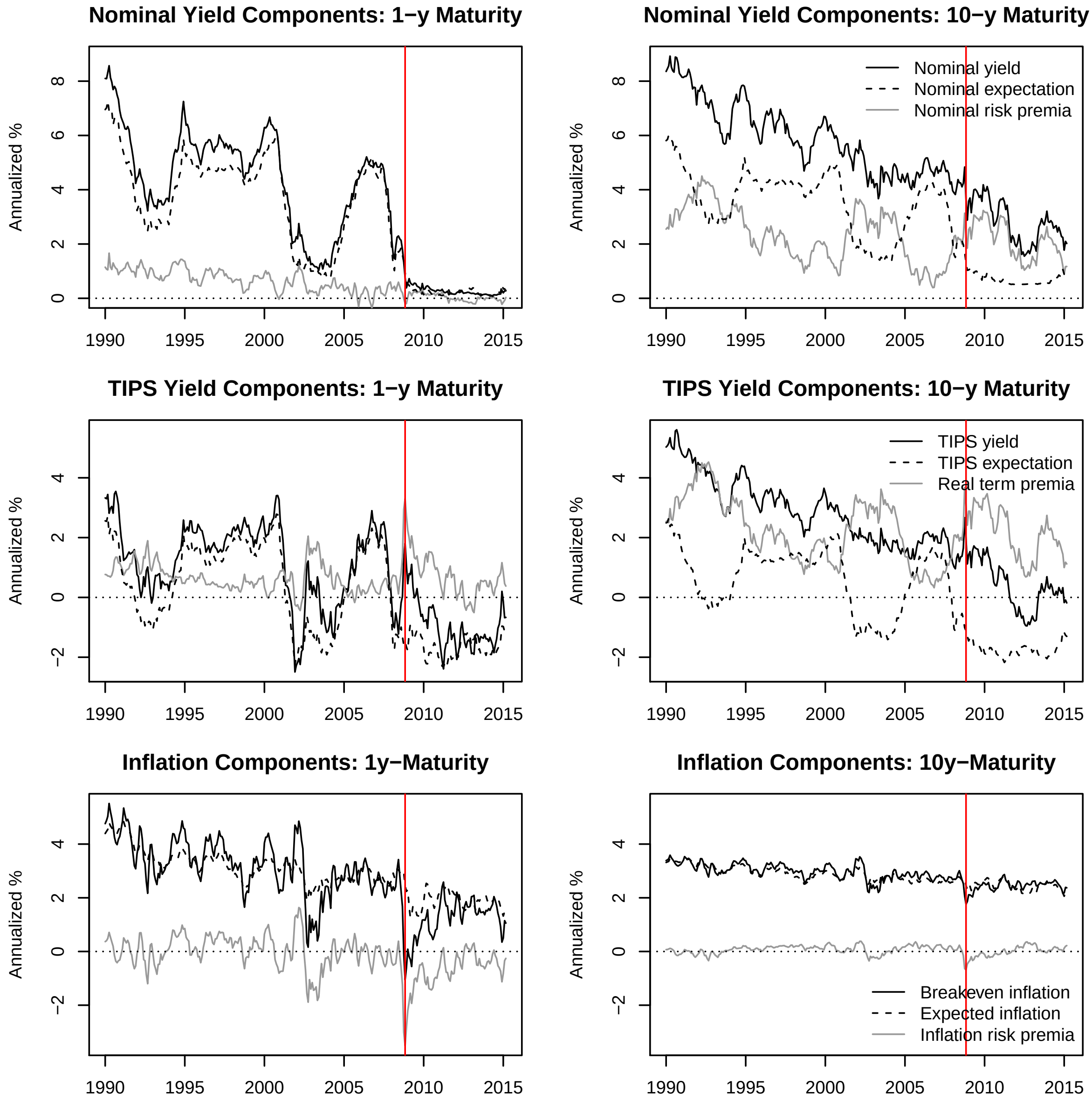
Notes: The black dots correspond to observed forecast data. The grey solid lines correspond to the model-implied forecasted values. Top graphs correspond respectively to the one-year ahead and 10-year ahead inflation average surveys. Bottom graphs correspond respectively to the three-months ahead and one-year ahead 10-year yield survey. Units are in annualized percentage points. Confidence intervals computed using the measurement errors standard deviations are plotted in grey dashed lines. Red vertical lines delimit the beginning of the zero lower bound period.

Figure 5: Marginal term structures



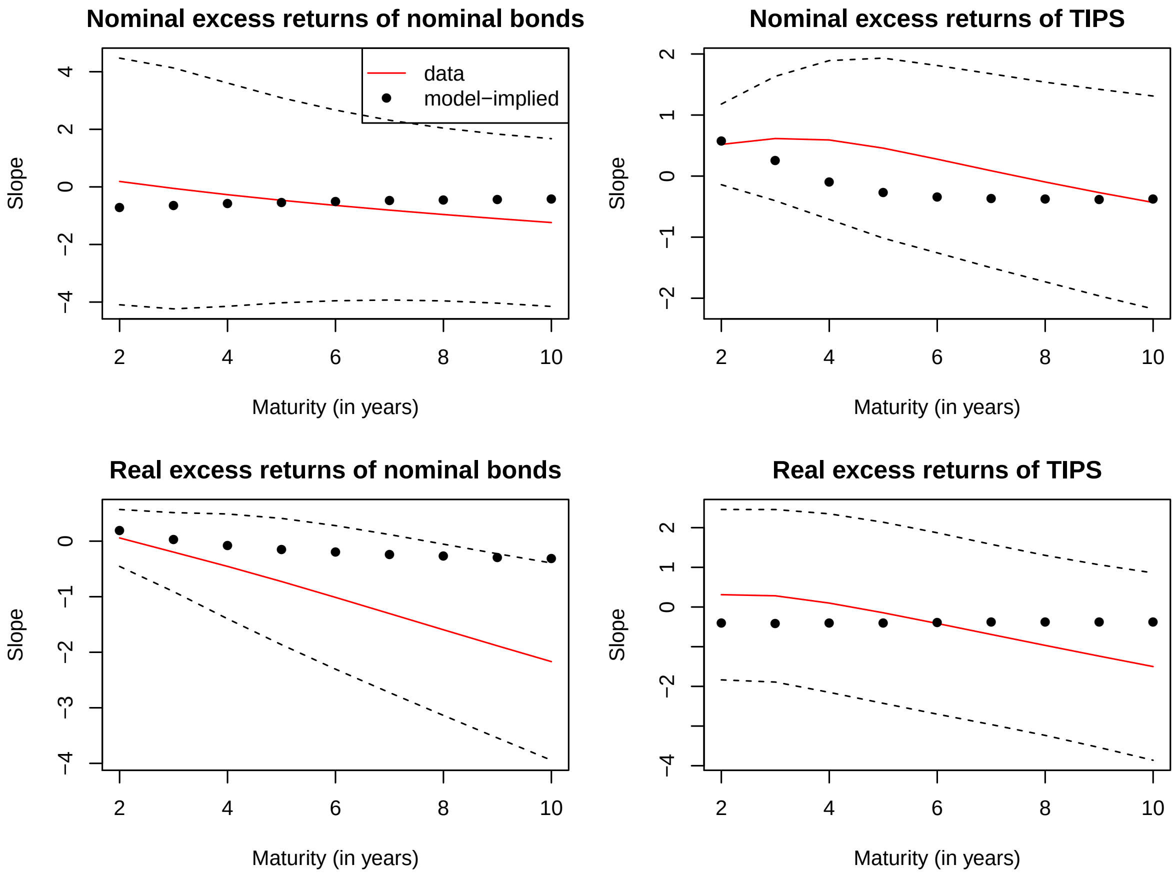
Notes: Black solid lines are observed data, black dashed lines are expected components and grey solid lines are risk premia components. The marginal term structure is obtained with the closed-form marginal mean of the transition equation (15).

Figure 6: Decomposition of interest rates



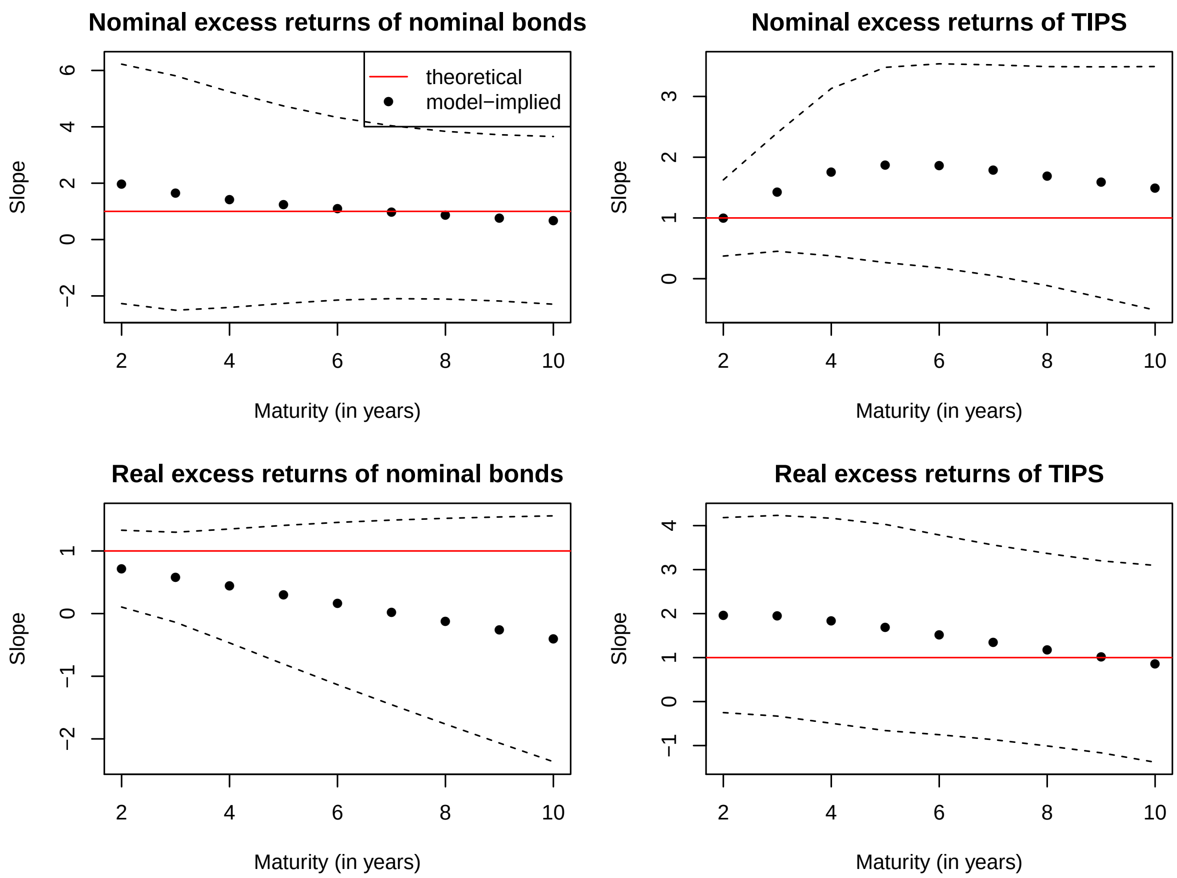
Notes: First column presents results for the 1-year maturity yields, whereas second column presents results for the 10-year maturity yields. The first row presents to the observed nominal yield (black solid line), the nominal term premia (grey solid line), and the expected component (black dashed line). The second row presents to the filtered TIPS yield (black solid line), the real term premia (grey solid line) and the expected real rate (black dashed line). The last row presents the filtered inflation breakeven rate (black solid line), the inflation risk premia (grey solid line) and the inflation expectation (black dashed line). Units are in annualized percentage points. Red vertical lines delimit the beginning of the zero lower bound period.

Figure 7: Campbell-Shiller regression slopes: LPY-I



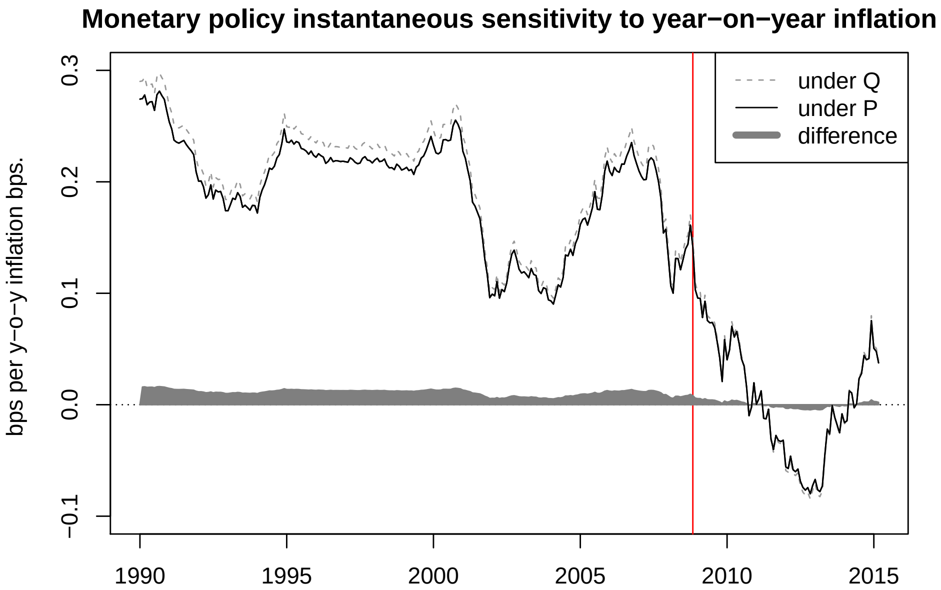
Notes: From left to right, from top to bottom, the graphs present the slopes of Campbell and Shiller regressions for (CS.1), (CS.3), (CS.4) and (CS.2) with a 12-months holding period. The red solid line gathers the slope estimates obtained with filtered yields data from January 1990 to August 2014. 95% Confidence intervals are computed using Newey-West robust estimators with automatically selected lag and are indicated with the dashed lines. Model-implied estimates are indicated with the black dots and computed with the yields and inflation expectation and variance formulas.

Figure 8: Campbell-Shiller regression slopes: LPY-II



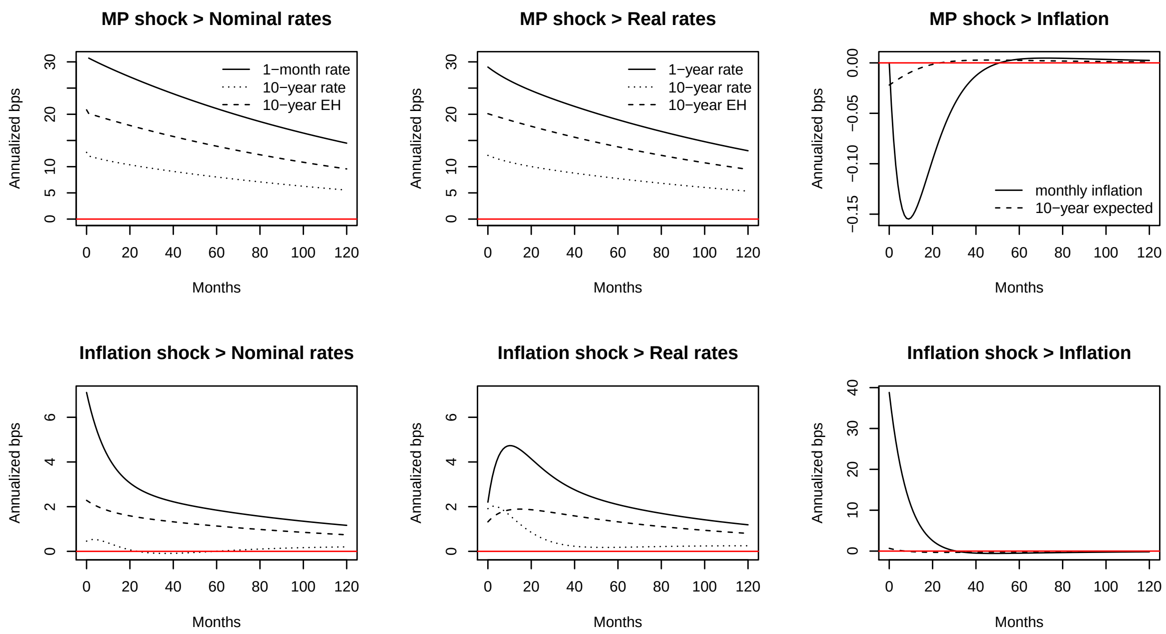
Notes: From left to right, from top to bottom, the graphs present the slopes of Campbell and Shiller regressions for (CS.1), (CS.3), (CS.4) and (CS.2) when regressors are adjusted by the corresponding model-implied expected excess return series. The red solid line represents the theoretical values of the regression, namely one for all maturities. Model-implied estimates are indicated with the black dots and computed performing the Campbell and Shiller regressions where the dependent variable is adjusted by the model-implied expected excess returns. 95% Confidence intervals are computed using Newey-West robust estimators with automatically selected lag and are indicated with the dashed lines.

Figure 9: Monetary policy reaction function



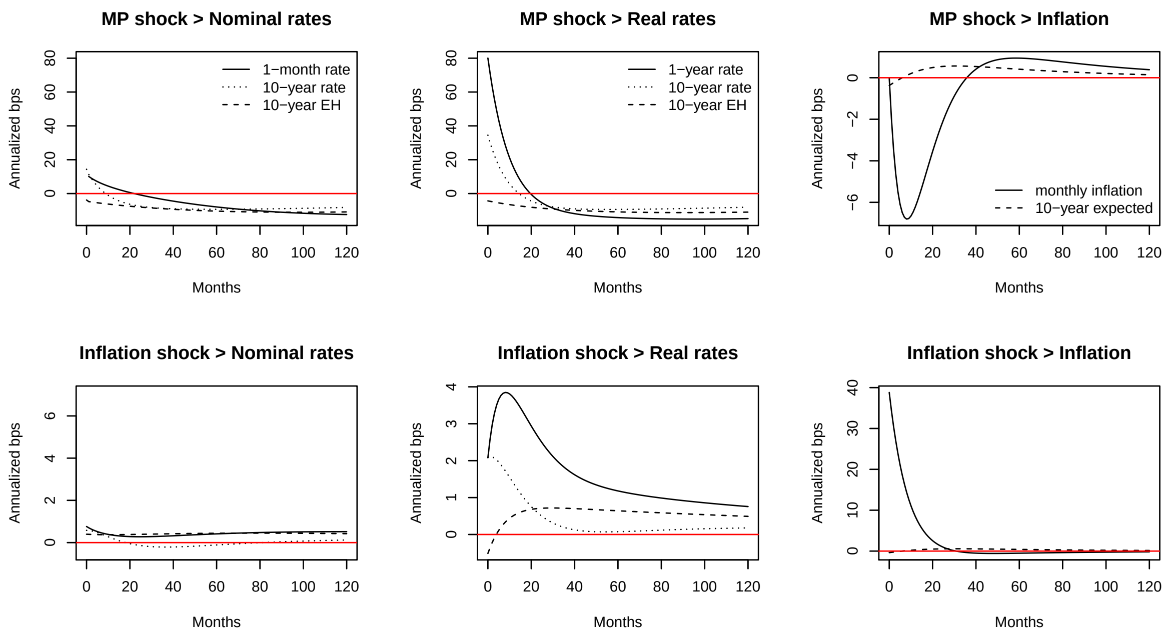
Notes: This graph represents the time-series of the inflation loadings in the Taylor-type rule representation of the short-term interest rate dynamics (see Equations (4) and (11)). The black solid line represents the physical loading of the inflation rate denoted as β_t in Equation (4) whereas the grey dashed line represents the risk-neutral loadings of the inflation rate denoted as $\beta_t^{\mathbb{Q}}$ in Equation (11). The grey shaded area is the difference between the risk-neutral and the physical coefficients. The red vertical line delimits the start of the zero lower bound period.

Figure 10: Impulse-response functions in the steady-state



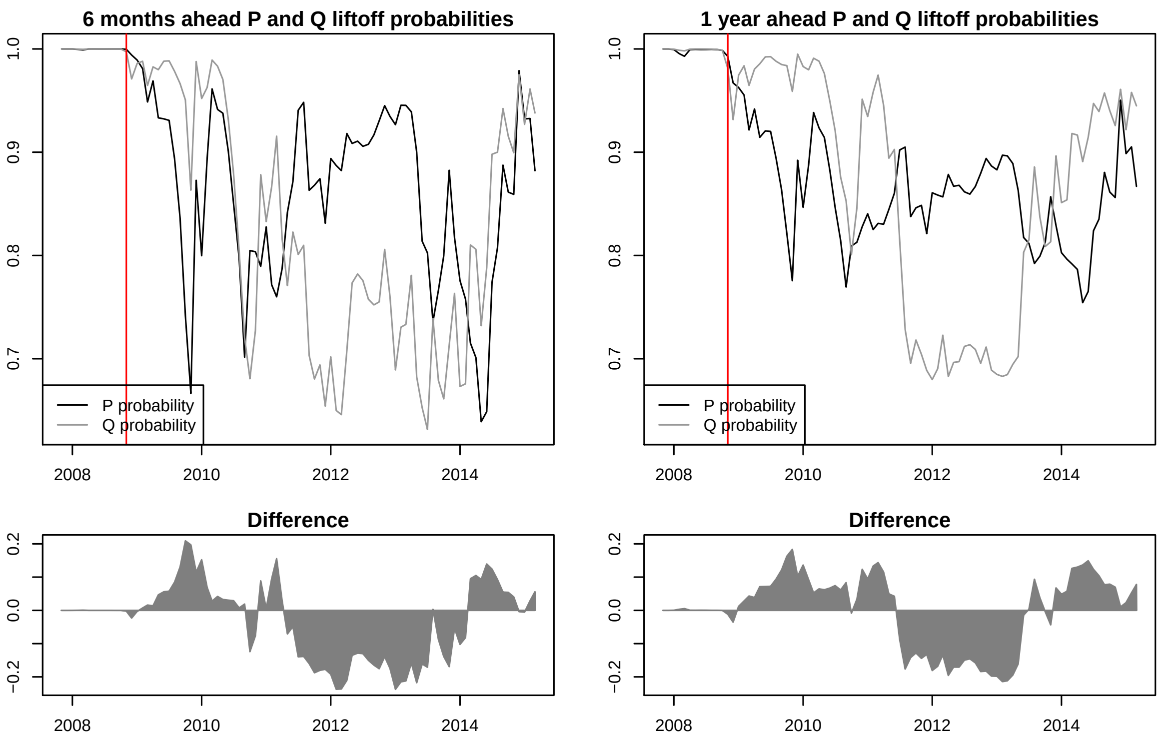
Notes: These graphs respectively present the effect of a monetary policy shock on the nominal yield curve (top-left), real yield curve (top-middle) and on the year-on-year inflation rate (top-right), and the effect of an inflation shock on the nominal yield curve (bottom-left), real yield curve (bottom-middle) and on the inflation rate (bottom-right), conditionally on being at the steady state (see Section 5.1 for the detailed procedure). The left column plots summarize the nominal yield curve with the 1-month rate (black solid line), the 10-year rate (black dotted line) and the 10-year expectation component (black dashed line). The right column plots represent both the observed inflation rate (black solid line) the its expectation at the 10-year horizon (black dashed line). Units are in annualized basis points.

Figure 11: Impulse-response functions at the zero lower bound



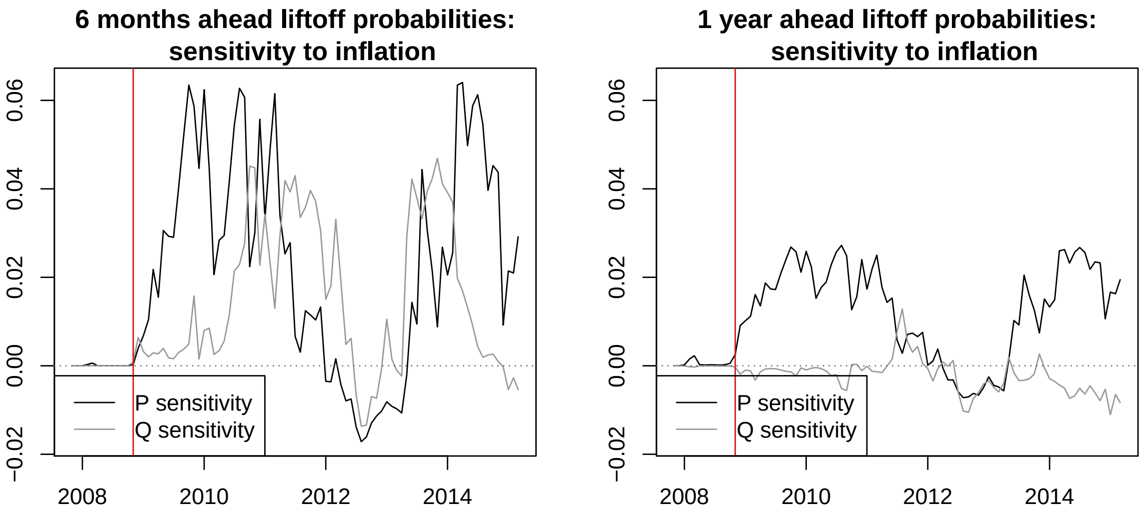
*Notes:*These graphs respectively present the effect of a monetary policy shock on the nominal yield curve (top-left), real yield curve (top-middle) and on the year-on-year inflation rate (top-right), and the effect of an inflation shock on the nominal yield curve (bottom-left), real yield curve (bottom-middle) and on the inflation rate (bottom-right), conditionally on being at the zero lower bound (see Section 5.1 for the detailed procedure). The left column plots summarize the nominal yield curve with the 1-month rate (black solid line), the 10-year rate (black dotted line) and the 10-year expectation component (black dashed line). The right column plots represent both the observed inflation rate (black solid line) the its expectation at the 10-year horizon (black dashed line). Units are in annualized basis points.

Figure 12: Physical and risk-neutral liftoff probabilities



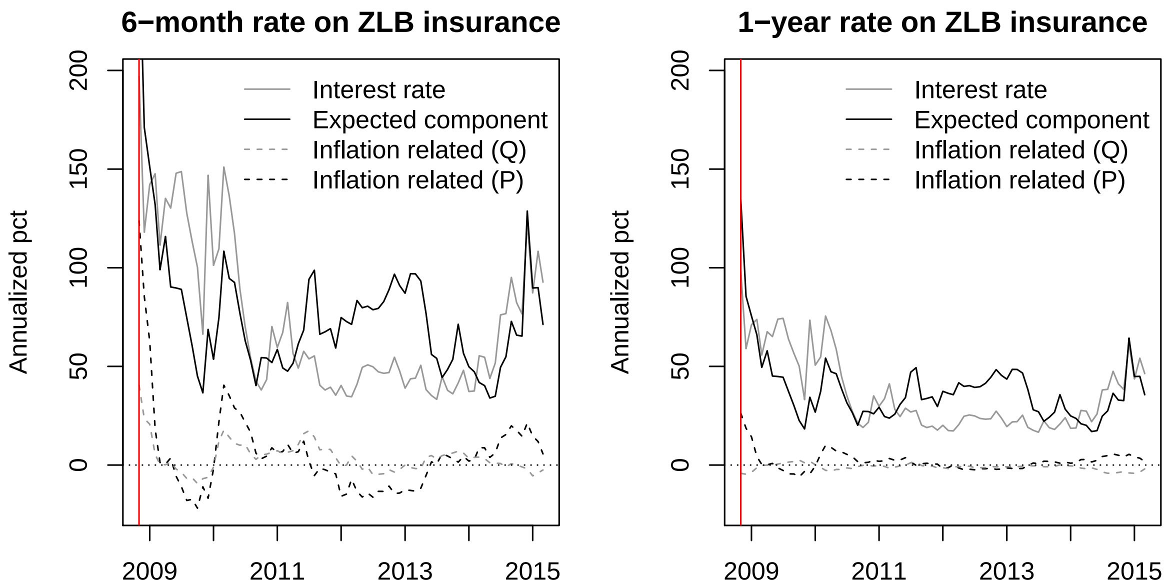
Notes: The two columns present respectively the 6-months ahead and 1-year ahead liftoff probabilities under the physical and the risk-neutral measure. $\mathbb{P}$- and $\mathbb{Q}$-probabilities are respectively represented with a black and a grey solid line. The second row presents the difference between the risk-neutral and the physical liftoff probabilities. Red vertical bars delimit the beginning of the zero lower bound period.

Figure 13: Liftoff probabilities sensitivity to inflation



Notes: The two columns present respectively the 6-months ahead and 1-year ahead liftoff probabilities sensitivity to inflation under the physical and the risk-neutral measure. $\mathbb{P}$- and $\mathbb{Q}$-sensitivities are respectively represented with a black and a grey solid line. These sensitivities are measured by the first partial derivative of liftoff probabilities with respect to the inflation rate. Units are as follows: on the last sample date, the 6-months $\mathbb{P}$-probability increases of approximately 3pps for each 1pp inflation increase. Red vertical bars delimit the beginning of the zero lower bound period.

Figure 14: Interest rates on a ZLB insurance: decomposition



Notes: These graphs present the ex-ante interest rate on a synthetic ZLB insurance bond paying only when the economy is at the ZLB. The interest rate is plotted with a grey solid line, its expectation component is plotted with a black solid line, and the contribution from inflation to these components are respectively plotted with a grey dashed and a black dashed line. Units are in Annualized percentages. Red vertical bars delimit the beginning of the zero lower bound period.

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