Where's the Risk? The Forward Premium Bias, the Carry-Trade Premium, and Risk-Reversals in General Equilibrium

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Abstract

This paper builds a two-country dynamic stochastic general equilibrium macro model to understand three empirical facts about international currency returns. They are the downward forward premium bias, the carry trade return, and the long-run risk reversal. A model with incomplete markets, country heterogeneity in productivity, and country heterogeneity in monetary policies is qualitatively consistent with these facts.

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Introduction

The downward forward premium bias, the carry trade excess return, and the long-run risk reversal are three distinct, but related empirical regularities that have come to characterize international currency returns. The downward forward premium bias has long been a topic of study. Academic interest in the carry trade has been growing for a little more than a decade now. The long-run risk reversal, identified and studied by Engel (2016), is relatively new. In this paper, we study how these empirical patterns in currency returns might be understood in the context of a two-country dynamic stochastic general equilibrium (DSGE) macroeconomic model.

The downward forward premium bias refers to regression evidence that uncovered interest rate parity (UIP) is violated in the data. UIP says the excess return earned by going short (borrowing) the low interest rate currency and going long (lending) the high interest rate currency should be exactly offset by a loss in value of the long currency. Econometrically, UIP predicts a zero constant and a unit slope coefficient in a regression of the future currency depreciation on today's interest rate differential. Bilson (1981) and Fama (1984) first ran these regressions, and many researchers have run them since. Almost always, the slope in the regression is less than 1 and often it is negative. This is what we refer to as the downward forward premium bias.[1] The downward forward premium bias is remarkably robust to the sample period and to the choice of base currency. The downward bias implies the bigger is the interest rate differential, the smaller is the subsequent currency depreciation rate, and the mainstream view is that this occurs because the two currencies have different risks and the downward bias results from the presence of a risk premium across currencies.

Recent advances in understanding the downward bias have generally been conducted with endowment models. Verdelhan (2010) employs habit persistence in utility following Campbell and Cochrane (1999). Bansal and Shaliastovich (2013) explain it with recursive utility of Epstein and Zin (1989) and Weil (1989) where exogenous consumption growth and inflation exhibit 'long-run risks.' Also, in an endowment setting, Backus et al. (2013) study risk induced by monetary policy heterogeneity across countries. An exception is Chinn and Zhang (2015), who study the forward premium bias at the zero-lower bound in a small-open economy New Keynesian framework.

The carry trade is a profitable, zero-net investment strategy for currencies. It says to go short the low interest currency and to go long the high interest currency. The carry trade, while related

[1] Before the global financial crisis in 2008, the covered interest arbitrage condition led the inter-bank interest differential to be equal to the forward contract premium on the spot exchange rate. Hence the forward premium and the interest rate differential were interchangeable. The imposition of new regulatory capital requirements on banks in 2008 caused covered interest parity to fail (Du et al. (2016) and Pinnington and Shamloo (2016)).

to the forward-premium bias, is not the same thing. One notable feature of the carry trade, when formed into portfolios, is its consistent profitability. An extensive and growing literature is devoted to its study. For example, assuming no transactions costs, Lustig et al. (2014) report an excess return between portfolios of the highest and lowest interest rate countries of 6.2 percent per annum. Research aimed at understanding the cross-section of returns–why the average excess return is increasing in the average size of the interest rate differential–includes Burnside et al. (2011), Della Corte et al. (2016), Lustig and Verdelhan (2007), Menkhoff et al. (2013), and Berg and Mark (2017a, 2017b). Employing the 'beta-risk' framework, these authors study the cross-sectional pricing of carry trade returns where the stochastic discount factors loads (depends) on risk factors constructed from macroeconomic data. This paper is more closely related to a smaller literature that studies the carry trade in general equilibrium macro models, such as Hassan (2013) who emphasizes differences in country size and Ready et al. (2017) who focus on cross-country differences in productive technology.

Engel's (2016) risk reversal begins with the observation that a country's currency strength is associated with it having high relative real interest rates. Classic articles by Dornbusch (1976) and Frankel (1978) established a theoretical link between real interest rates and currency value. Empirical evidence for the link is found in Engel and West (2006), Alquist and Chinn (2008), and Mark (2009). Moreover, the idea has been put into practice by central banks to defend their currencies in times of crisis. For example, in September 1992, in an attempt to maintain the krona, Sweden's Riksbank briefly raised its marginal lending rate to 500 percent per annum. The International Monetary Fund has historically advised its members to defend their currencies by raising interest rates—advice heeded by South Korea during the 1997 Asian Financial Crisis. Engel's (2016) argument then, is that the high interest rate country is the (relatively) safe one. It is safe because it has a strong currency. Being safe, it should pay out a negative risk premium. But observations on the carry trade and the forward premium bias says the high interest country pays a positive risk premium so it must be risky. To reconcile these contradictory predictions, the high interest rate country must undergo a risk reversal over time. In the short run, it is risky and pays a positive risk premium through the carry trade, but over time, that risk premium turns negative. Engel (2016) reports empirical evidence of these risk reversals in a sample of the G-7 countries. He then argues that the current generation of international finance models are unlikely to explain the risk reversal and what may be missing from those models is a non-pecuniary liquidity return on assets.

Is there a common source of risk that gives rise to these empirical currency return patterns? We address this question in a two-country New Keynesian DSGE model that features local currency pricing (LCP) by exporters. We evaluate both a complete markets version of the model and an incomplete markets version. Productivity is nonstationary and is driven by a stochastic trend. A central point to emerge from studies on the carry trade is that heterogeneity across countries is key to understanding currency excess returns.[2]

Our model gives prominence to two sources of country heterogeneity. The first is heterogeneity in cross-country productivity. A central feature of total factor productivity (TFP) across countries is that they are stochastically trending. Significantly, there is little evidence of TFP convergence across countries. Divergent TFP represents a potentially significant risk factor that could be priced into currency excess returns. Our solution technique is perturbation with pruning of a third-order approximation around a nonstochastic steady state, so we cannot literally have divergent random walks in the country TFPs. Our specification of TFP features near unit root in the error-correction term so they are technically cointegrated, but they approximate divergence. The TFP of one country can stay well above the other for 400 quarters or more.

The second source of country heterogeneity lies in cross-country differences in monetary policy. Differences in the cyclical response of the interest rate, and differences in accommodation to inflation can be a source of currency risk. Other researchers have incorporated inflation into their analyses of currency risk, but have done so in endowment frameworks. In Bansal and Shaliastovich (2012), inflation and consumption growth are jointly governed by an exogenous long-run risk process, but people in their economy care about inflation only to the extent that exogenous inflation and consumption growth are correlated. While inflation is endogenously determined in Backus et al. (2013), inflation has no effect on welfare in their endowment economy model. In our general equilibrium model, inflation and associated price dispersion do have effects on welfare.

To summarize our main results, when the only heterogeneity across countries are differences in productivity, the incomplete markets model generates the downward forward premium bias where the implied slope coefficient in the Fama regression is less than 1 (but not negative), a carry-trade excess return of 2.1 percent per annum and Engel's risk-reversal. The complete markets model can generate none of these return patterns. When cross-country differences in monetary policies are incorporated, both the complete markets and incomplete markets model generate the downward forward premium bias and produce sizable carry trade excess returns, however only the incomplete markets model generates the risk reversal.

The remainder of the paper is structured as follows. The next section develops the two-country New Keynesian DSGE model with nominal rigidities and recursive utility. In Section 2, we discuss the parameterization of the model. Section 3 discusses each of the three currency

[2]For example, Hassan (2013) exploits differences in country size, whereas Lustig et al. (2014) exploit differences in risk factor loadings across country stochastic discount factors (SDFs) .

return empirical regularities in some detail and assesses the model's ability to explain the return patterns. Section 4 considers alternative specifications of productivity, Section 5 discusses the impulse response functions (IRFs), and Section 6 concludes.

1 A Two-Country Macroeconomic Model

In this section, we outline the two-country DSGE model used in our analysis. Prices are sticky and adjust through a Calvo (1983) mechanism.[3] Exporters set nominal export prices in advance and in terms of the foreign currency–a practice called local currency pricing (LCP). Households have recursive utility (Epstein and Zin (1989) and Weil (1989)). These preferences have gained popularity in macroeconomic and financial economics research.[4] In addition to making the current utility flow dependent on expected future utility, recursive utility generalizes power utility by regulating the intertemporal elasticity of substitution and the degree of risk aversion through separate parameters. The exogenous variables are shocks to country productivity, which themselves are unit-root processes. Productivity is cointegrated across countries, but not strongly, in the sense that the error correction term has a near unit root.

Unless it is necessary to distinguish between countries $i = \{1, 2\}$, we will suppress the country subscript. Because a country's productivity, A_t, has a stochastic trend, level variables (except labor), will trend with A_t which causes the model to be nonstationary. The model solution requires an approximation around a non-stochastic steady state. Hence, we need a stationary representation, which we obtain by dividing one-period lagged productivity into the trending variables. Variables normalized in this way are denoted by a tilde (e.g., normalized consumption is $\tilde{c}_t \equiv c_t/A_{t-1}$). The gross growth rate in technology is denoted by $G_t \equiv A_t/A_{t-1}$. The normalized model is presented in the text.

1.1 Households

We assume a particular functional form of recursive utility employed by Swanson (2016).[5] Let V_t be current utility, c_t be the household's real consumption and ℓ_t be its labor supply. Households

[3] A complete description of the model is contained in the appendix.

[4] In international finance, Bansal and Shaliastovich (2012) extend the long-run risk model of Bansal and Yaron (2004) with recursive utility to study international bond and currency markets, while Colacito and Croce (2011) adapt that framework to study international equity pricing. Backus et al. (2013) also employ recursive utility in their analysis. In international macroeconomics, Kollmann (2017) studies real exchange rate volatility with recursive utility.

[5] This is a monotone transformation of the more familiar recursive utility specification, with the intertemporal elasticity of substitution (IES) set to 1. See Karantounias (2017).

in both countries want to maximize the transformed utility

$$\tilde{V}_t = \left\{ (1-\beta) \left[\ln(\tilde{c}_t) - \eta \frac{\ell_t^{1+\chi}}{1+\chi} \right] - \frac{\beta}{\alpha} \ln \left[E_t e^{-\alpha \tilde{V}_{t+1}} \right] \right\} \tag{1}$$

where $\beta \in (0,1)$ is the subjective discount factor and $\eta > 0$, $\chi > 0$, $\alpha \in \mathbb{R}$ are also parameters. The log form of the current utility flow of consumption fixes the intertemporal elasticity of substitution (IES) to be 1. The Frisch elasticity of labor supply is $1/\chi$. Swanson (2016) shows that relative risk aversion (RRA) is

$$RRA = \alpha + \left(\frac{1}{1+\frac{\eta}{\chi}} \right).$$

The intertemporal marginal rate of substitution (IMRS) is, $\beta M_{t,t+1}$ where

$$M_{t,t+1} = \left(\frac{\tilde{c}_t}{\tilde{c}_{t+1}} \right) \left(\frac{e^{-\alpha \tilde{V}_{t+1}}}{E_t \left(e^{-\alpha \tilde{V}_{t+1}} \right)} \right). \tag{2}$$

As is the convention in asset-pricing research, we refer to the IMRS as the stochastic discount factor (SDF). If π_{t+1} is the inflation rate from t to $t+1$, the nominal SDF is $\beta N_{t,t+1}$ where

$$N_{t,t+1} = M_{t,t+1} e^{-\pi_{t+1}} \tag{3}$$

We consider both a complete markets and an incomplete markets environment. Engel (2016) expresses doubt that complete market models can be made consistent with the risk reversal. We include an analysis under complete markets to benchmark the results.

1.1.1 Complete markets

Denote the current state of the world by ω_t and the state history by $\omega^t = \{\omega_t, \omega_{t-1}, \ldots\}$. Households in both countries have access to a full set of nominal state-contingent securities. These securities pay one unit of currency 1 if the state occurs. Making explicit, the functional dependence on the state, let $\tilde{B}\left(\omega^t\right)$ be the number of state ω^t contingent bonds held by the household (recall that the tilde means the variable is normalized by the lagged level of technology). The price of a bond that pays off in state ω_{t+1} is $p_\omega\left(\omega_{t+1}|\omega^t\right)$. Shares of firms are not internationally traded and are entirely owned by domestic households. Let $P\left(\omega^t\right)$ be the price level, $\tilde{W}\left(\omega^t\right)$ be the nominal wage, and $\tilde{\Pi}\left(\omega^t\right)$ be nominal firm profits. There is no physical capital in the model. Households obtain flow resources from labor income, firm profits, and state-contingent bond payoffs. Those resources are spent on consumption and a portfolio of state-contingent bonds. In country 1, the household budget constraint is,

$$\tilde{c}_1\left(\omega^t\right) = \frac{\tilde{W}_1\left(\omega^t\right)}{P_1\left(\omega^t\right)} \ell_1\left(\omega^t\right) + \frac{\tilde{\Pi}_1\left(\omega^t\right)}{P_1\left(\omega^t\right)} + \frac{\tilde{B}_1\left(\omega^t\right)}{P_1\left(\omega^t\right)} - \sum_{\omega^{t+1}} \frac{p_\omega\left(\omega_{t+1}|\omega^t\right) \tilde{B}_1\left(\omega^{t+1}\right)}{P_1\left(\omega^t\right)} G_1\left(\omega^t\right). \tag{4}$$

If $\pi\left(\omega_{t+1}|\omega^t\right)$ is the conditional probability of state ω_{t+1}, the optimality conditions for the household give the Euler equation for the state-contingent bond and the labor supply equation,

$$p_\omega\left(\omega_{t+1}|\omega^t\right) = \beta\pi\left(\omega_{t+1}|\omega^t\right)\left(\frac{M_1\left(\omega_{t+1}|\omega^t\right)}{G_1\left(\omega^t\right)}\right)\left(\frac{P_1\left(\omega^t\right)}{P_1\left(\omega^{t+1}\right)}\right), \tag{5}$$

$$\eta\tilde{c}_1\left(\omega^t\right)\ell_1\left(\omega^t\right)^\chi = \frac{\tilde{W}_1\left(\omega^t\right)}{P_1\left(\omega^t\right)}. \tag{6}$$

Summing over the prices of all state-contingent bonds gives the price of the nominally risk-free bond,

$$\frac{1}{1+i_1\left(\omega^t\right)} = \left(\frac{\beta}{G_1\left(\omega^t\right)}\right)E_t\left(M_1\left(\omega_{t+1}|\omega^t\right)e^{-\pi_1\left(\omega^{t+1}\right)}\right). \tag{7}$$

The country 2 household faces a similar environment, except the state contingent bonds are denominated in currency 1. To get real contingent bond holdings, country 2's household revalues by the exchange rate. If $S_{1,2}\left(\omega^t\right)$ is the nominal exchange rate (the price of currency 2), then real country 2 contingent bond holdings are $\tilde{B}_2\left(\omega^t\right)\left(S_{1,2}\left(\omega^t\right)P_2\left(\omega^t\right)\right)^{-1}$. The Euler equation for the country 2 state-contingent bond and the labor supply equation are,

$$p_\omega\left(\omega^{t+1}|\omega^t\right) = \beta\pi\left(\omega_{t+1}|\omega^t\right)\left(\frac{M_2\left(\omega_{t+1}|\omega^t\right)}{G_2\left(\omega^t\right)}\right)\left(\frac{S_{1,2}\left(\omega^t\right)P_2\left(\omega^t\right)}{S_{1,2}\left(\omega^{t+1}\right)P_2\left(\omega^{t+1}\right)}\right), \tag{8}$$

$$\eta\tilde{c}_2\left(\omega^t\right)\ell_2\left(\omega^t\right)^\chi = \frac{\tilde{W}_2\left(\omega^t\right)}{P_2\left(\omega^t\right)}. \tag{9}$$

Equating equations (5) and (8) and rearranging, gives the gross nominal depreciation of country 1's currency,

$$\frac{S_{1,2}\left(\omega^{t+1}\right)}{S_{1,2}\left(\omega^t\right)} = \left(\frac{M_2\left(\omega_{t+1}|\omega^t\right)e^{-\pi_2\left(\omega^{t+1}\right)}}{M_1\left(\omega_{t+1}|\omega^t\right)e^{-\pi_1\left(\omega^{t+1}\right)}}\right)\left(\frac{G_1\left(\omega^t\right)}{G_2\left(\omega^t\right)}\right), \tag{10}$$

and rearrangement of equation (10) gives the gross real depreciation,

$$\frac{Q_{1,2}\left(\omega^{t+1}\right)}{Q_{1,2}\left(\omega^t\right)} = \left(\frac{M_2\left(\omega_{t+1}|\omega^t\right)}{M_1\left(\omega_{t+1}|\omega^t\right)}\right)\left(\frac{G_1\left(\omega^t\right)}{G_2\left(\omega^t\right)}\right), \tag{11}$$

where $Q_{1,2}\left(\omega^t\right) = \frac{S_{1,2}\left(\omega^t\right)P_2\left(\omega^t\right)}{P_1\left(\omega^t\right)}$ is the real exchange rate. Exchange rates are defined so that a decline in the value of currency 1 is reflected by an increase in $S_{1,2}$ in nominal terms and an increase in $Q_{1,2}$ in real terms. In anticipation of future use, we define the real exchange rate from country 2's perspective as $Q_{2,1} = Q_{1,2}^{-1}$, and for the nominal exchange rate, $S_{2,1} = S_{1,2}^{-1}$. Equations (10) and (11) form the basic building blocks in the stochastic discount factor approach to the exchange rate.[6]

[6] Equation (11) was first derived by Backus and Smith (1993). Backus et al. (2001) further developed and refined the SDF approach to the exchange rate. See also, Lustig and Verdelhan (2013) for an exposition and survey of the approach.

1.1.2 Incomplete markets

In the incomplete markets version of the model we can suppress the state-dependent notation. Here, each country issues a nominal non-state contingent bond denominated in their own currency. International asset trade is restricted to this pair of nominal bonds. Country 1 issues its bond at a price of 1 unit of currency 1. It pays off $1 + i_{1,t}$ units of currency 1 next period. Country 2 issues its bond at a price of 1 unit of currency 2, which pays off $1 + i_{2,t}$ units of currency 2 next period. $B_{i,j,t}$ is the number of bonds issued in currency j and held by people of country i, $(i = \{1, 2\}, j = \{1, 2\})$. Even when productivity shocks are stationary, formulating incomplete markets this way causes net foreign bond positions to be non-stationary. To induce stationarity in international bond positions, we follow Schmidtt-Grohe and Uribe (2003) and impose a small fee on the foreign bond position. We let τ be the fee paid by households for taking a long or short position in the foreign currency bond. The real cost to a country i household for taking a position in the currency j bond $(i \neq j)$ of size $B_{i,j,t}$, is $\frac{\tau}{2A_{i,t-1}} \left(\frac{S_{i,j,t} B_{i,j,t}}{P_{i,t}} \right)^2$. Costs for people to take positions, $B_{i,i,t}$, in their domestic currency bonds are zero.[7]

Shares of the firms continue to be domestically owned and not internationally traded. The household draws flow resources from labor income, firm profits, and net payoffs from positions in the domestic and foreign currency bonds. Those resources are spent on consumption, new net positions in domestic and foreign currency bonds, and the transaction fee on net foreign currency bond positions. Upon making the stationary transformation, the country $i = \{1, 2\}$ $(i \neq j)$ household budget constraint is,

$$\tilde{c}_{i,t} + \frac{\tilde{B}_{i,i,t}}{P_{i,t}} + \frac{Q_{i,jt}\tilde{B}_{i,j,t}}{P_{j,t},} + \frac{\tau}{2}\left(\frac{Q_{i,jt}\tilde{B}_{i,j,t}}{P_{j,t}}\right)^2 = \frac{\tilde{W}_{i,t}\ell_{i,t}}{P_{j,t}} + \frac{\tilde{\Pi}_{i,t}}{P_{i,t}} + \frac{(1+i_{i,t-1})\,\tilde{B}_{i,i,t-1}}{P_{i,t}G_{i,t-1}} + \frac{(1+i_{j,t-1})\,Q_{i,jt}\tilde{B}_{i,j,t-1}}{P_{j,t}G_{i,t-1}} \tag{12}$$

As long as $\tau > 0$, households will want $\tilde{B}_{i,j} = 0$ in the steady state.

The Euler equations associated with optimal bond holdings for country i are,

$$\frac{1}{1+i_{i,t}} = \frac{\beta}{G_{i,t}} E_t\left(M_{i,t,t+1} e^{-\pi_{i,t+1}}\right), \tag{13}$$

$$\left(\frac{1 + \tau\left(Q_{i,jt}\tilde{B}_{i,j,t}/P_{j,t}\right)}{1+i_{j,t}}\right) = \frac{\beta}{G_{i,t}} E_t\left(M_{i,t,t+1}\frac{Q_{i,j,t+1}}{Q_{i,j,t}} e^{-\pi_{j,t+1}}\right). \tag{14}$$

Looking at equation (14), the effect of the transactions fee is to raise the net price paid for the foreign currency bond when the household has a long position, $\tilde{B}_{i,j} > 0$, and to lower the return. The price is increasing in the long position. Conversely, if the household has a short position,

[7] Borrowing is a short position in which $B < 0$, and lending is a long position in which $B > 0$. The transaction fee is treated as a deadweight loss.

$\tilde{B}_{i,j} < 0$, the effect of the transaction fee is to lower the net issue price and to increase the cost of the foreign currency loan.

The optimality conditions for the labor-leisure choice are unaffected by the change to incomplete markets and continue to be described by equations (6) and (9).

1.2 Goods Demand

In each country, a continuum of firms, indexed by $f \in [0,1]$ each produce a differentiated product. $c_{i,j,t}$ is the consumption good produced in country j and consumed in country i. The aggregate consumption by country i households of goods produced in country j is

$$\tilde{c}_{i,j,t} = \left[\int_0^1 \tilde{c}_{i,j,t}(f)^{\frac{\sigma-1}{\sigma}} df \right]^{\frac{\sigma}{\sigma-1}}, \tag{15}$$

where σ is the elasticity of substitution between varieties f. When $i = j$, this is 'home' goods consumption of 'domestically' produced goods, and for $i \neq j$, $c_{i,j,t}$ are imports. The price index associated with the bundle $c_{i,j,t}$ is

$$P_{i,j,t} = \left[\int_0^1 p_{i,j,t}(f)^{1-\sigma} df \right]^{\frac{1}{1-\sigma}}. \tag{16}$$

Aggregate consumption in country i is the constant elasticity of substitution (CES) index,

$$\tilde{c}_{i,t} = \left(d^{\frac{1}{\mu}} \tilde{c}_{i,i,t}^{\frac{\mu-1}{\mu}} + (1-d)^{\frac{1}{\mu}} \tilde{c}_{i,j,t}^{\frac{\mu-1}{\mu}} \right)^{\frac{\mu}{\mu-1}}. \tag{17}$$

The elasticity of substitution between 'home' and 'foreign' goods is μ, and home-bias in consumption is represented by $d > 1/2$. The aggregate price level associated with equation (17) is

$$P_{i,t} = \left[dP_{i,i,t}^{1-\mu} + (1-d) P_{i,j,t}^{1-\mu} \right]^{\frac{1}{1-\mu}}. \tag{18}$$

1.3 Intermediate Goods Production

Firm $f \in [0,1]$ is able to distinguish between domestic and foreign shoppers and can charge them different prices. Country i firms set prices of their exports in country j's currency. The (normalized) production function for a firm is,

$$\tilde{y}_{i,t}(f) = G_{i,t} \ell_{i,t}(f). \tag{19}$$

The firm's total costs are

$$\frac{\tilde{W}_{i,t}}{P_{i,t}} \ell_{it}(f).$$

Output is demand determined, $\tilde{y}_{i,t}(f) = \tilde{c}_{i,i,t}(f) + \tilde{c}_{i,j,t}(f)$, where home and foreign demands are, respectively,

$$\tilde{c}_{i,i,t}(f) = d\left(\frac{p_{i,i,t}(f)}{P_{i,i,t}}\right)^{-\sigma}\left(\frac{P_{i,i,t}}{P_{i,t}}\right)^{-\mu}\tilde{c}_{i,t}, \tag{20}$$

$$\tilde{c}_{j,i,t}(f) = (1-d)\left(\frac{p_{j,i,t}(f)}{P_{j,i,t}}\right)^{-\sigma}\left(\frac{P_{j,i,t}}{P_{j,t}}\right)^{-\mu}\tilde{c}_{j,t}\left(\frac{A_{j,t-1}}{A_{i,t-1}}\right). \tag{21}$$

It follows that labor employed by firm f is

$$\ell_{i,t}(f) = \frac{\tilde{c}_{i,i,t}(f) + \tilde{c}_{j,i,t}(f)}{G_{i,t}}. \tag{22}$$

Prices are sticky in the sense of Calvo (1983). Each period, the firm is allowed to change its price with probability $1-\alpha_c$. If the firm is chosen to reset prices, it adjusts both the price for domestic market, $p_{i,i,t}(f)$, which is set in country i's currency, and the price of exports, $p_{j,i,t}(f)$, set in units of country j's currency. These prices are set to maximize expected present value of future profits with prices fixed at the optimal choices. Formally, the problem is to maximize

$$E_t\sum_{s=0}^{\infty}(\alpha_c\beta)^s M_{i,t,t+s}\left[\frac{p_{i,i,t}(f)}{P_{i,t+s}}\tilde{c}_{i,i,t+s}(f) + \frac{Q_{i,j,t+s}p_{j,i,t}(f)}{P_{j,t+s}}\tilde{c}_{j,i,t+s}(f) - \frac{\tilde{W}_{i,t+s}}{P_{i,t+s}}\ell_{i,t+s}(f)\right], \tag{23}$$

subject to the output demand equations (20) and (21) and the labor demand equation (22).

1.4 Aggregation, Equilibrium, and Monetary Policy

We obtain aggregate domestic demand for domestically produced goods in country i by equating firm f's supply to demand,

$$G_{i,t}\ell_{i,t}(f) = d\left(\frac{P_{i,i,t}}{P_{i,t}}\right)^{-\mu}\left(\frac{p_{i,i,t}(f)}{P_{i,i.t}}\right)^{-\sigma}\tilde{c}_{i,t} + (1-d)\left(\frac{p_{j,i,t}(f)}{P_{j,i,t}}\right)^{-\sigma}\left(\frac{P_{j,i,t}}{P_{j,t}}\right)^{-\mu}\tilde{c}_{j,t}\left(\frac{A_{j,t-1}}{A_{i,t-1}}\right), \tag{24}$$

then integrating equation (24) to get,

$$G_{i,t}\ell_{i,t} = \tilde{c}_{i,i,t}v^p_{i,i,t} + \tilde{c}_{j,i,t}v^p_{j,i,t}, \tag{25}$$

where $\ell_{i,t} = \int_0^1 \ell_{i,t}(f)\,df$ is total country 1 employment, and

$$\tilde{c}_{i,i,t} = d\left(\frac{P_{i,i,t}}{P_{i,t}}\right)^{-\mu}\tilde{c}_{i,t} = \left(\int_0^1 \tilde{c}_{i,i,t}(f)^{\frac{\sigma-1}{\sigma}}df\right)^{\frac{\sigma}{\sigma-1}}, \tag{26}$$

$$\tilde{c}_{j,i,t} = (1-d)\left(\frac{P_{j,i,t}}{P_{j,t}}\right)^{-\mu}\tilde{c}_{j,t}\left(\frac{A_{j,t-1}}{A_{i,t-1}}\right) = \left(\int_0^1 \tilde{c}_{j,i,t}(f)^{\frac{\sigma-1}{\sigma}}df\right)^{\frac{\sigma}{\sigma-1}}, \tag{27}$$

are aggregate domestic demand and export demand.

In equation (25), $v^p_{i,i,t} \equiv \int_0^1 \left(\frac{p_{i,i,t}(f)}{P_{i,i,t}}\right)^{-\sigma} df$ is a measure of price dispersion for goods in the domestic market, and $v^p_{j,i,t} \equiv \int_0^1 \left(\frac{p_{j,i,t}(f)}{P_{j,i,t}}\right)^{-\sigma} df$ is import price dispersion in foreign country j. The recursive representation for the price dispersion terms, $v^p_{i,j,t}$ ($i = \{1,2\}$, $j = \{1,2\}$), is obtained by noting that a fraction α_c of these firms are stuck with last period's price, $p_{i,j,t-1}(f)$. Since there are a large number of firms charging what they charged last period, it will also be the case that $\int_0^{\alpha_c} p_{i,j,t-1}(f)^{-\sigma} df = \alpha_c P^{-\sigma}_{i,j,t-1}$.[8] The complementary measure of firms $(1-\alpha_c)$ are able to reset price for exports and the domestic market. They all reset to the same price, $p^*_{i,j,t}$. The result is the recursive representation,

$$v^p_{i,j,t} = (1-\alpha_c)\left(\frac{p^*_{i,j,t}}{P_{i,j,t}}\right)^{-\sigma} + \alpha_c \left(\frac{P_{i,j,t-1}}{P_{i,j,t}}\right)^{-\sigma} v^p_{i,j,t-1}. \tag{28}$$

Finally, to close the model, we specify the interest rate rule followed by the monetary authorities. The natural level of output is an infinite-dimensional moving average of past output,

$$\ln(\bar{y}_{j,t}) = \rho_y \ln(\bar{y}_{j,t-1}) + (1-\rho_y)\ln(\tilde{y}_{j,t}).$$

We take the deviation $(\ln(\tilde{y}_{j,t}) - \ln(\bar{y}_{j,t}))$ to measure the output gap. Let $\bar{\pi}_j$ be country j's inflation target. The monetary authorities in country j set the short-term interest rate according to a Taylor (1993) type feedback rule with interest rate smoothing,

$$i_{j,t} = (1-\delta_j)\bar{\imath} + \delta_j i_{j,t-1} + (1-\delta_j)\left(\xi_j(\pi_{j,t} - \bar{\pi}_j) + \zeta_j(\ln(\tilde{y}_{j,t}) - \ln(\bar{y}_{j,t}))\right), \tag{29}$$

where $\bar{\imath} = -\ln(\beta)$ is the steady state real interest rate.

[8] We have, as definition of the price index, $P_{i,j,t} = \left[\int_0^1 p_{i,j,t}(f)^{1-\sigma} df\right]^{\frac{1}{1-\sigma}}$, which can be represented as

$$P^{1-\sigma}_{i,j,t} = (1-\alpha_c)\, p^*_{i,j,t} + \alpha_c P^{1-\sigma}_{i,j,t-1}.$$

Now the price dispersion term is defined to be

$$\begin{aligned}
v^p_{i,j,t} &= \int_0^1 \left(\frac{p_{i,j,t}(f)}{P_{i,j,t}}\right)^{-\sigma} df \\
&= \int_0^{1-\alpha_c} \left(\frac{p^*_{i,j,t}}{P_{i,j,t}}\right)^{-\sigma} df + \int_{1-\alpha_c}^1 \left(\frac{p_{i,j,t-1}(f)}{P_{i,j,t}}\right)^{-\sigma} df \\
&= (1-\alpha_c)\left(\frac{p^*_{i,j,t}}{P_{i,j,t}}\right)^{-\sigma} + \int_{1-\alpha_c}^1 \left(\frac{p_{i,j,t-1}(f)}{P_{i,j,t-1}}\right)^{-\sigma} \left(\frac{P_{i,j,t-1}}{P_{i,j,t}}\right)^{-\sigma} df \\
&= (1-\alpha_c)\left(\frac{p^*_{i,j,t}}{P_{i,j,t}}\right)^{-\sigma} + \alpha_c \left(\frac{P_{i,j,t-1}}{P_{i,j,t}}\right)^{-\sigma} v^p_{i,j,t-1}
\end{aligned}$$

2 The Productivity Process and Parameter Values

The model does not have price indexation or non-zero trend inflation. As such, the model is more suitable to similar pairs of low-inflation developed economies. The exogenous shocks driving the model are from productivity. As motivation for our productivity processes, we construct quarterly total factor productivity observations for Australia, Japan, and the US, from GDP, investment, and employment. Observations for Australia and the US extend from 1973Q1 through 2014Q4. Observations for Japan begin in 1979Q4. GDP and investment are from *Datastream.* Employment is also from *Datastream* for Japan and the US. Australian employment from 1973Q1-1978Q1 is from FRED, and from 1978Q2-2014Q4 it is from *Datastream.* Capital is imputed by the perpetual inventory method. A 4-quarter backward looking moving average is used to seasonally adjust the observations. Investment and GDP for Australia and Japan were converted to real 2013 US dollars to facilitate comparison across countries. One reason for looking at Japan and Australia is that they form a typical country pair in the carry trade with Japan serving as the funding source and Australia as the destination.[9]

Figure 1 plots log TFP for the US, Japan, and Australia. To facilitate comparison, the Australian and US series are normalized to be 1 in 1973, and the Japanese series normalized to be 1 in 1979 when its series begins. Japan's TFP grew rapidly in the 1980s but slowed down significantly in the 1990s, following the collapse of the Japanese stock and housing markets. A less pronounced slowdown for the US, and a more pronounced slowdown for Australia occurs in the early 2000s. Notably, log TFP ($a_t = \ln(A_t)$) for all three countries appear to be stochastically trending within our observational time-frame, and show no evidence of converging toward each other. To capture these features in productivity in our two-country model, we assume that each country's log TFP has a unit root, and that the tendency for them to converge is weak. We cannot, however, let the TFP series diverge from each other. A specification that achieves this is,[10]

$$\Delta a_{1,t} = -\psi_1 \left(a_{1,t-1} - a_{2,t-1}\right) + \sigma_1 \epsilon_{1,t} \tag{30}$$

$$\Delta a_{2,t} = -\psi_2 \left(a_{1,t-1} - a_{2,t-1}\right) + \sigma_2 \epsilon_{2,t} \tag{31}$$

where $\epsilon_{i,t} \overset{i.i.d}{\sim} N(0,1)$ and $\sigma_i > 0$, for $i = \{1,2\}$, and $0 < \psi_2 < \psi_1 < 1$. Setting ψ_1 to be a small

[9]Ready et al. (2016) emphasize Australia and Japan, whereas Backus et al. (2013) focus on Australia and the US.

[10]Let $z_t = a_{1,t} - a_{2,t}$ be the error-correction term. Subtracting equation (31) from equation (30) gives

$$z_t = \left(1 + \psi_2 - \psi_1\right) z_{t-1} + \left(\sigma_1 \epsilon_{1,t} - \sigma_2 \epsilon_{2,t}\right)$$

The autoregressive coefficient, $1 + \psi_2 - \psi_1$, is close to, but less than 1.

positive number, and setting ψ_2 slightly below ψ_1 gives persistence to deviations between a_1 and a_2, while maintaining the technical requirements of cointegration.[11]

Figure 2 plots a realization of the simulated process against the data. We set $\psi_1 = 0.01$, $\psi_2 = 0.009$, and the innovation standard deviations $\sigma_1 = \sigma_2 = 0.01$, which approximately matches the volatility of TFP growth (0.0073 for Australia and 0.0110 for Japan) in the data. Figure 3 shows a long realization of the simulated TFP processes. The log levels of the joint process are systematically different from each other. They do not cross each other frequently (or at all in this realization) and are not cointegrated, but they do not diverge either.

The Calvo (1983) probability is set at $\alpha_c = 0.7$, which implies an average contract duration of 3 quarters. Home bias is assumed to be $d = 2/3$, and $\sigma = 10$ which implies a a mark up of 11 percent. The elasticity of substitution between domestic goods and imports is $\mu = 1.5$.

In parameterizing the utility function, we follow Swanson (2016) in setting $\chi = 3$, which implies a Frisch elasticity of labor supply of 1/3 and set η to generate a steady state labor supply of $\bar{\ell}_1 = \bar{\ell}_2 = 1$. We consider a range of relative risk aversion values of 10, 20, 30, and 60. High degrees of risk aversion are typically needed to explain asset returns data.

The log consumption part of the utility function implies that the intertemporal elasticity of substitution is 1. This is lower than the values, ranging between 1.5 and 2, typically assumed in asset pricing research (Bansal and Yaron (2004), Colacito and Croce (2011), Bansal and Shaliastovich (2012)). Empirical estimates of recursive utility functions that parameterize the intertemporal elasticity of substitution and employ both consumption and asset price data (e.g., Chen et al. (2007)), estimate the elasticity to lie between 1.11 and 2.22.

In our benchmark monetary rule, the parameters are symmetric across countries. The coefficient on the lagged interest rate is $\delta_j = 0.9$, and the inflation and output gap response coefficients, $\xi_j = 1.5$ and $\zeta_j = 0.5$, conform exactly to the Taylor (1983) rule.

A third-order approximation of the model to its non-stochastic steady state is numerically solved with pruning, using Dynare 4.3.3. The third-order approximation is necessary in order to generate time-variation in risk premia, and pruning is required for non-explosive simulations. Kim et al. (2005) discuss how recursively built observations in second-ordered approximations introduce higher-ordered terms in the expansion that do not correspond to higher-order coefficients in the Taylor expansion. These higher-ordered terms generate explosive time paths in simulations, and a stable solution is obtained by pruning the extraneous higher order terms. These explosive elements are also present in third-order approximated simulations.

[11] Kollmann (2016) works with a similar process in a two-country endowment economy model.

3 International Currency Returns

This section discusses the three empirical regularities on international currency returns in some detail, and reports the model's contribution towards understanding them. Subsection 3.1 discusses the long-standing issue of the downward forward premium bias. Subsection 3.2 analyzes the carry trade return and Subsection 3.3 takes up Engel's risk-reversal.

3.1 The Downward Forward Premium Bias

UIP says excess currency returns are zero in expectation. Because the difference between bond yields across countries is expected to be offset by a loss in the value of the high interest rate currency, UIP also says the interest rate differential is an unbiased predictor of the future change in the spot exchange rate. UIP implies $\alpha_0 = 0$ and $\beta_0 = 1$, in the regression of the depreciation rate of currency 1 on the interest rate differential,

$$\Delta \ln \left(S_{1,2,t+1}\right) = \alpha_0 + \beta_0 \left(i_{1,t} - i_{2,t}\right) + \epsilon_{t+1}. \tag{32}$$

Equation (32) is referred to as the Fama regression. Fama (1984) ran these regressions and reported estimates of β_0 that not only differed from 1, but were negative. This empirical pattern has been found to be pervasive and robust over time. The near universal estimates of $\beta_0 < 1$ is what we are calling the downward forward premium bias. Froot and Thaler (1990) distinguish between the forward premium puzzle (or anomaly), when $\beta_0 < 0$, and the forward premium bias, when $0 < \beta_0 < 1$. The 'forward premium' terminology stems from the epoch before the global financial crisis (2008), when the covered interest parity arbitrage condition held. At that time, there was an equivalence between the interest rate differential and the forward premium.[12]

If the forward premium puzzle is present, an excess return from going short the low interest rate currency and going long the high interest rate might be expected to be enhanced by an increase in the value of the long currency. If only a negative bias is present $0 < \beta_0 < 1$, the excess yield differential might be expected to be less than fully offset by a loss in the long currency value. The dominant hypothesis for the downward forward premium bias is that excess currency returns are available to investors as compensation for differential currency risk.

Since the downward forward premium bias is well-known, has been shown to be remarkably robust over time, and has been extensively documented, it is perhaps not necessary to report here.[13] Nevertheless, Table 1 shows estimates of equation (32) using low-inflation and developed

[12] The forward premium is the percentage difference between the forward price of the foreign currency and the current spot price. Here, we use inter-bank interest rates. Interest rates are from *Datastream*. Exchange rates are from *Bloomberg*.

[13] See the surveys by Engel (1996, 2014) and Lewis (1995).

countries with the US, Australia, and Japan alternately serving as the base country. Observations are quarterly and span from 1973Q1 to 2014Q4. The forward premium puzzle is, by no means universal. With the US as the base country, negative estimates of β_0 are obtained for 6 cases and significantly negative only once. When Japan is the base country, the puzzle is present in only 2 of 9 cases and never significantly negative. In 2 cases (Norway and Sweden), the slope exceeds 1. Similarly, with Australia as the base country, the puzzle is present in 5 cases but never significantly negative. However, the downward bias is pervasive. For each choice of base country, slope estimates are significantly less than 1 in 5 cases with the US as base country, 6 cases for Australia as base country, and 5 cases with Japan as base country.

Two other features of the table are worthy of note. The first is that R^2 are nearly zero. The second is that many of the regressions show evidence consistent with the exchange rate following a random walk ($\alpha_0 = \beta_0 = 0$). For example, with Japan as the base country, neither the constant (α_0) nor slope (β_0) are ever individually significantly different from zero.

It is also worth mentioning that the forward premium puzzle does not say that a positive country 1-2 interest rate differential predicts an appreciation of currency 1. It says that the higher is $i_{1,t} - i_{2,t}$, the smaller is the depreciation in currency 1. The exchange rate can still be increasing, but does so at a decreasing rate as the interest differential rises. It does not necessarily imply that the exchange rate is expected to decline. To infer the predicted direction of change, one needs also to properly account for the size and sign of the constant.

Table 2 shows the implied slope coefficient (β_0) in the Fama regression generated by the model. In Panel A, both countries follow the Taylor rule where the response coefficient on inflation is 1.5 and 0.5 on the output gap. Under complete markets, increasing risk aversion has no effect. Whether risk aversion is 10 or 60, the complete markets model does not generate the downward forward premium bias. In the incomplete markets model, on the other hand, a forward premium bias emerges as risk aversion is increased, but it does not generate the forward premium puzzle. The implied slope on the Fama regression is 0.66 when risk aversion is 60.[14]

In the model results shown in Panel A, the only differences between countries is in productivity. An important insight from international currency return research (e.g. Lustig and Verdelhan (2007) and Lustig et al. (2011)), is that *differences* across countries are essential for understanding risk premiums. It appears that differences between country productivity processes alone are not sufficient to generate a sizable downward forward premium bias in the complete markets model.

[14]In the incomplete markets model, symmetric bond holding costs are assumed, $\tau_{1,2} = \tau_{2,1} = \tau$. The results are robust to alternative settings of the international bond holding cost of $\tau = 0.005$, 0.01, 0.03, and 0.05. In the reported calculations, we set $\tau = 0.01$.

Backus et al. (2013) study the effect of heterogeneous monetary policies in generating risk and the forward premium bias in a complete markets endowment model with recursive preferences and with endogenous inflation. In their endowment economy framework, the nominal interest rate is generated by a feedback rule from inflation and exogenous consumption growth. The nominal interest rate also has to equal (minus) the log of the expected nominal SDF, which follows from the household's Euler equation. They obtain a closed form solution for the model from imposing the restrictions implied by these two equations. Even though the model's agents do not care about inflation per se, in the sense that it has no effect on welfare, inflation is generated endogenously in their framework, and the correlation between inflation and consumption growth and hence, the characteristics of currency returns, can be altered by varying the coefficients in the interest rate rule. They find that a combination of relative inflation accommodation (small ξ_i) and relative procyclicality (large ζ_i) generates currency risk which produces the downward forward premium bias.[15]

In Panel B, we examine the effect of introducing cross-country differences in monetary policies. Risk aversion is set at 60. In the complete markets model, just by making country 1 more accommodating to inflation, $\xi_1 = 1.2$, lowers the implied Fama slope to 0.384. Making country 1 also relatively more procyclical lowers the implied Fama slope to 0.313. In the incomplete markets model, the downward bias (implied slope of 0.530) is most pronounced when country 2 adopts the risky monetary policy of accommodation and procyclicality.

Creating differences across countries through different monetary policies generates a downward forward premium bias in both the complete markets and incomplete markets model. The complete markets model generates more of a bias than the incomplete markets model.

3.2 The Carry Trade Return

The carry trade is a rule that says to go short the low interest currency (e.g. the yen) and to go long the high interest currency (e.g. the Australian dollar). The carry trade places primary emphasis on interest rates. Exchange rate considerations are secondary. It can be seen how the carry would generate positive returns on average if the exchange rate followed a random walk, since on average, the carry trader is earning the interest rate differential. If a forward premium puzzle is present, the subsequent exchange rate movements might be expected to further enhance the return from the interest differential, adding to the positive expected excess return. If a downward forward premium bias is present, the carry trade might still yield positive profits because the yield

[15] In Bansal and Shaliastovich (2012), exogenous inflation and consumption growth are governed by a joint process which they estimate from data. But because inflation is exogenous, it has no effect on welfare in their model.

differential is expected to be only partially offset by subsequent exchange rate movements.

An expanding literature has advanced our understanding of the carry trade. Many recent empirical studies focus on portfolios of the carry trade and investigate the cross-sectional variation of carry trade returns in relation to their exposure to risk factors (see Burnside et al. (2011), Lustig and Verdelhan (2007), Lustig et al. (2011), Menkhoff et al. (2013), Della Corte et al. (2013), and Berg and Mark (2017b)). Formation of portfolios enhances identification of systematic components of the returns by diversifying away idiosyncratic risks

An example of the kinds of excess returns found among developed countries with similar (and relatively low) inflationary experiences is given in Table 3, which we take from Berg and Mark (2017a). Each period, they sort countries by interest rate from low to high and compute excess currency returns using the USD as the funding currency. The excess returns are divided into 6 portfolios and the average of the equally weighted portfolio returns are shown in the table. Also shown, are the mean interest differentials between the portfolios and the US, and the currency appreciation on the portfolio of foreign currencies.[16]

Countries with the lowest interest rates P_1, pay a carry return of -1.19 percent per annum. Their interest rates lie 2.9 percent below the US interest rate. If UIP held, the -2.9 percent interest differential would be offset by an +2.9 percent loss on the destination currencies, but instead, they gain on average 1.7 percent. The high interest rate portfolio P_5, pays an average carry currency excess return of 3.2 percent. The 2.6 percent interest rate differential is enhanced by an additional 67 basis point appreciation of the foreign currency. There is a forward premium puzzle present in the P_5 portfolio of currencies. In the portfolio of the highest interest currencies (P_6), the 6.7 percent interest rate differential is partially offset by a 2.9 percent loss on the exchange rate. The highest interest rate currencies depreciate on average, against the USD.

The carry trade return and the downward forward premium bias are related, but distinct phenomena. The point was made empirically by Hassan and Mano (2014). We illustrate the distinction using the complete markets specification outlined in Subsection 1.1.1. Here, we temporarily ignore the unit root in productivity (i.e., set $G_1 = G_2 = 1$) and β. Let $m_{t+1} = \ln(M_{t+1})$ be the log real SDF. Similarly, let the log nominal SDF be $n_{t+1} = \ln(N_{t+1}) = m_{t+1} - \pi_{t+1}$. Representing currency 1's depreciation with equation (10) and interest rates with equation (7), the Fama regression can be expressed in terms of the SDFs as

$$n_{2,t+1} - n_{1,t+1} = \alpha_0 + \beta_0 \left(E_t \left(n_{2,t+1} - n_{1,t+1}\right)\right) + \epsilon_{t+1}.$$

The forward premium puzzle is why the correlation between the relative log nominal SDFs are negatively correlated with expected relative log nominal SDFs. $E_t n_{t+1}$ is the conditional entropy

[16] Positive mean exchange rate returns mean the portfolios of currencies is rising in value relative to the US dollar.

of N_{t+1}. If the nominal SDFs are log-normally distributed, the conditional entropy of N_{t+1} is $E_t(n_{t+1}) + \frac{1}{2}Var_t(n_{t+1})$.[17] Exploiting these results, the country 1 currency risk premium can be represented as

$$(i_{1,t} - i_{2,t} - E_t \Delta \ln(S_{1,2,t+1})) = \frac{1}{2}(Var_t(n_{2,t+1}) - Var_t(n_{1,t+1})).$$

The carry trade return then is, an issue about differences in SDF volatility across countries. If country 1 systematically pays a positive carry trade return, it has the smoother, lower variance nominal SDF. Country 1 pays the excess return presumably because it is the risky country. What is an explanation that reconciles country 1 being risky and also having the less volatile SDF? One explanation is the riskiness of country 1 induces its residents to save heavily through the precautionary motive. Over time, they have accumulated a large buffer stock of saving which they use to insulate the SDF from shocks. Countries with high interest rates also exhibit high saving rates, so the empirical patterns seem to fit the story.

Table 4 shows the model results for the carry trade. The gross carry currency return at $t+1$ is the excess return from going long currency 1 and going short currency 2 if $i_{1,t} > i_{2,t}$. If, at t, country 2 has the higher interest rate, the excess return is calculated as shorting currency 1 and going long currency 2. Panel A shows the mean gross carry currency excess returns generated under symmetric monetary policy, for different values of risk aversion. The only differences here between countries, is in the process driving productivity. This asymmetry is not enough for the complete markets model to generate much of a carry trade return. With risk aversion at 60, the average excess return is a measly 4.5 basis points. The gross carry return is substantially higher under incomplete markets. That model generates a carry excess return of nearly 1 percent with a risk aversion of 30 and 2 percent with a risk aversion of 60.

Under incomplete markets, people are subject to transactions costs involving international borrowing and lending. The gross carry calculations ignore these costs. The net carry figures are carry trade excess returns for country 1 and country 2 individuals after accounting for the international bond positional fees. Net carry returns are obtained as follows: A country 1 individual who goes long currency 1 by shorting currency 2 realizes the net excess return

$$(1 + i_{1,t}) - \left(\frac{1 + i_{2,t}}{1 + \tau \tilde{B}_{1,2,t} Q_{1,2,t}}\right)\left(\frac{S_{1,2,t+1}}{S_{1,2,t}}\right),$$

whereas a country 2 individual realizes the net excess return

$$\left(\frac{1 + i_{1,t}}{1 + \tau \tilde{B}_{2,1,t} Q_{2,1,t}}\right)\left(\frac{S_{2,1,t+1}}{S_{2,1,t}}\right) - (1 + i_{2,t}).$$

[17]See Backus at al. (2001).

The net carry excess return is gently increasing in risk aversion and is slightly larger for country 2 (see Net Carry 2). Risk aversion of 60 generates a net carry of 51 basis points for country 1 and 75 basis points for country 2, which is positive but down from the gross carry rate of 2.1 percent.

How might monetary policy heterogeneity across countries create systematic risks that are compensated? To economize on notation, let $\sigma_t^2(x_{t+1}) = Var_t(x_{t+1})$ denote the conditional variance of x_{t+1} and $\sigma_t(x_{t+1}, u_{t+1})$ be the conditional covariance between x_{t+1} and u_{t+1}. Under complete markets, the nominal carry trade return is

$$i_{1,t} - i_{2,t} - E_t \Delta \ln(S_{1,2,t+1}) = \frac{1}{2} \left\{ \begin{array}{c} \left[\sigma_t^2(m_{2,t+1}) + \sigma_t^2(\pi_{2,t+1})\right] \\ -\left[\sigma_t^2(m_{1,t+1}) + \sigma_t^2(\pi_{1,t+1})\right] \\ +2\left[\sigma_t(m_{1,t+1}, \pi_{1,t+1}) - \sigma_t(m_{2,t+1}, \pi_{2,t+1})\right] \end{array} \right\}$$

Monetary policy can increase the carry return paid by country 1 by lowering the conditional variance of the log real SDF, the conditional variance of inflation, and by increasing the conditional covariance between the log real SDF and inflation. In the case of log utility, this would mean generating a negative covariance between consumption growth and inflation.

In Panel B, we again show the results from introducing monetary policy heterogeneity. A positive gross carry can now be generated under complete markets. The carry trade return ranges between 0.79 percent and 1.8 percent, depending on the policy specification. The riskiest configuration that we considered is when country 2 is both relatively accommodating to inflation and relatively procyclical ($\xi_1 = 1.5, \zeta_1 = 0, \xi_2 = 1.2, \zeta_2 = 0.5$), which gives a gross carry return of 1.6 percent. Similarly, under incomplete markets, a positive gross carry is generated under heterogeneous monetary policies, with the carry premium larger when country 2 accommodates inflation. Net carry returns are also relatively large when country 1 is accommodative to inflation, but is not more procyclical ($\xi_1 = 1.2, \zeta_1 = 0.5, \xi_2 = 1.5, \zeta_2 = 0.5$).

Under incomplete markets, a higher carry return (gross and net) is generated when country 2 is relatively accommodating to inflation. Whether country 1 or 2 is more or less pro-cyclical has only a minor effect. An accommodating country 2 can generate a gross carry of 5.6 percent, a net carry of 4.2 percent for country 2 people and 3.7 percent for country 1 people.

3.3 Risk-Reversals

The conventional wisdom is that the high real-interest rate country should have a strong currency. IMF advice that countries defend against currency depreciation during foreign exchange crises by raising interest rates are founded on this view. This view is an implication of classic exchange rate models, such as Dornbusch (1976) and Frankel (1979). Engel (2016) observes that the positive relation between the real interest rate and currency strength seems to contradict the empirical evidence on the forward premium bias.

The argument, according to Engel (2016), is this. Say the real interest rate in country 2 is high. Country 2 should have the strong currency, and a strong currency means country 2 is safe. Its risk is low and the risk premium paid out over time should be negative. But if the downward forward premium bias is present (or if there is a carry trade return) in the short run, country 2 pays a positive risk premium to those who go long currency 2 and short currency 1. The question is how can country 2 be the risky country in the short run and the safe country in the long run? Engel's answer is that there must be a risk reversal for country 2 over time.

Engel (2016) characterizes the issue in terms of real interest rate differentials and the real exchange rate. Let the ex post real excess return on a long position in currency 2 be

$$\rho_{t+1} = r_{2,t} - r_{1,t} + \Delta \ln \left(Q_{1,2,t+1}\right)$$

He characterizes the downward forward premium bias as a positive correlation between the ex ante excess return and the real interest rate differential,

$$E_S \equiv Corr\left(\left(E_t \rho_{t+1}\right), \left(r_{2,t} - r_{1,t}\right)\right) > 0.$$

When country 2 has the higher real interest rate, it is expected to pay a positive currency excess return in the short run.

The long-run risk premium paid out by country 2 is $\sum_{j=0}^{\infty} \rho_{t+1+j} = \sum_{j=0}^{\infty} \left(r_{2,t+j} - r_{1,t+j}\right) + \ln\left(Q_{1,2,\infty}\right) - \ln\left(Q_{1,2,t}\right)$, where $Q_{1,2,\infty}$ denotes the long-run real exchange rate.[18] If country 2, being the high real interest rate country, is safe in the long-run, the long-run risk premium should be negatively correlated with the current real interest rate differential,

$$E_L \equiv Corr\left(E_t\left(\sum_{j=0}^{\infty} \rho_{t+1+j}\right), \left(r_{2,t} - r_{1,t}\right)\right) < 0.$$

Engel (2016) estimates a three-variable vector error correction model and uses the estimated model to compute $E_t \rho_{t+1}$, $E_t\left(\sum_{j=0}^{\infty} \rho_{t+1+j}\right)$, and $Q_{1,2,\infty}$ for the G7 countries with the USD serving as the base currency. In every instance, he finds $E_L < 0$.[19] He then studies log-linearized versions of long-run risk models under complete markets, considering differences across countries arising from differences in country-specific shocks and by considering heterogeneous dependence on the same global shocks. He concludes that these models are consistent with the forward premium

[18] Engel (2016) characterizes the puzzle in terms of covariances. Here, we work with correlations.

[19] He also finds a downward forward premium bias for Canada and Italy, and a forward premium puzzle for France, Germany, Japan, and the UK.

bias, $E_S > 0$, but they cannot generate the risk reversal, $E_L < 0$. His suggestion is to introduce non-pecuniary liquidity returns on assets as an avenue to explain the risk-reversal. Valchev (2015) pursued just such a suggestion.

Table 5 reports the E_S and E_L correlations implied by our model. Monetary policies are symmetric and set at the benchmark Taylor rule values in Panel A. Risk aversion varies from 10 to 60. Under complete markets, E_S has the wrong sign. This is consistent with the inability of the complete markets model to generate a sizable downward forward premium bias under symmetric monetary policies. Except when risk aversion is 30, the long-run correlation, E_L, is also the wrong sign under complete markets. The incomplete markets model gets the sign of both the short-run and long-run correlations correct.

In Panel B, we again create country heterogeneity in monetary policies under risk aversion of 60. Under complete markets, monetary policy heterogeneity that we considered, generally is insufficient to get E_S to be positive and E_L to be negative. The one exception is when country 2 is relatively accommodating to inflation and relatively procyclical. Under incomplete markets, the short-run Engel correlation is systematically positive and the long-run correlation is systematically negative.

4 Symmetrically Cointegrated TFP and Stationary TFP

The previous section shows that the general equilibrium model is able to qualitatively produce the three international currency return facts if there are incomplete markets, there are differences across countries in the process driving TFP, and differences in monetary policies. Country heterogeneity of some sort appears to be crucial. In this section, we ask how important is it for the model to display TFP heterogeneity, and how important is it for TFP to be non-stationary.

In Table 6, risk aversion is 60 and the error-correction coefficients in equations (30) and (31) are $\psi_1 = 0.02$ and $\psi_2 = -0.02$. This gives a symmetric log TFP processes with AR(1) coefficient on the error correction term of 0.96. Symmetric monetary policy is assumed in column 2. Country 2 accommodates inflation in column 3. Country 2 both accommodates inflation and is relatively more procyclical in column 4. Here, we see that the complete markets model does not explain any of the three currency return facts. The incomplete markets model generates a modest downward forward premium bias and the risk reversal, but does not generate a sizable carry trade return.

In Table 7, log TFP is the symmetric stationary AR(1) process, $a_t = \rho_A a_{t-1} + \epsilon_t$. Risk aversion is 60 and monetary policy is such that country 2 accommodates inflation and is more procyclical in setting its interest rate. In the complete markets model, the carry trade return increases as we increase ρ_A and TFP approaches non-stationarity but does not generate a forward premium bias and generates the wrong sign on the short-run Engel correlation. Very similar results are generated by the incomplete markets model. Stationarity in log TFP is inconsistent with the data on TFP and the model does not explain the currency return facts when TFP is assumed to be stationary.[20]

5 Impulse Response Functions

To illustrate some of the workings of the model, this section plots impulse responses to exogenous shocks to TFP growth in the incomplete markets model. Risk aversion is 60 and monetary policies are symmetric with coefficients set according to the Taylor rule. Recall that the log TFP are unit-root processes so the shocks result in a permanent change in productivity.

Figure 4 plots log TFP, log output, and labor responses. A positive country 1 growth shock initially increases TFP and output in country 1. In country 2, TFP and output gradually adjusts downward. The initial shock raises $a_{1,t}$ but has no initial effect on $a_{2,t}$. At $t+1$, the difference between $a_{1,t+1} - a_{2,t+1} = \sigma_1 > 0$, the standard deviation of the $\Delta a_{1,t}$ process. From $t+1$ onward, Δa_1 and Δa_2 decline at the same rate. A positive shock to $a_{2,t}$ causes $a_1 - a_2 < 0$, which causes a_1 and a_2 to increase over time. The impulse response patterns for output are similar to TFP. A country 1 shock causes labor to initially decline in country 1 and to increase in country 2. The pattern is reversed following a positive shock to country 2 productivity.

Figure 5 plots consumption components of aggregate consumption responses. A positive growth and LRR shock causes aggregate consumption to jump up on impact, with country 1 showing a bigger response. Consumption declines over time, following the path of TFP. A positive shock to a_2 raises country 2 consumption more than country 1, and both consumption continue a gradual increase afterwards. The country 1 productivity shock raises both components of country 1 consumption. Country 1 consumption of its own domestically produced goods and imports increase on impact. The initial impact on country 2 consumption components is relatively small. A symmetric response follows a shock to country 2 productivity.

Figure 6 shows inflation, interest rate, and exchange rate responses. A country 1 productivity shock causes inflation to fall in both countries, but a country 2 shock causes inflation to increase in both countries. Prices continue to fall after the country 1 shock because productivity declines

[20] We also tried specifying a long-run risks model for TFP growth with stochastic volatility in the growth innovations and in the long-run risk components. Estimation of the parameters of this process with Bayesian methods revealed the serial correlation of the latent long-run risk component to be relatively small–in the neighborhood of 0.2. Using the estimated long-run risk model for TFP growth, the model performance was nearly identical to results reported in Section 3.

and hence demand declines. The reverse is at work following a country 2 productivity shock.

The country 2 productivity shock causes both interest rates to increase, but the country 1 shock initially causes only its interest rate to increase, while country 2's interest rate declines. A positive shock to country 1 productivity generates an initial real and nominal depreciation of country 1's currency.

Figure 7 shows the responses of sub-component price dispersion. A country 1 productivity shock has its largest effect on country 2 import price dispersion $v^p_{2,1}$. A country 2 productivity shock also has its largest effect on country 2 import price dispersion. It also has a relatively large effect on country 1's import price dispersion v^p_{11}.

Finally, a country 1 productivity shock makes country 1 risky in the short run. The impact effect is to generate a positive expected excess return from a long position in country 1's currency, but this expected excess return turns negative shortly afterward. A country 2 productivity shock makes country 2 risky, generating a positive expected excess return from a long position in country 2's currency.

6 Conclusion

This paper has shown how a two-country dynamic stochastic general equilibrium macro model can explain three empirical facts that characterize international currency returns–the downward forward premium bias, the carry trade return, and the long-run risk reversal. Previous research has typically employed endowment models with exogenous consumption and has not jointly addressed all three aspects of returns. Our model lays the foundation for a unified framework for understanding empirical patterns in international currency returns.

Some sort of heterogeneity across countries is an essential element in understanding international currency returns. We allowed heterogeneity in the exogenous process driving productivity and in the execution of monetary policy. Under these kinds of heterogeneity, the model under complete markets can be shown to be consistent with the downward forward premium bias and the carry trade return but, as Engel (2016) conjectured, it does not generate the risk reversal. The incomplete markets model is consistent with all three return patterns.

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Table 1: Fama Regression–Forward Premium Puzzle/Bias in the Data

Country 2	α_0	t-stat	β_0	t-stat	R^2
United States is Country 1					
Australia	-0.005	-1.035	-0.487	-1.160	0.005
Canada	-0.001	-0.318	-0.430	-0.573	0.003
Great Britain	-0.006	-1.014	-0.636	-0.854	0.006
Japan	0.007	1.244	-0.171	-0.411	0.001
Korea	0.003	0.357	0.558	1.173	0.009
Norway	0.003	0.562	0.346	0.363	0.002
New Zealand	-0.013	-1.893	-1.027	-2.607	0.027
Switzerland	0.015	2.414	-1.022	-1.801	0.016
Sweden	0.006	0.816	0.819	0.520	0.011
Australia is Country 1					
Canada	0.004	1.131	-0.507	-1.385	0.005
Great Britain	0.000	-0.012	0.047	0.100	0.000
Japan	0.005	1.074	0.278	1.088	0.003
Korea	-0.006	-0.927	-0.109	-0.203	0.000
Norway	-0.003	-0.622	1.318	1.850	0.021
New Zealand	-0.001	-0.301	-0.176	-0.347	0.001
Switzerland	0.010	1.030	-0.059	-0.071	0.000
Sweden	-0.004	-0.603	1.645	1.535	0.038
United States	0.005	1.035	-0.487	-1.160	0.005
Japan is Country 1					
Australia	-0.005	-1.074	0.278	1.088	0.003
Canada	-0.006	-0.960	0.056	0.138	0.000
Great Britain	-0.008	-1.340	0.040	0.115	0.000
Korea	0.006	0.413	0.697	1.106	0.011
Norway	0.021	1.935	2.081	1.884	0.037
New Zealand	-0.003	-0.255	0.422	0.699	0.004
Switzerland	0.002	0.330	-0.059	-0.176	0.000
Sweden	0.020	1.252	2.533	1.363	0.046
United States	-0.007	-1.244	-0.171	-0.411	0.001

Notes: The regression is $\Delta \ln(S_{1,2,t+1}) = \alpha_0 + \beta_0(i_{1,t} - i_{2,t}) + \epsilon_{t+1}$. Data is quarterly, from 1973Q1 through 2014Q4. T-ratios are constructed with Newey-West (1983) standard errors.

Table 2: Implied Slope in Fama Regression–Forward Premium Puzzle/Bias

A. Symmetric Monetary Policies				
Risk Aversion	10	20	30	60
Complete	0.999	1.004	1.009	1.020
Incomplete	1.059	0.874	0.762	0.655
B. Heterogenous Monetary Policies				
Policy Parameters				
ξ_1	1.2	1.2	1.5	1.5
ξ_2	1.5	1.5	1.2	1.2
ζ_1	0.5	0.5	0.5	0
ζ_2	0.5	0	0.5	0.5
Risk Aversion is 60				
Complete	0.384	0.313	0.750	0.659
Incomplete	0.917	1.018	0.850	0.530

Notes: The regression is $\Delta \ln(S_{1,2,t+1}) = \alpha_0 + \beta_0(i_{1,t} - i_{2,t}) + \epsilon_{t+1}$. Under symmetric monetary policy, inflation response coefficients are $\xi_1 = \xi_2 = 1.5$ and output-gap response coefficients are $\zeta_1 = \zeta_2 = 0.5$.

Table 3: Monthly Currency Excess Return Summary Statistics (1973.04–2014.12): Developed Countries

	P_1	P_2	P_3	P_4	P_5	P_6
Mean Currency Excess Return	-1.188	-0.482	1.311	0.828	3.263	3.849
Mean Interest Rate Differential	-2.904	-1.297	0.024	1.144	2.590	6.736
Mean Exchange Rate Return	1.716	0.816	1.287	-0.316	0.674	-2.886

Notes: This table is taken from Berg and Mark (2017a). Developed countries include Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Israel, Italy, Japan, Netherlands, New Zealand, Norway, Poland, Portugal, Singapore, South Korea, Spain, Sweden, Switzerland, Taiwan, United Kingdom, and United States.

Table 4: Carry Trade Returns

A. Symmetric Monetary Policies				
	Complete Markets			
Risk Aversion	10	20	30	60
Gross Carry	0.001	0.002	0.004	0.045
	Incomplete Markets			
Risk Aversion	10	20	30	60
Gross Carry	0.297	0.638	0.986	2.068
Net Carry 1	0.043	0.119	0.205	0.511
Net Carry 2	0.092	0.212	0.340	0.745
B. Heterogeneous Monetary Policies (Risk Aversion is 60)				
	Policy Parameters			
ξ_1	1.2	1.2	1.5	1.5
ξ_2	1.5	1.5	1.2	1.2
ζ_1	0.5	0.5	0.5	0
ζ_2	0.5	0	0.5	0.5
	Complete Markets			
Gross Carry	1.807	0.794	1.226	1.554
	Incomplete Markets			
Gross Carry	1.226	1.206	5.541	5.622
Net Carry 1	-0.374	-0.379	3.668	3.747
Net Carry 2	-0.992	-1.010	3.871	4.152

Table 5: Risk Reversal

A. Symmetric Monetary Policies				
Complete Markets				
Risk Aversion	10	20	30	60
E_S	-0.866	-0.553	-0.124	-0.375
E_L	0.407	0.004	-0.094	0.009
Incomplete Markets				
Risk Aversion	10	20	30	60
E_S	0.261	0.491	0.620	0.746
E_L	-0.261	-0.467	-0.586	-0.703
B. Heterogeneous Monetary Policies (Risk Aversion is 60)				
Policy Parameters				
ξ_1	1.2	1.2	1.5	1.5
ξ_2	1.5	1.5	1.2	1.2
ζ_1	0.5	0.5	0.5	0
ζ_2	0.5	0	0.5	0.5
Complete Markets				
E_S	-0.050	-0.082	0.001	0.019
E_L	-0.023	0.000	0.006	-0.038
Incomplete Markets				
E_S	0.205	0.297	0.517	0.512
E_L	-0.113	-0.228	-0.444	-0.456

Table 6: Symmetrically Cointegrated TFP with Monetary Policy Heterogeneity (Risk Aversion is 60)

	Policy Parameters		
ξ_1	1.5	1.5	1.5
ξ_2	1.5	1.2	1.2
ζ_1	0.5	0.5	0
ζ_2	0.5	0.5	0.5
	Complete Markets		
Fama	1.050	0.998	1.125
Gross Carry	-0.009	0.051	-0.072
E_S	-0.821	-0.825	-0.852
E_L	0.437	0.341	0.544
	Incomplete Markets		
Fama	0.733	0.827	0.824
GrossCarry	0.086	-0.005	-0.052
NetCarry1	0.019	-0.023	-0.019
NetCarry2	0.008	-0.031	-0.010
E_S	0.713	0.569	0.575
E_L	-0.711	-0.585	-0.585

Note: $\psi_1 = 0.02, \psi_2 = -0.02$.

Table 7: Stationary TFP with Monetary Policy Heterogeneity (Risk Aversion is 60)

ρ_A	0.9	0.96	0.99
	Policy Parameters		
ξ_1	1.5	1.5	1.5
ξ_2	1.2	1.2	1.2
ζ_1	0	0	0
ζ_2	0.5	0.5	0.5
	Complete Markets		
Fama	1.004	1.023	1.009
Carry	0.090	0.362	1.780
E_S	-0.223	-0.422	-0.585
E_L	-0.434	-0.325	-0.091
	Incomplete Markets		
Fama	1.03	1.06	1.05
Gross Carry	0.07	0.48	1.56
Net Carry 1	0.09	0.41	0.94
Net Carry 2	0.10	0.49	1.50
E_S	0.07	-0.07	-0.05
E_L	-0.07	0.03	-0.01

Figure 1: Log TFP

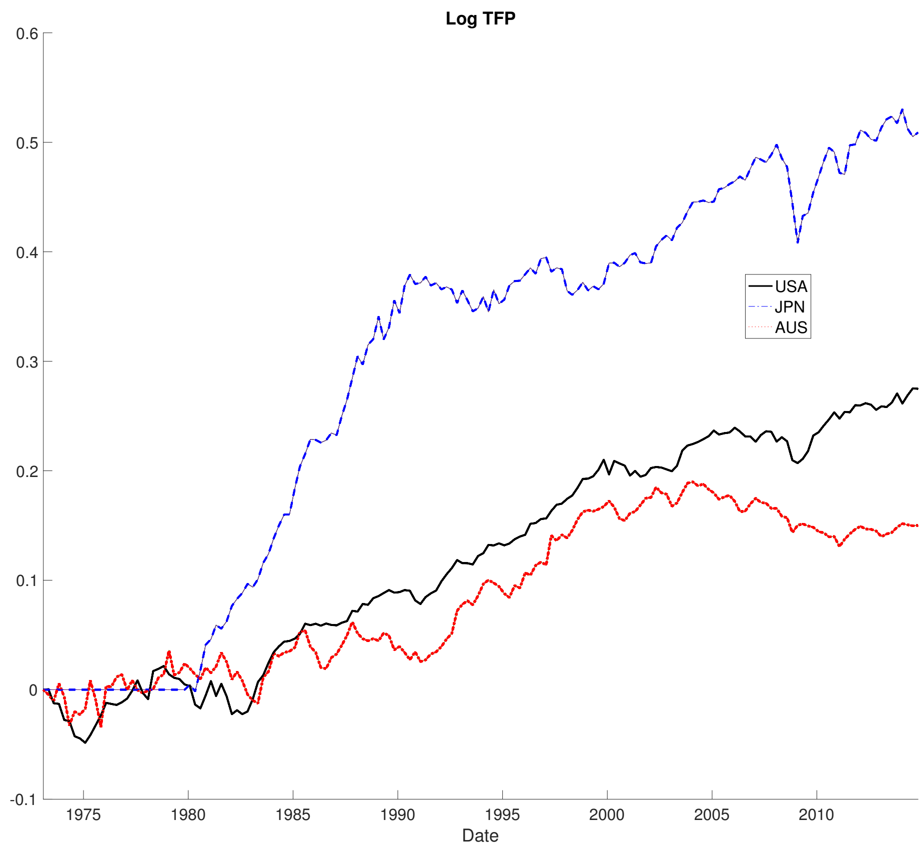


Figure 2: Log TFP: Data and Simulated

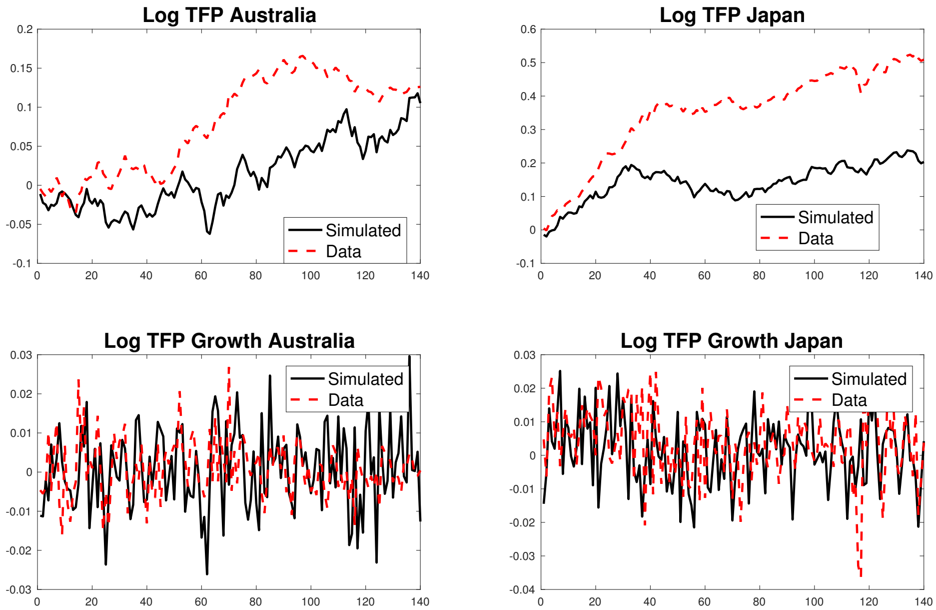


Figure 3: Long Series of Simulated Log TFP

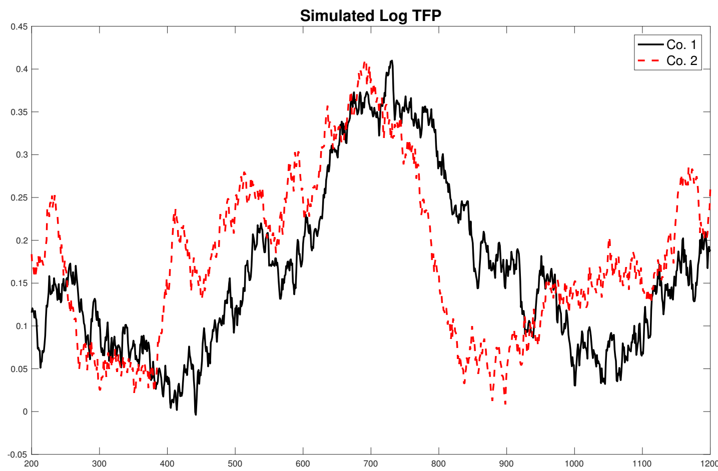


Figure 4: Log TFP, log output and labor responses. Country 1 responses shown with solid lines and Country 2 responses with dashed lines.

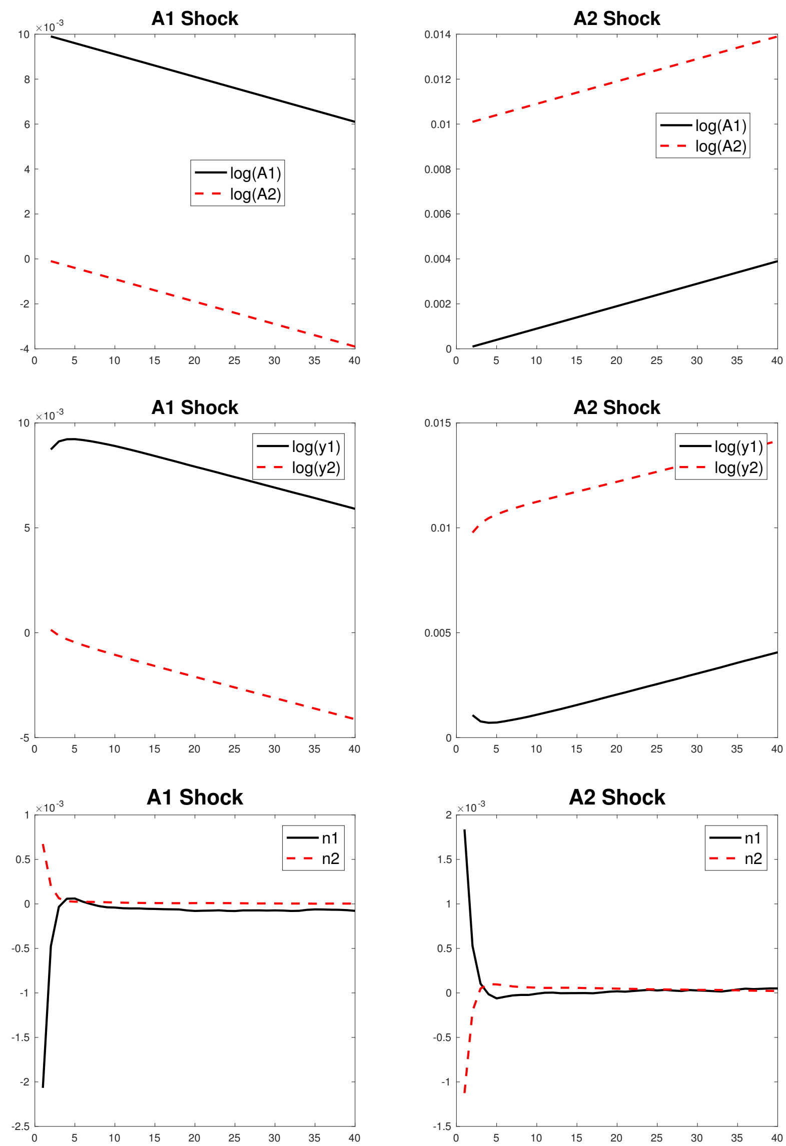


Figure 5: Consumption responses. Aggregates and components.

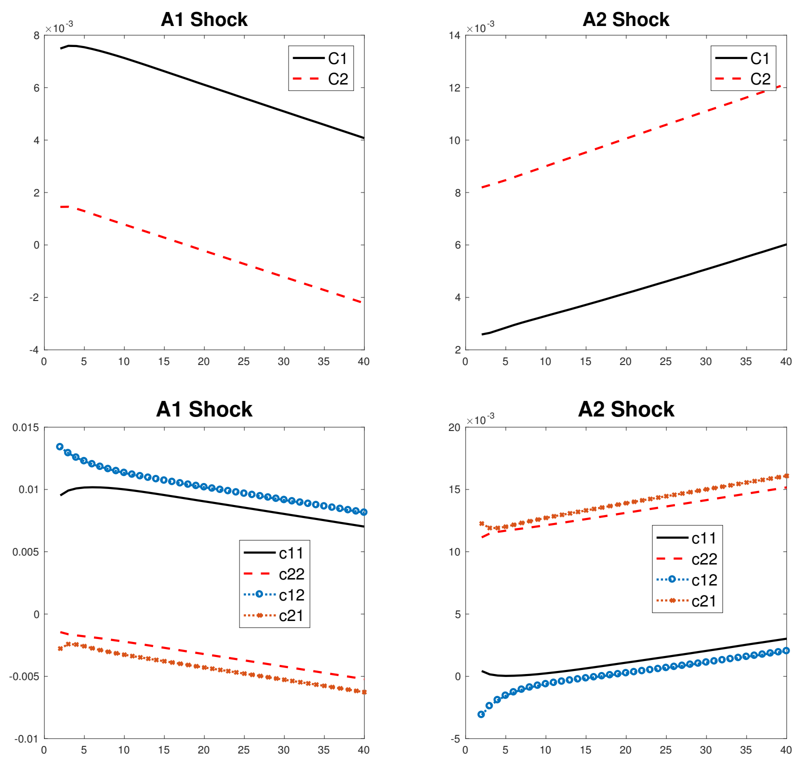


Figure 6: Inflation, Interest Rate and Exchange Rate Responses

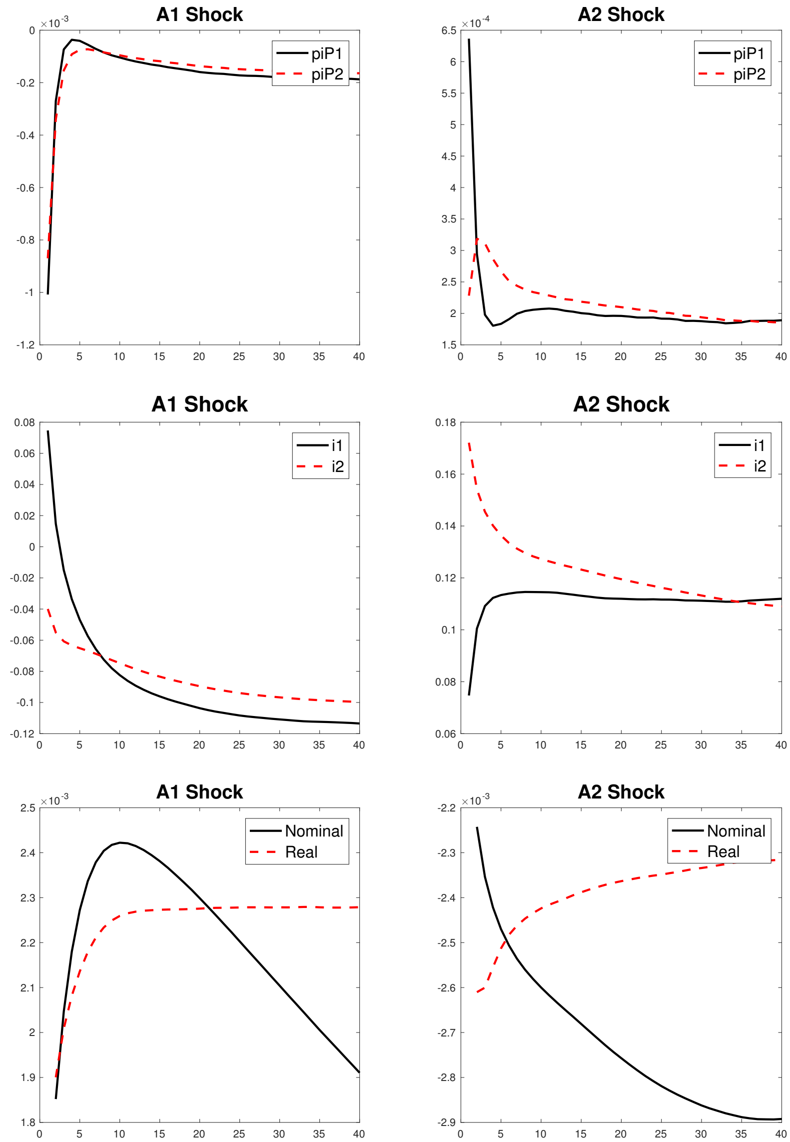
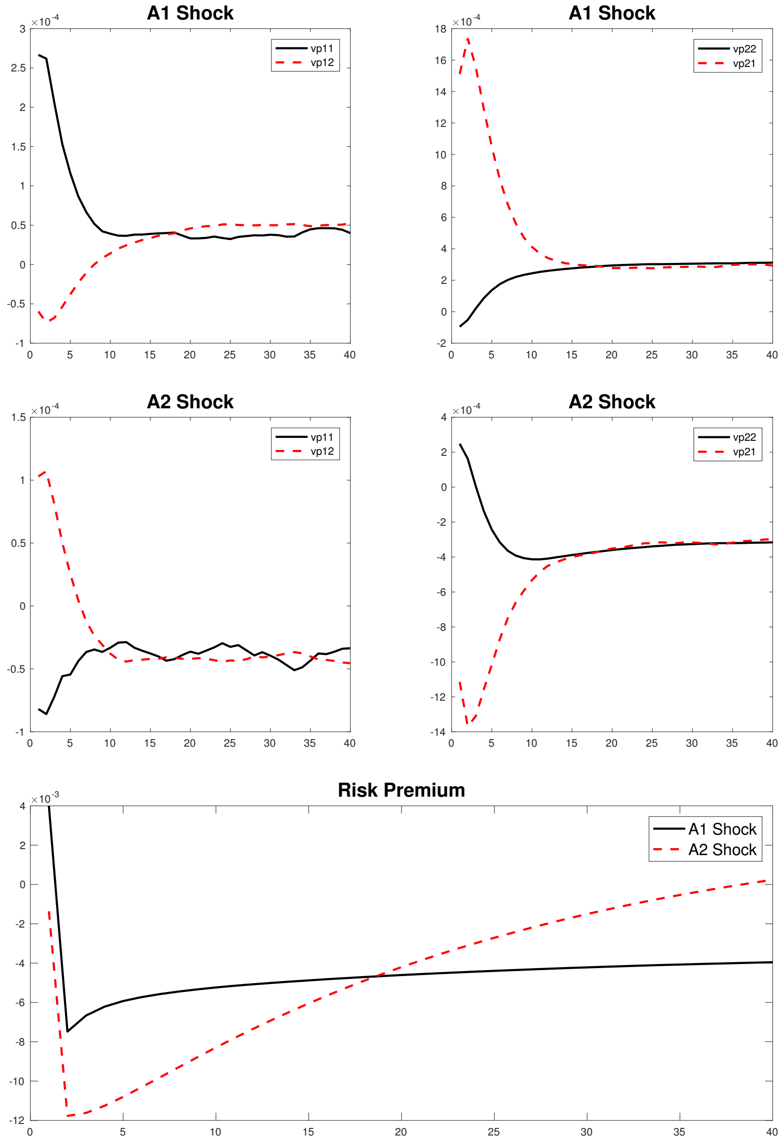


Figure 7: Price Dispersion and the Risk Premium



A Appendix

This appendix fully develops the two-country model described in the text.

A.1 Recursive Utility Formulation

Recursive utility was introduced to macroeconomics and finance by Epstein and Zin (1989) and Weil (1989). We employ a functional form of recursive utility considered by Swanson (2016). Let c_t be the household's real consumption and ℓ_t be its labor supply. The utility function is,

$$V_t = (1-\beta)\left(\ln(c_t) - \eta\frac{\ell_t^{1+\chi}}{1+\chi}\right) - \frac{\beta}{\alpha}\ln\left[E_t\left(e^{-\alpha V_{t+1}}\right)\right], \tag{A.1}$$

where β is the subjective discount factor and η, χ, α are also parameters. The Epstein and Zin (1989) and Weil (1989) formulations separate the coefficient of relative risk aversion and the elasticity of intertemporal substitution. In (A.1), the elasticity of intertemporal substitution is 1 and risk aversion (RRA) is

$$RRA = \alpha + \left(\frac{1}{\left(1+\frac{\eta}{\chi}\right)}\right). \tag{A.2}$$

Also, the Frisch elasticity of labor supply is $1/\chi$.

A.2 Stationarity Transformation of Utility

Productivity has a stochastic trend which causes the model to be nonstationary. Except for labor ℓ_t, the other variables in the model will inherit the stochastic trend from productivity, so the model must be transformed to induce stationarity. We do this by normalizing (dividing) the variables that grow over time, by the lagged level of total factor productivity, A_{t-1}. To transform (A.1), (i) multiply and divide c_t by A_{t-1}, where $\tilde{c}_t \equiv c_t/A_{t-1}$, (ii) subtract $\ln(A_{t-1})$ from both sides of (A.1), (iii) add and subtract $\ln(A_t)$ to V_{t+1}, and define $\tilde{V}_t \equiv V_t - \ln(A_{t-1})$ and $G_t \equiv A_t/A_{t-1}$. These steps give the stationary form of utility,

$$\tilde{V}_t = \left\{(1-\beta)\left[\ln(\tilde{c}_t) - \eta\frac{\ell_t^{1+\chi}}{1+\chi}\right] - \frac{\beta}{\alpha}\ln\left[E_t e^{-\alpha\tilde{V}_{i,,t+1}}\right]\right\} + \beta\ln(G_t) \tag{A.3}$$

The **real stochastic discount factor** implied by (A.3) is $\beta M_{t,t+1}$,where

$$M_{t,t+1} = \left(\frac{\tilde{c}_t}{\tilde{c}_{t+1}}\right)\left(\frac{e^{-\alpha\tilde{V}_{t+1}}}{E_t\left(e^{-\alpha\tilde{V}_{t+1}}\right)}\right) \tag{A.4}$$

The nominal stochastic discount factor is $N_{t,t+1} = \beta M_{t,t+1}e^{-\pi_{t+1}}$, where π_t is the inflation rate.

A.3 Households

Country $i = 1, 2$ households want to maximize

$$\tilde{V}_{i,t} = \left\{ (1-\beta) \left[\ln(\tilde{c}_{i,t}) - \eta \frac{\ell_{i,t}^{1+\chi}}{1+\chi} \right] - \frac{\beta}{\alpha} \ln \left[E_t e^{-\alpha \tilde{V}_{i,,t+1}} \right] \right\}. \tag{A.5}$$

The household splits its total labor supply across production of the domestically consumed and exported goods. For traded goods, the first subscript tells where the good is consumed and the second subscript tells where it is produced. Hence, $\ell_{i,j,t}$ is labor to produce the good in j which is consumed in i.

A.3.1 Complete markets

Denote the current state of the world by ω_t and the state history by $\omega^t = \{\omega_t, \omega_{t-1}, \ldots\}$. Households in Countries 1 and 2 have access to a full set of nominal state-contingent securities that pay one unit of Country 1's currency if the state occurs. Making the dependence on the state explicit, let $\tilde{B}_1(\omega^t)$ be the number of state ω^t contingent bonds held by the household (recall the tilde ˜means the variable is normalized by the level of technology). The price of a bond that pays off in state ω_{t+1} is $p_\omega(\omega_{t+1}|\omega^t)$. Country 1's household takes the payoffs from state-contingent bonds, its wages and firm profits to pay for consumption and its portfolio of state-contingent securities. Let $\tilde{W}_1(\omega^t)$ be the nominal wage, $P_1(\omega^t)$ be the price level and $\tilde{\Pi}_1(\omega^\tau)$ firm profits. Shares of Country 1 firms are not traded and are entirely owned by by Country 1 households. The household takes the flow resources of labor income, firm profits, and state-contingent bond payoffs to purchase consumption and a portfolio of state-contingent bonds. There is no physical capital in the model. Country 1's household budget constraint is,

$$c_1(\omega^t) = \frac{W_1(\omega^t)}{P_1(\omega^t)} \ell_1(\omega^t) + \frac{\Pi_1(\omega^t)}{P_1(\omega^t)} + \frac{B_1(\omega^t)}{P_1(\omega^t)} - \sum_{\omega^{t+1}} \frac{Q(\omega_{t+1}|\omega^t) B_1(\omega^{t+1})}{P_1(\omega^t)}. \tag{A.6}$$

To obtain a stationary representation for (A.6), divide both sides by $A_1(\omega^{t-1})$,

$$\frac{c_1(\omega^t)}{A_1(\omega^{t-1})} = \frac{W_1(\omega^t)}{A_1(\omega^{t-1}) P_1(\omega^t)} \ell_1(\omega^t) + \frac{\Pi_1(\omega^t)}{A_1(\omega^{t-1}) P_1(\omega^t)} + \frac{B_1(\omega^t)}{A_1(\omega^{t-1}) P_1(\omega^t)} - \sum_{\omega^{t+1}} \frac{Q(\omega_{t+1}|\omega^t) B_1(\omega^{t+1})}{P_1(\omega^t) A_1(\omega^{t-1})} \frac{A_1(\omega^t)}{A_1(\omega^t)}, \tag{A.7}$$

and rewrite (A.7) by indicating those variables divided by $A_1(\omega^{t-1})$ with a tilde,

$$\tilde{c}_1(\omega^t) = \frac{\tilde{W}_1(\omega^t)}{P_1(\omega^t)} \ell_1(\omega^t) + \frac{\tilde{\Pi}_1(\omega^t)}{P_1(\omega^t)} + \frac{\tilde{B}_1(\omega^t)}{P_1(\omega^t)} - \sum_{\omega^{t+1}} \frac{p_\omega(\omega_{t+1}|\omega^t) \tilde{B}_1(\omega^{t+1})}{P_1(\omega^t)} G_1(\omega^t). \tag{A.8}$$

where $G_1\left(\omega^t\right) = A_1\left(\omega^t\right)/A_1\left(\omega^{t-1}\right)$.

If $\pi\left(\omega_{t+1}|\omega^t\right)$ is the conditional probability of state ω_{t+1},the optimality conditions for the household give the Euler equations for the state-contingent bond and labor supply,

$$p_\omega\left(\omega^{t+1}|\omega^t\right) = \beta\pi\left(\omega_{t+1}|\omega^t\right)\frac{M_1\left(\omega^{t+1}\right)}{G_1\left(\omega^t\right)}\frac{P_1\left(\omega^t\right)}{P_1\left(\omega^{t+1}\right)}, \tag{A.9}$$

$$\eta\tilde{c}_1\left(\omega^t\right)\ell_1\left(\omega^t\right)^\chi = \frac{\tilde{W}_1\left(\omega^t\right)}{P_1\left(\omega^t\right)}. \tag{A.10}$$

Summing over the prices of all state-contingent bonds gives the price of the nominally risk-free bond,

$$\frac{1}{1+i_1\left(\omega^t\right)} = \left(\frac{\beta}{G_1\left(\omega^t\right)}\right)E_t\left(M_1\left(\omega^{t+1}\right)e^{-\pi_1\left(\omega^{t+1}\right)}\right). \tag{A.11}$$

The transformation of Country 2's budget constraint follows analgously, and gives the stationary form,

$$\tilde{c}_2\left(\omega^t\right) = \frac{\tilde{W}_2\left(\omega^t\right)\ell_2\left(\omega^t\right)}{P_2\left(\omega^t\right)} + \frac{\tilde{\Pi}_2\left(\omega^t\right)}{P_2\left(\omega^t\right)} + \frac{\tilde{B}_2\left(\omega^t\right)}{S\left(\omega^t\right)P_2\left(\omega^t\right)} - \sum_{\omega^{t+1}}\frac{p_\omega\left(\omega^{t+1}|\omega^t\right)\tilde{B}_2\left(\omega^{t+1}\right)}{S\left(\omega^t\right)P_2\left(\omega^t\right)} \tag{A.12}$$

where $S\left(\omega^t\right)$ is the nominal exchange rate (price of Country 2's currency). The optimality conditions for the foreign household gives the Euler equations for the state-contingent bond and labor supply,

$$p_\omega\left(\omega^{t+1}|\omega^t\right) = \beta\pi\left(\omega_{t+1}|\omega^t\right)\frac{M_2\left(\omega^{t+1}\right)}{G_2\left(\omega^t\right)}\frac{S\left(\omega^t\right)P_2\left(\omega^t\right)}{S\left(\omega^{t+1}\right)P_2\left(\omega^{t+1}\right)} \tag{A.13}$$

$$\eta\tilde{c}_2\left(\omega^t\right)\ell_2\left(\omega^t\right)^\chi = \frac{\tilde{W}_2\left(\omega^t\right)}{P_2\left(\omega^t\right)} \tag{A.14}$$

Let $Q\left(\omega^t\right) = \frac{S\left(\omega^t\right)P_2\left(\omega^t\right)}{P_1\left(\omega^t\right)}$ be the real exchange rate. Equating (A.9) to (A.13) gives the real exchange rate depreciation,

$$\frac{Q\left(\omega^{t+1}\right)}{Q\left(\omega^t\right)} = \left(\frac{M_2\left(\omega^{t+1}\right)}{M_1\left(\omega^{t+1}\right)}\right)\left(\frac{G_1\left(\omega^t\right)}{G_2\left(\omega^t\right)}\right), \tag{A.15}$$

and the nominal depreciation,

$$\frac{S\left(\omega^{t+1}\right)}{S\left(\omega^t\right)} = \left(\frac{M_2\left(\omega^{t+1}\right)e^{-\pi_2\left(\omega^{t+1}\right)}}{M_1\left(\omega^{t+1}\right)e^{-\pi_1\left(\omega^{t+1}\right)}}\right)\left(\frac{G_1\left(\omega^t\right)}{G_2\left(\omega^t\right)}\right).$$

A.3.2 Incomplete markets

Under incomplete markets, we can suppress the functional dependence on the state notation. Each country issues a non-state contingent nominal bond that is internationally traded. The country 1 bond is issued at a price of 1 and pays a gross return $1+i_{1,t}$ units of country 1 currency next period. Similarly, country 2 issues a bond, priced at 1 unit of currency 2 which pays $1+i_{2,t}$ units of currency 2 next period. Let $B_{i,j,t}$ be currency j bonds issued by j and held by i.[21] This way of formulating incomplete markets renders bond holdings to be non-stationary even when productivity shocks are stationary. Following Schmidtt-Grohe and Uribe (2003), we impose a small fee on net foreign bond positions to induce stationarity in these positions. Let $\tau_{1,2}$ be the fee paid by country 1 households for holding country 2 bonds. The real cost of holding 'foreign' bonds valued at $S_t B_{1,2,t}$ is $\frac{\tau_{1,2}}{2A_{1,t-1}}\left(\frac{S_t B_{1,2,t}}{P_{1,t}}\right)^2$. As long as $\tau_{1,2,t}>0$, the Country 1 household will want $B_{1,2}=0$ in the steady state. The budget constraint facing the country 1 household is,

$$c_{1,t}+\frac{B_{1,1,t}}{P_{1,t}}+\frac{S_t B_{1,2,t}}{P_{1,t}}+\frac{\tau_{1,2}}{2A_{1,t-1}}\left(\frac{S_t B_{1,2,t}}{P_{1,t}}\right)^2=\frac{W_{1,t}\ell_{1,t}}{P_{1,t}}+\frac{\Pi_{1,t}}{P_{1,t}}+\frac{\left(1+i_{1,t-1}\right)B_{1,1,t-1}}{P_{1,t}}+\frac{\left(1+i_{2,t-1}\right)S_t B_{1,2,t-1}}{P_{1,t}}. \tag{A.16}$$

The stationary transformation of (A.16) is obtained by dividing both sides by $A_{1,t-1}$, then re-arranging to get

$$\tilde{c}_{1,t}+\frac{\tilde{B}_{1,1,t}}{P_{1,t}}+\frac{Q_t\tilde{B}_{1,2,t}}{P_{2,t}}+\frac{\tau_{1,2}}{2}\left(\frac{q_t\tilde{B}_{1,2,t}}{P_{2,t}}\right)^2=\frac{\tilde{W}_{1,t}\ell_{1,t}}{P_{1,t}}+\frac{\tilde{\Pi}_{1,t}}{P_{1,t}}+\frac{\left(1+i_{1,t-1}\right)\tilde{B}_{1,1,t-1}}{P_{1,t}G_{1,t-1}}+\frac{\left(1+i_{2,t-1}\right)Q_t\tilde{B}_{1,2,t-1}}{P_{2,t}G_{1,t-1}}, \tag{A.17}$$

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[21] Here are the first order conditions:

$$\frac{\partial\tilde{V}_{1,t}}{\partial\tilde{c}_{1,t}}=0 \iff (1-\beta)\left(\frac{1}{\tilde{c}_{1,t}}\right)=\lambda_{1,t}$$

$$\frac{\partial\tilde{V}_{1,t}}{\partial n_{1,t}}=0 \iff (1-\beta)\,\eta n_{1,t}^{\chi}=\lambda_{1,t}\left(\frac{\tilde{W}_{1,t}}{P_{1,t}}\right)$$

$$\frac{\partial\tilde{V}_{1,t}}{\partial\tilde{B}_{1,1,t}}=0 \iff \lambda_{1,t}\left(\frac{1}{P_{1,t}}\right)=\beta\left(\frac{e^{-\alpha\tilde{V}_{1,t+1}}}{E_t\left(e^{-\alpha\tilde{V}_{1,t+1}}\right)}\right)\left(\frac{\partial\tilde{V}_{1,t+1}}{\partial B_{1,1,t}}\right)$$

$$\left(\frac{\partial\tilde{V}_{1,t}}{\partial B_{1,1,t-1}}\right)=\left(1+i_{1,t-1}\right)\left(\frac{1}{P_{1,t}}\right)\left(\frac{\Gamma_{1,t-2}}{\Gamma_{1,t-1}}\right)\lambda_{1,t}$$

$$\frac{\partial\tilde{V}_{1,t}}{\partial\tilde{B}_{1,2,t}}=0 \iff -\lambda_{1,t}\left(\frac{q_t}{P_{2,t}}\right)-\frac{\beta}{\alpha}\left(\frac{(-\alpha)\,e^{-\alpha\tilde{V}_{1,t+1}}}{E_t e^{-\alpha\tilde{V}_{1,t+1}}}\right)\left(\frac{\partial\tilde{V}_{1,t+1}}{\partial\tilde{B}_{1,2,t}}\right)-\lambda_{1,t}\tau_{1,2}\left(\frac{q_t^2\tilde{B}_{1,2,t}}{P_{2,t}^2}\right)$$

$$\frac{\partial\tilde{V}_{1,t}}{\partial B_{1,2,t-1}}=\lambda_{1,t}\left(1+i_{2,t-1}\right)\left(\frac{q_t}{P_{2,t}}\right)\left(\frac{\Gamma_{1,t-2}}{\Gamma_{1,t-1}}\right)$$

$$\rightarrow\left(\frac{1}{1+i_{2,t}}\right)\left(1+\tau_{1,2}\left(\frac{q_t\tilde{B}_{1,2,t}}{P_{2,t}}\right)\right)=\frac{\beta}{G_{1,t}}E_t m_{1,t+1}\left(\frac{q_{t+1}}{q_t}\right)\left(\frac{P_{2,t}}{P_{2,t+1}}\right)$$

where the tilde'd variables are divided by the lagged country 1 productivity (e.g., $\tilde{B}_{1,2,t-1} \equiv B_{1,2,t-1}/A_{1,t-2}$). The Euler equations associated with optimal bond holdings are, for country 1,

$$\frac{1}{1+i_{1,t}} = \frac{\beta}{G_{1,t}} E_t \left(M_{1,t+1} \frac{P_{1,t}}{P_{1,t+1}} \right), \tag{A.18}$$

$$\left(\frac{1+\tau_{1,2}\left(q_t \tilde{B}_{1,2,t}/P_{2,t} \right)}{1+i_{2,t}} \right) = \frac{\beta}{G_{1,t}} E_t \left(M_{1,t+1} \frac{Q_{t+1}}{Q_t} \frac{P_{2,t}}{P_{2,t+1}} \right). \tag{A.19}$$

Similarly, country 2 begins with the budget constraint

$$c_{2,t} + \frac{B_{2,2,t}}{P_{2,t}} + \frac{B_{2,1,t}}{S_t P_{2,t}} + \frac{\tau_{2,1}}{2A_{2,t-1}} \left(\frac{B_{2,1,t}}{S_t P_{2,t}} \right)^2 = \frac{W_{2,t}\ell_{2,t}}{P_{2,t}} + \frac{\Pi_{2,t}}{P_{2,t}} + \frac{(1+i_{2,t-1})\, B_{2,2,t-1}}{P_{2,t}} + \frac{(1+i_{1,t-1})\, B_{2,1,t-1}}{P_{2,t} S_t}. \tag{A.20}$$

Divide both sides of (A.20) by $A_{2,t-1}$ and re-arrange to get,

$$\tilde{c}_{2,t} + \frac{\tilde{B}_{2,2,t}}{P_{2,t}} + \frac{\tilde{B}_{2,1,t}}{Q_t P_{1,t}} + \frac{\tau_{2,1}}{2} \left(\frac{\tilde{B}_{2,1,t}}{Q_t P_{1,t}} \right)^2 = \frac{\tilde{W}_{2,t}\ell_{2,t}}{P_{2,t}} + \frac{\tilde{\Pi}_{2,t}}{P_{2,t}} + \frac{(1+i_{2,t-1})\, \tilde{B}_{2,2,t-1}}{P_{2,t} G_{2,t-1}} + \frac{(1+i_{1,t-1})\, \tilde{B}_{2,1,t-1}}{Q_t P_{1,t} G_{2,t-1}}. \tag{A.21}$$

The Euler equations associated with optimal bond holdings for country 2 are,

$$\frac{1}{1+i_{2,t}} = \frac{\beta}{G_{2,t}} E_t \left(M_{2,t+1} \frac{P_{2,t}}{P_{2,t+1}} \right), \tag{A.22}$$

$$\frac{1+\tau_{2,1}\left(\tilde{B}_{2,1,t}/\left(Q_t P_{1,t} \right) \right)}{1+i_{1,t}} = \frac{\beta}{G_{2,t}} E_t \left(\tilde{m}_{2,t+1} \frac{Q_t}{Q_{t+1}} \frac{P_{1,t}}{P_{1,t+1}} \right). \tag{A.23}$$

The optimality conditions for the labor-leisure choice is unaffected by the change to incomplete markets and continue to be described by (A.10) and (A.14).

A.4 Demand functions

In each country, there are a continuum of firms, indexed by $f \in [0,1]$ each producing a differentiated product. Our convention on subscripts is $c_{i,j,t}$ is made in country j and consumed in country i. Let σ be the elasticity of substitution between varieties f. In country 1, consumption of 'home' produced goods and of imports are,

$$\tilde{c}_{1,1,t} = \left[\int_0^1 \tilde{c}_{1,1,t}(f)^{\frac{\sigma-1}{\sigma}} \, df \right]^{\frac{\sigma}{\sigma-1}}, \tag{A.24}$$

$$\tilde{c}_{1,2,t} = \left[\int_0^1 \tilde{c}_{1,2,t}(f)^{\frac{\sigma-1}{\sigma}} \, df \right]^{\frac{\sigma}{\sigma-1}}, \tag{A.25}$$

and the price indices for the $\tilde{c}_{1,1}$ and $\tilde{c}_{1,2}$ bundles associated with (A.24) and (A.25) are,

$$P_{1,1,t} = \left[\int_0^1 p_{1,1,t}(f)^{1-\sigma}\, df\right]^{\frac{1}{1-\sigma}}, \tag{A.26a}$$

$$P_{2,1,t} = \left[\int_0^1 p_{2,1,t}(f)^{1-\sigma}\, df\right]^{\frac{1}{1-\sigma}}. \tag{A.26b}$$

Aggregate consumption in country 1 is the constant elasticity of substitution (CES) index,

$$\tilde{c}_{1,t} = \left[d^{\frac{1}{\mu}} \tilde{c}_{1,1,t}^{\frac{\mu-1}{\mu}} + (1-d)^{\frac{1}{\mu}} \tilde{c}_{1,2,t}^{\frac{\mu-1}{\mu}}\right]^{\frac{\mu}{\mu-1}}, \tag{A.27}$$

of goods produced in country 1, $\tilde{c}_{1,1,t} \equiv c_{1,1,t}/A_{1,t-1}$, and imports from country 2, $\tilde{c}_{1,2,t} \equiv c_{1,2,t}/A_{1,t-1}$. The elasticity of substitution between 'home' and 'foreign' goods is μ, and home-bias in consumption is represented by $d > 1/2$. The aggregate price level assocated with (A.27) is

$$P_{1,t} = \left[dP_{1,1,t}^{1-\mu} + (1-d)\, P_{1,2,t}^{1-\mu}\right]^{\frac{1}{1-\mu}}. \tag{A.28}$$

The countries are symmetrical. Bundles of country 2 'domestic' consumption and its imports are,

$$\tilde{c}_{2,2,t} = \left[\int_0^1 \tilde{c}_{2,2,t}(f)^{\frac{\sigma-1}{\sigma}}\, df\right]^{\frac{\sigma}{\sigma-1}}, \tag{A.29}$$

$$\tilde{c}_{2,1,t} = \left[\int_0^1 \tilde{c}_{2,1,t}(f)^{\frac{\sigma-1}{\sigma}}\, df\right]^{\frac{\sigma}{\sigma-1}}, \tag{A.30}$$

where $\tilde{c}_{2,2,t} \equiv c_{2,2,t}/A_{2,t-1}$ and $\tilde{c}_{2,1,t} \equiv c_{2,1,t}/A_{2,t-1}$. The associated price indices are

$$P_{2,2,t} = \left[\int_0^1 p_{2,2,t}(f)^{1-\sigma}\, df\right]^{\frac{1}{1-\sigma}}, \tag{A.31}$$

$$P_{1,2,t} = \left[\int_0^1 p_{2,2,t}(f)^{1-\sigma}\, df\right]^{\frac{1}{1-\sigma}}. \tag{A.32}$$

Aggregate country 2 consumption and price level are,

$$\tilde{c}_{2,t} = \left[d^{\frac{1}{\mu}} \tilde{c}_{2,2,t}^{\frac{\mu-1}{\mu}} + (1-d)^{\frac{1}{\mu}} \tilde{c}_{2,1,t}^{\frac{\mu-1}{\mu}}\right]^{\frac{\mu}{\mu-1}}, \tag{A.33}$$

$$P_{2,t} = \left[dP_{2,2,t}^{1-\mu} + (1-d)\, P_{2,1,t}^{1-\mu}\right]^{\frac{1}{1-\mu}}. \tag{A.34}$$

A.5 Intermediate Goods Firm Problem

In country 1, output of firm $f \in [0,1]$ is demand determined. Firm f can distinguish between domestic and foreign shoppers and is ablt to charge them different prices. Country 1 export prices are set in country 2's currency (local currency pricing, or LCP).

Labor is the only input of production. The production function for firm f is

$$y_{1t}(f) = A_{1,t}\ell_{1,t}(f),$$

where $A_{1,t}$ is the level of productivity. Normalize by dividing both sides by $A_{1,t-1}$ to get

$$\tilde{y}_{1,t}(f) = G_{1,t}\ell_{1,t}(f). \tag{A.35}$$

Total costs are

$$\frac{\tilde{W}_{1,t}}{P_{1,t}}\ell_{1t}(f),$$

where $W_{1,t}$ is the nominal wage and $\tilde{W}_{1,t} = W_{1,t}/A_{1,t-1}$. Output is demand determined, $\tilde{y}_{1,t}(f) = \tilde{c}_{1,1,t}(f) + \tilde{c}_{2,1,t}(f)$. The firm can always adjust its labor input. A Lagrangian for the firm is,

$$L = -\frac{\tilde{W}_{1,t}}{P_{1,t}}\ell_{1,t}(f) + \varphi_{1,t}(f)\left\{G_{1,t}\ell_{1,t}(f) - \tilde{c}_{1,1,t}(f) - \tilde{c}_{2,1,t}(f)\right\}. \tag{A.36}$$

The Euler equation implied by choosing labor is,

$$\frac{\tilde{W}_{1,t}}{P_{1,t}} = \varphi_{1,t}(f)\, G_{1,t}. \tag{A.37}$$

To determine optimal price setting, note that firm f faces these (normalized) domestic and foreign demands for its good

$$\tilde{c}_{1,1,t}(f) = d\left(\frac{p_{1,1,t}(f)}{P_{1,1,t}}\right)^{-\sigma}\left(\frac{P_{1,1,t}}{P_{1,t}}\right)^{-\mu}\tilde{c}_{1,t}, \tag{A.38}$$

$$\tilde{c}_{2,1,t}(f)\Omega_{t-1} = (1-d)\left(\frac{p_{2,1,t}(f)}{P_{2,1,t}}\right)^{-\sigma}\left(\frac{P_{2,1,t}}{P_{2,t}}\right)^{-\mu}\tilde{c}_{2,t}\Omega_{t-1}, \tag{A.39}$$

where $\Omega_t \equiv A_{2,t}/A_{1,t}$ is the ratio of country 2 to country 1 productivity. Since output is demand determined, we have

$$\tilde{y}_{1,t}(f) = G_{1,t}\ell_{1,t}(f) = \tilde{c}_{1,1,t}(f) + \tilde{c}_{2,1,t}(f)\,\Omega_{t-1}, \tag{A.40}$$

from which it follows that firm f employment is,

$$\ell_{1,t}(f) = \frac{\tilde{c}_{1,1,t}(f) + \tilde{c}_{2,1,t}\Omega_{t-1}}{G_{1,t}}. \tag{A.41}$$

Let $\tilde{w}_{1,t}=\tilde{W}_{1,t}/P_{1,t}$ be the real wage. Current profit is,

$$\tilde{\Pi}_{1,t}\left(f\right)=\frac{p_{1,1,t}\left(f\right)}{P_{1}}\tilde{c}_{1,1}\left(f\right)+\frac{Q_{t}p_{2,1,t}\left(f\right)}{P_{2,t}}\tilde{c}_{2,1,t}\left(f\right)\Omega_{t-1}-\tilde{w}_{1,t}\ell_{1,t}(f). \tag{A.42}$$

Prices are sticky in the sense of Calvo (1983). Each period, the probability that the firm is allowed to change prices is $1-\alpha_{c}$. In periods when the firm does reset prices, it adjusts both the price for domestic and foreign markets, $p_{1,1,t}\left(f\right)$ and $p_{2,1,t}\left(f\right)$, to maximize

$$E_{t}\sum_{s=0}^{\infty}(\alpha_{c}\beta)^{s}M_{1,t,t+s}\left[\frac{p_{1,1,t}(f)}{P_{1,t+s}}\tilde{c}_{1,1,t+s}(f)+\frac{Q_{t+s}p_{2,1,t}(f)}{P_{2,t+s}}\tilde{c}_{2,1,t+s}(f)\Omega_{t-1+s}-\tilde{w}_{1,t+s}\ell_{1,t+s}\left(f\right)\right], \tag{A.43}$$

subject to the demand functions (A.38), (A.39) and labor demand function (A.41). Noting that

$$\begin{aligned}
\frac{\partial\tilde{c}_{1,1}\left(f\right)}{\partial p_{1,1}\left(f\right)}&=-\sigma p_{1,1}\left(f\right)^{-\sigma-1}dP_{1,1}^{\sigma}\left(\frac{P_{1,1}}{P_{1}}\right)^{-\mu}\tilde{c}_{1},\\
\frac{\partial\left(p_{1,1}\left(f\right)\tilde{c}_{1,1}\left(f\right)\right)}{\partial p_{1,1}\left(f\right)}&=p_{1,1}\left(f\right)^{-\sigma}\left(1-\sigma\right)dP_{1,1}^{\sigma}\left(\frac{P_{1,1}}{P_{1}}\right)^{-\mu}\tilde{c}_{1},\\
\frac{\partial\tilde{c}_{1,1}\left(f\right)}{\partial p_{1,1}\left(f\right)}&=-\sigma p_{1,1}\left(f\right)^{-\sigma-1}P_{1,1}^{\sigma}\left(\frac{P_{1,1}}{P_{1}}\right)^{-\mu}d\tilde{c}_{1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\tilde{c}_{2,1}\left(f\right)}{\partial p_{2,1}\left(f\right)}&=-\sigma p_{2,1}\left(f\right)^{-\sigma-1}\left(1-d\right)P_{2,1}^{\sigma}\left(\frac{P_{2,1}}{P_{2}}\right)^{-\mu}\tilde{c}_{2},\\
\frac{\partial\left(p_{2,1}\left(f\right)\tilde{c}_{2,1}\left(f\right)\right)}{\partial p_{2,1}\left(f\right)}&=p_{2,1}\left(f\right)^{-\sigma}\left(1-\sigma\right)\left(1-d\right)P_{2,1}^{\sigma}\left(\frac{P_{2,1}}{P_{2}}\right)^{-\mu}\tilde{c}_{2},\\
\frac{\partial\tilde{c}_{2,1}\left(f\right)}{\partial p_{2,1}\left(f\right)}&=-\sigma p_{2,1}\left(f\right)^{-\sigma-1}P_{2,1}^{\sigma}\left(\frac{P_{2,1}}{P_{2}}\right)^{-\mu}\left(1-d\right)\tilde{c}_{2},
\end{aligned}$$

when the firm chooses $p_{1,1,t}^{*}\left(f\right)$, the first-order condition can be re-arranged as,

$$p_{1,1,t}^{*}\left(f\right)=\frac{\sigma}{\left(\sigma-1\right)}\frac{a_{1,1,t}}{b_{1,1t}}, \tag{A.44}$$

where

$$a_{1,1t}=E_{t}\sum_{s=0}^{\infty}(\alpha_{c}\beta)^{s}M_{1,t,t+s}\frac{\tilde{w}_{1,t+s}}{G_{1,t+s}}P_{1,1,t+s}^{\sigma}\left(\frac{P_{1,1,t+s}}{P_{1,t+s}}\right)^{-\mu}d\tilde{c}_{1,t+s}, \tag{A.45}$$

$$b_{1,1,t}=E_{t}\sum_{s=0}^{\infty}(\alpha_{c}\beta)^{s}M_{1,t,t+s}\frac{dP_{1,1,t+s}^{\sigma}\left(\frac{P_{1,1,t+s}}{P_{1,t+s}}\right)^{-\mu}\tilde{c}_{1,t+s}}{P_{1,t+s}}. \tag{A.46}$$

$a_{1,1,t}$ and $b_{1,1,t}$ can be represented recursively as,

$$a_{1,1,t}=\tilde{w}_{1,t}\frac{d\tilde{c}_{1,t}}{G_{1,t}}P_{1,1,t}^{\sigma}\left(\frac{P_{1,1,t}}{P_{1,t}}\right)^{-\mu}+\left(\alpha_{c}\beta\right)E_{t}\left(M_{1,t,t+1}a_{1,1,t+1}\right), \tag{A.47}$$

$$b_{1,1,t}=\frac{dP_{1,1,t}^{\sigma}\left(\frac{P_{1,1,t}}{P_{1,t}}\right)^{-\mu}\tilde{c}_{1,t}}{P_{1,t}}+\left(\alpha_{c}\beta\right)E_{t}\left(M_{1,t,t+1}b_{1,1,t+1}\right). \tag{A.48}$$

Now multiply both sides of (A.47) andby $P^{\mu-\sigma}_{1,1,t}P^{-\mu}_{1,t}$, and multiply both sides of (A.48) by $P^{\mu-\sigma}_{1,1,t}P^{1-\mu}_{1,t}$.This gives

$$a_{1,1,t}P^{\mu-\sigma}_{1,1,t}P^{-\mu}_{1,t} = d\frac{\tilde{W}_{1,t}}{P_{1,t}}\frac{\tilde{c}_{1,t}}{G_{1,t}} + \alpha_c\beta E_t\left(M_{1,t,t+1}\left(\frac{P^{\mu-\sigma}_{1,1,t}P^{-\mu}_{1,t}}{P^{\mu-\sigma}_{1,1,t+1}P^{-\mu}_{1,t+1}}\right)a_{1,1,t+1}P^{\mu-\sigma}_{1,1,t+1}P^{-\mu}_{1,t+1}\right), \tag{A.49}$$

$$b_{1,1,t}P^{\mu-\sigma}_{1,1,t}P^{1-\mu}_{1,t} = d\tilde{c}_{1,t} + \alpha_c\beta E_t\left(M_{1,t+1}\frac{P^{\mu-\sigma}_{1,1,t}P^{1-\mu}_{1,t}}{P^{\mu-\sigma}_{1,1,t+1}P^{1-\mu}_{1,t+1}}b_{1,1,t+1}P^{\mu-\sigma}_{1,1,t+1}P^{1-\mu}_{1,t+1}\right). \tag{A.50}$$

Because of the separable form of the recursive utility function, we can simplify further by dividing both sides of (A.49) and (A.50) by $\tilde{c}_{1,t}$. The price-reset can be rewritten as

$$\left(\frac{p^*_{1,1,t}(f)}{P_{1,1,t}}\right)\left(\frac{P_{1,1,t}}{P_{1,t}}\right) = \frac{\sigma}{(\sigma-1)}\frac{z^n_{1,1,t}}{z^d_{1,1,t}}, \tag{A.51}$$

where

$$z^n_{1,1,t} = \frac{d}{G_{1,t}}\tilde{w}_{1,t} + \alpha_c\beta E_t\Lambda_{1,t,t+1}\left(\frac{P_{1,1,t}}{P_{1,1,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{1,t}}{P_{1,t+1}}\right)^{-\mu}z^n_{1,1,t+1}, \tag{A.52}$$

$$z^d_{1,1,t} = d + \alpha_c\beta E_t\Lambda_{1,t+1}\left(\frac{P_{1,1,t}}{P_{1,1,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{1,t}}{P_{1,t+1}}\right)^{1-\mu}z^d_{1,1,t+1}, \tag{A.53}$$

and

$$\Lambda_{1,t,t+1} \equiv \left(\frac{e^{-\alpha\tilde{V}_{t+1}}}{E_t\left(e^{-\alpha\tilde{V}_{t+1}}\right)}\right). \tag{A.54}$$

To determine the optimal price reset for exports $p^*_{2,1,t}(f)$, the firm faces the same Calvo probability as in adjusting the domestic good price. Re-arrangement of the first-order condition for choosing $p^*_{2,1,t}(f)$ gives

$$p^*_{2,1}(f) = \left(\frac{\sigma}{(\sigma-1)}\right)\frac{E_t\sum_{s=0}^{\infty}(\alpha_c\beta)^s M_{1,t,t+s}\tilde{w}_{1,t+s}\frac{\Omega_{t-1+s}}{G_{1,t+s}}P^{\sigma}_{2,1,t+s}\left(\frac{P_{2,1,t+s}}{P_{2,t+s}}\right)^{-\mu}(1-d)\,\tilde{c}_{2,t+s}}{E_t\sum_{s=0}^{\infty}(\alpha_c\beta)^s M_{1,t,t+s}\frac{\Omega_{t-1+s}Q_{t+s}}{P_{2,t+s}}P^{\sigma}_{2,1,t+s}\left(\frac{P_{2,1,t+s}}{P_{2,t+s}}\right)^{-\mu}(1-d)\,\tilde{c}_{2,t+s}} \tag{A.55}$$

$$= \left(\frac{\sigma}{(\sigma-1)}\right)\frac{a_{2,1,t}}{b_{2,1,t}} \tag{A.56}$$

where

$$a_{2,1,t} = E_t\sum_{s=0}^{\infty}(\alpha_c\beta)^s M_{1,t,t+s}\tilde{w}_{1,t+s}\frac{\Omega_{t-1+s}}{G_{1,t+s}}P^{\sigma}_{2,1,t+s}\left(\frac{P_{2,1,t+s}}{P_{2,t+s}}\right)^{-\mu}(1-d)\,\tilde{c}_{2,t+s}, \tag{A.57}$$

$$b_{2,1,t} = E_t\sum_{s=0}^{\infty}(\alpha\beta)^s M_{1,t,t+s}\frac{\Omega_{t-1+s}Q_{t+s}}{P_{2,t+s}}P^{\sigma}_{2,1,t+s}\left(\frac{P_{2,1,t+s}}{P_{2,t+s}}\right)^{-\mu}(1-d)\,\tilde{c}_{2,t+s}. \tag{A.58}$$

Now multiply both sides of (A.57) by $P_{2,1,t}^{\mu-\sigma}P_{2,t}^{-\mu}$ and multiply both sides of (A.58) by $P_{2,1,t}^{\mu-\sigma}P_{2,t}^{1-\mu}$,where

$$a_{2,1,t}P_{2,1,t}^{\mu-\sigma}P_{2,t}^{-\mu} = \tilde{w}_{1,t}\left(1-d\right)\frac{\Omega_{t-1}\tilde{c}_{2,t}}{G_{1,t}} + \alpha_c\beta E_t M_{1,t+1}\frac{P_{2,1,t}^{\mu-\sigma}P_{2,t}^{-\mu}}{P_{2,1,t+1}^{\mu-\sigma}P_{2,t+1}^{-\mu}}a_{2,1,t+1}P_{2,1,t+1}^{\mu-\sigma}P_{2,t+1}^{-\mu},$$

$$b_{2,1,t}P_{2,1,t}^{\mu-\sigma}P_{2,t}^{1-\mu} = \Omega_{t-1}Q_t\left(1-d\right)\tilde{c}_{2,t} + \alpha_c\beta E_t M_{1,t+1}\frac{P_{2,1,t}^{\mu-\sigma}P_{2,t}^{1-\mu}}{P_{2,1,t+1}^{\mu-\sigma}P_{2,t+1}^{1-\mu}}b_{2,1,t+1}P_{2,1,t+1}^{\mu-\sigma}P_{2,t+1}^{1-\mu}.$$

Now divide both equations by $\tilde{c}_{2,t}$. This gives,

$$\frac{p_{2,1,t}^{*}\left(f\right)}{P_{2,1,t}}\frac{P_{2,1,t}}{P_{2,t}} = \frac{\sigma}{\left(\sigma-1\right)}\frac{z_{2,1,t}^{n}}{z_{2,1,t}^{d}}, \tag{A.59}$$

where

$$z_{2,1,t}^{n} = \left(1-d\right)\tilde{w}_{1,t}\frac{\Omega_{t-1}}{G_{1,t}} + \alpha_c\beta E_t\Lambda_{1,t+1}\left(\frac{P_{2,1,t}}{P_{2,1,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{2,t}}{P_{2,t+1}}\right)^{-\mu}z_{2,1,t+1}^{n}, \tag{A.60}$$

$$z_{2,1,t}^{d} = \left(1-d\right)\Omega_{t-1}Q_t + \alpha_c\beta E_t\Lambda_{1,t+1}\left(\frac{P_{2,1,t}}{P_{2,1,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{2,t}}{P_{2,t+1}}\right)^{1-\mu}z_{2,1,t+1}^{d}. \tag{A.61}$$

Firm $f \in [0,1]$ in country 2 faces an analgous and symmetrical environment. It has the same Calvo probability for price-reset opportunities, $1-\alpha_c$. The production functions for its domestic and foreign markets are,

$$y_{2,t}\left(f\right) = A_{2,t}\ell_{2,t}\left(f\right).$$

Normalize by dividing both sides by $A_{2,t-1}$ to get

$$\tilde{y}_{2,t}\left(f\right) = G_{2,t}\ell_{2,t}\left(f\right). \tag{A.62}$$

Total costs are

$$\frac{\tilde{W}_{2,t}}{P_{2,t}}\ell_{2,t}\left(f\right).$$

Let $\tilde{w}_{2,t} = \tilde{W}_{2,t}/P_{2,t}$ be the real wage. The firm can always adjust its labor input. A Lagrangian for the firm is,

$$L = -\tilde{w}_{2,t}\ell_{2,t}\left(f\right) + \varphi_{2,t}\left(f\right)\left\{G_{2,t}\ell_{2,t}\left(f\right) - \tilde{c}_{2,2,t}\left(f\right) - \tilde{c}_{1,2,,t}\left(f\right)\right\} \tag{A.63}$$

where $\tilde{c}_{1,2,t}\left(f\right) = c_{1,2,t}\left(f\right)/A_{2,t-1}$The Euler equation implied by choosing labor input yield the Euler equation,

$$\tilde{w}_{2,t} = \varphi_{2,t}\left(f\right)G_{2,t}. \tag{A.64}$$

To determine optimal price setting, note that firm f faces these (normalized) domestic and foreign demands for its good

$$\tilde{c}_{2,2,t}(f) = d\left(\frac{p_{2,2,t}\left(f\right)}{P_{2,2,t}}\right)^{-\sigma}\left(\frac{P_{2,2,t}}{P_{2,t}}\right)^{-\mu}\tilde{c}_{2,t}, \tag{A.65}$$

$$\frac{\tilde{c}_{1,2,,t}(f)}{\Omega_{t-1}} = \left(1-d\right)\left(\frac{p_{1,2,t}(f)}{P_{1,2,t}}\right)^{-\sigma}\left(\frac{P_{1,2,t}}{P_{2,t}}\right)^{-\mu}\frac{\tilde{c}_{1,t}}{\Omega_{t-1}}. \tag{A.66}$$

Since output is demand determined, it follows that

$$\tilde{y}_{2,t}(f) = G_{2,t}\ell_{2,t}(f) = \tilde{c}_{2,2,t}(f) + \frac{\tilde{c}_{1,2,t}(f)}{\Omega_{t-1}}, \tag{A.67}$$

where $\tilde{c}_{1,2,t}(f) \equiv c_{1,2,t}(f)/A_{2,t-1}$, and $\tilde{c}_{1,t} \equiv c_{1,t}/A_{1,t-1}$. Firm $f \in [0,1]$ in country 2 firm wants to maximize

$$E_t \sum_{s=0}^{\infty} (\alpha_c\beta)^s M_{2,t,t+s}\left[\frac{p^*_{2,2,t}(f)\tilde{c}_{2,2,t+s}(f)}{P_{2,t+s}} + \frac{p_{1,2,t}(f)\tilde{c}_{1,2,t+s}(f)}{\Omega_{t-1+s}q_{t+s}P_{1,t+s}} - \tilde{w}_{2,t+s}\left[\frac{\tilde{c}_{2,2,t+s}(f) + \tilde{c}_{1,2,t}(f)/\Omega_{t-1}}{G_{2,t+s}}\right]\right]. \tag{A.68}$$

By inspection, the choice for $p^*_{2,2,t}(f)$ is analagous and symmetrical to the choice for $p^*_{1,1,t}(f)$. The first-order condition can be re-arranged to give

$$\frac{p^*_{2,2,t}(f)}{P_{2,2,t}}\frac{P_{2,2,t}}{P_{2,t}} = \frac{\sigma}{(\sigma-1)}\frac{z^n_{2,2t}}{z^d_{2,2t}}, \tag{A.69}$$

where

$$z^n_{2,2,t} = \frac{d\tilde{w}_{2,t}}{G_{2,t}} + \alpha_c\beta E_t\Lambda_{2,t,t+1}\left(\frac{P_{2,2,t}}{P_{2,2,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{2,t}}{P_{2,t+1}}\right)^{-\mu} z^n_{2,2,t+1}, \tag{A.70}$$

$$z^d_{2,2,t} = d + \alpha_c\beta E_t\Lambda_{2,t+1}\left(\frac{P_{2,2,t}}{P_{2,2,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{2,t}}{P_{2,t+1}}\right)^{1-\mu} z^d_{2,2,t+1}. \tag{A.71}$$

To choose $p^*_{1,2,t}(f)$, note that

$$\frac{\partial\tilde{c}_{2,2}(f)}{\partial p_{2,2}(f)} = -\sigma p_{2,2}(f)^{-\sigma-1} dP^{\sigma}_{2,2}\left(\frac{P_{2,2}}{P_2}\right)^{-\mu}\tilde{c}_2,$$

$$\frac{\partial\left(p_{2,2}(f)\tilde{c}_{2,2}(f)\right)}{\partial p_{2,2}(f)} = p_{2,2}(f)^{-\sigma}(1-\sigma)\, dP^{\sigma}_{2,2}\left(\frac{P_{2,2}}{P_2}\right)^{-\mu}\tilde{c}_2,$$

$$\frac{\partial\left[\tilde{c}_{2,2}(f)\right]}{\partial p_{2,2}(f)} = -\sigma p_{2,2}(f)^{-\sigma-1} P^{\sigma}_{2,2}\left(\frac{P_{2,2}}{P_2}\right)^{-\mu} d\tilde{c}_2,$$

$$\frac{\partial\tilde{c}_{1,2}(f)}{\partial p_{1,2}(f)} = -\sigma p_{1,2}(f)^{-\sigma-1}(1-d)\, P^{\sigma}_{1,2}\left(\frac{P_{1,2}}{P_1}\right)^{-\mu}\tilde{c}_1,$$

$$\frac{\partial\left(p_{1,2}(f)\tilde{c}_{1,2}(f)\right)}{\partial p_{1,2}(f)} = p_{1,2}(f)^{-\sigma}(1-\sigma)(1-d)\, P^{\sigma}_{1,2}\left(\frac{P_{1,2}}{P_1}\right)^{-\mu}\tilde{c}_1,$$

$$\frac{\partial\left[\tilde{c}_{1,2}(f)\right]}{\partial p_{1,2}(f)} = -\sigma p_{1,2}(f)^{-\sigma-1} P^{\sigma}_{1,2}\left(\frac{P_{1,2}}{P_1}\right)^{-\mu}(1-d)\,\tilde{c}_1.$$

Re-arrangement of the first-order conditions for choosing $p^*_{1,2,t}(f)$ gives,

$$p^*_{1,2,t}(f) = \frac{\sigma}{(\sigma-1)}\frac{a_{1,2,t}}{b_{1,2,t}}, \tag{A.72}$$

where

$$a_{1,2,t}=E_t\sum_{s=0}^{\infty}(\alpha_c\beta)^s M_{2,t,t+s}\tilde{w}_{2,t+s}P_{1,2,t+s}^{\sigma}\left(\frac{P_{1,2,t+s}}{P_{1,t+s}}\right)^{-\mu}(1-d)\,\frac{\tilde{c}_{1,t+s}}{\Omega_{t-1}G_{2,t,t+s}},$$

$$b_{1,2,t}=E_t\sum_{s=0}^{\infty}(\alpha_c\beta)^s M_{2,t,t+s}\frac{(1-d)\,P_{1,2,t+s}^{\sigma}\left(\frac{P_{1,2,t+s}}{P_{1,t+s}}\right)^{-\mu}\tilde{c}_{1,t+s}}{\Omega_{t-1+s}q_{t+s}P_{1,t+s}},$$

or in terms of the recursive form,

$$a_{1,2,t}P_{1,2,t}^{\mu-\sigma}P_{1,t}^{-\mu}=\tilde{w}_{2,t}\,(1-d)\,\frac{\tilde{c}_{1,t}}{\Omega_{t-1}G_{2,t}}+\alpha_c\beta E_t M_{2,t+1}\frac{P_{1,2,t}^{\mu-\sigma}P_{1,t}^{-\mu}}{P_{1,2,t+1}^{\mu-\sigma}P_{1,t+1}^{-\mu}}a_{1,2,t+1}P_{1,2,t+1}^{\mu-\sigma}P_{1,t+1}^{-\mu}, \tag{A.73}$$

$$b_{1,2,t}P_{1,2,t}^{\mu-\sigma}P_{1,t}^{1-\mu}=(1-d)\,\frac{\tilde{c}_{1,t}}{\Omega_{t-1}Q_t}+\alpha_c\beta E_t M_{2,t+1}\frac{P_{1,2,t}^{\mu-\sigma}P_{1,t}^{1-\mu}}{P_{1,2,t+1}^{\mu-\sigma}P_{1,t+1}^{1-\mu}}b_{1,2,t+1}P_{1,2,t+1}^{\mu-\sigma}P_{1,t+1}^{1-\mu}. \tag{A.74}$$

Dividing both sides of (**??**) and (**??**) by $\tilde{c}_{1,t}$, the optimal reset price is

$$\frac{p_{1,2,t}^{*}(f)}{P_{1,2,t}}\left(\frac{P_{1,2,t}}{P_{1,t}}\right)=\frac{\sigma}{(\sigma-1)}\frac{z_{1,2,t}^{n}}{z_{1,2,t}^{d}}, \tag{A.75}$$

$$z_{1,2,t}^{n}=\frac{(1-d)\,\tilde{w}_{2,t}}{\Omega_{t-1}G_{2,t}}+\alpha_c\beta E_t\Lambda_{2,t+1}\left(\frac{P_{1,2,t}}{P_{1,2,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{1,t}}{P_{1,t+1}}\right)^{-\mu}z_{1,2,t+1}^{n}, \tag{A.76}$$

$$z_{1,2,t}^{d}=\frac{(1-d)}{\Omega_{t-1}Q_t}+\alpha\beta E_t\Lambda_{2,t+1}\left(\frac{P_{1,2,t}}{P_{1,2,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{1,t}}{P_{1,t+1}}\right)^{1-\mu}z_{1,2,t+1}^{d}. \tag{A.77}$$

A.6 Equilibrium conditions

Equating country 1's firm $f's$ supply to its demand gives,

$$G_{1,t}\ell_{1,t}(f)=d\left(\frac{P_{1,1,t}}{P_{1,t}}\right)^{-\mu}\left(\frac{p_{1,1,t}(f)}{P_{1,1.t}}\right)^{-\sigma}\tilde{c}_{1,t},+(1-d)\left(\frac{p_{2,1,t}(f)}{P_{2,1,t}}\right)^{-\sigma}\left(\frac{P_{2,1,t}}{P_{2,t}}\right)^{-\mu}\tilde{c}_{2,t}\Omega_{t-1}, \tag{A.78}$$

and integrating (A.78) gives,

$$G_{1,t}\ell_{1,t}=\left(\frac{P_{1,1,t}}{P_{1,t}}\right)^{-\mu}d\tilde{c}_{1,t}v_{1,1,t}^{p}+\left(\frac{P_{2,1,t}}{P_{2,t}}\right)^{-\mu}(1-d)\,\tilde{c}_t\Omega_{t-1}v_{2,1,t}^{p}, \tag{A.79}$$

where $\ell_{1,t}=\int_0^1\ell_{1,t}(f)\,df$ is total employment at firm f, $v_{1,1,t}^{p}\equiv\int_0^1\left(\frac{p_{1,1,t}(f)}{P_{1,1,t}}\right)^{-\sigma}df$ is a measure of price dispersion for domestic goods in the domestic market, and $v_{2,1,t}^{p}=\int_0^1\left(\frac{p_{2,1,t}(f)}{P_{2,1,t}}\right)df$ is import price dispersion in country2. A fraction α_c of firms are stuck with last period's price $p_{1,1,t-1}(f)$, and $p_{2,1,t-1}(f)$. Since there are a large number of firms charging what they charged last period, it will also be the case that $\int_0^{\alpha_c} p_{1,1,t-1}(f)^{-\sigma} df = \alpha_c P_{1,1,t-1}^{-\sigma}$.[22] Similarly, $\int_0^{\alpha_x} p_{2,1,t-1}(f)^{-\sigma} df = \alpha_c P_{2,1,t-1}^{-\sigma}$.The complementaryfraction $(1-\alpha_c)$ are able to reset price, and they all reset to the same price $p_{1,1,t}^*$. This gives the recursive representation for price dispersion,

$$v_{1,1,t}^p = (1-\alpha_c)\left(\frac{p_{1,1,t}^*}{P_{1,1,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{1,1,t-1}}{P_{1,1,t}}\right)^{-\sigma} v_{1,1,t-1}^p, \tag{A.80}$$

$$v_{2,1,t}^p = (1-\alpha_c)\left(\frac{p_{2,1,t}^*}{P_{2,1,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{2,1,t-1}}{P_{2,1,t}}\right)^{-\sigma} v_{2,1,t-1}^p. \tag{A.81}$$

Price dynamics also have a recursive formulation. To obtain, we illustrate with $P_{1,1,t}$.

$$\begin{aligned}
P_{1,1,t}^{1-\sigma} &= \int_0^1 p_{1,1,t}(f)^{1-\sigma} df \\
&= \int_0^{1-\alpha_c} p_{1,1,t}^{*(1-\sigma)} df + \int_{1-\alpha_c}^1 p_{1,1,t-1}(f)^{1-\sigma} df \\
&= (1-\alpha_c)\, p_{1,1,t}^{*(1-\sigma)} + \underbrace{\alpha_c \int_0^1 p_{1,1,t-1}(f)^{1-\sigma} df}_{\alpha_c P_{1,1,t-1}^{1-\sigma}} \\
&= (1-\alpha_c)\, p_{1,1,t}^{*(1-\sigma)} + \alpha_c P_{1,1,t-1}^{1-\sigma}.
\end{aligned}$$

Analagous calculations hold for country 2.

[22]We have, as definition of the price index, $P_{1,1,t} = \left[\int_0^1 p_{1,1,t}(f)^{1-\sigma} df\right]^{\frac{1}{1-\sigma}}$, which we know can be represented as

$$P_{1,1,t}^{1-\sigma} = (1-\alpha_c)\, p_{1,1,t}^* + \alpha_c P_{t-1}^{1-\sigma}.$$

Now the price dispersion term is defined to be

$$\begin{aligned}
v_{1,1,t}^p &= \int_0^1 \left(\frac{p_{1,1,t}(f)}{P_{1,1,t}}\right)^{-\sigma} df \\
&= \int_0^{1-\alpha_c} \left(\frac{p_{1,1,t}^*}{P_{1,1,t}}\right)^{-\sigma} df + \int_{1-\alpha_c}^1 \left(\frac{p_{1,1,t-1}(f)}{P_{1,1,t}}\right)^{-\sigma} df \\
&= (1-\alpha_c)\left(\frac{p_{1,1,t}^*}{P_{1,1,t-1}}\right)^{-\sigma} + \int_{1-\alpha_c}^1 \left(\frac{p_{1,1,t-1}(f)}{P_{1,1,t-1}}\right)^{-\sigma} \left(\frac{P_{1,1,t-1}}{P_{1,1,t}}\right)^{-\sigma} df \\
&= (1-\alpha_c)\left(\frac{p_{1,1,t}^*}{P_{1,1,t-1}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{1,1,t-1}}{P_{1,1,t}}\right)^{-\sigma} v_{1,1,t-1}^p
\end{aligned}$$

A.6.1 Collection of Equilibrium Conditions

Let us collect the equilibrium conditions here.[23] Consumption bundles:

$$\tilde{c}_{1,1} = \left(\frac{P_{1,1}}{P_1}\right)^{-\mu} d\tilde{c}_1 = \left(\int \tilde{c}_{1,1}(f)^{\frac{\sigma-1}{\sigma}} df\right)^{\frac{\sigma}{\sigma-1}} \tag{A.82}$$

$$\tilde{c}_{1,2} = \left(\frac{P_{1,2}}{P_1}\right)^{-\mu} (1-d)\,\tilde{c}_1 = \left(\int \tilde{c}_{1,2}(f)^{\frac{\sigma-1}{\sigma}} df\right)^{\frac{\sigma}{\sigma-1}} \tag{A.83}$$

$$\tilde{c}_{2,2} = \left(\frac{P_{2,2}}{P_2}\right)^{-\mu} d\tilde{c}_2 = \left(\int \tilde{c}_{2,2}(f)^{\frac{\sigma-1}{\sigma}} df\right)^{\frac{\sigma}{\sigma-1}} \tag{A.84}$$

$$\tilde{c}_{2,1} = \left(\frac{P_{2,1}}{P_2}\right)^{-\mu} (1-d)\,\tilde{c}_2 = \left(\int \tilde{c}_{2,1}(f)^{\frac{\sigma-1}{\sigma}} df\right)^{\frac{\sigma}{\sigma-1}} \tag{A.85}$$

Output:

$$G_{1,t}\ell_{1,t} = \tilde{c}_{1,1,t}v^p_{1,1,t} + \tilde{c}_{2,1,t}v^p_{2,1,t} \tag{A.86}$$

$$G_{2,t}\ell_{2,t} = \tilde{c}_{2,2,t}v^p_{2,2,t} + \tilde{c}_{1,2,t}v^p_{1,2,t} \tag{A.87}$$

Price dispersion:

$$v^p_{1,1,t} = (1-\alpha_c)\left(\frac{p^*_{1,1,t}}{P_{1,1,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{1,1,t-1}}{P_{1,1,t}}\right)^{-\sigma} v^p_{1,1,t-1} = \int_0^1 \left(\frac{p_{1,1,t}(f)}{P_{1,1,t}}\right)^{-\sigma} df \tag{A.88}$$

$$v^p_{2,1,t} = (1-\alpha_c)\left(\frac{p^*_{2,1,t}}{P_{2,1,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{2,1,t-1}}{P_{2,1,t}}\right)^{-\sigma} v^p_{2,1,t-1} = \int_0^1 \left(\frac{p_{2,1,t}(f)}{P_{2,1,t}}\right)^{-\sigma} df \tag{A.89}$$

$$v^p_{2,2,t} = (1-\alpha_c)\left(\frac{p^*_{2,2,t}}{P_{2,2,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{2,2,t-1}}{P_{2,2,t}}\right)^{-\sigma} v^p_{2,2,t-1} = \int_0^1 \left(\frac{p_{2,2,t}(f)}{P_{2,2,t}}\right)^{-\sigma} df \tag{A.90}$$

$$v^p_{1,2,t} = (1-\alpha_c)\left(\frac{p^*_{1,2,t}}{P_{1,2,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{1,2,t-1}}{P_{1,2,t}}\right)^{-\sigma} v^p_{1,2,t-1} = \int_0^1 \left(\frac{p_{1,2,t}(f)}{P_{1,2,t}}\right)^{-\sigma} df \tag{A.91}$$

Profits:

$$\tilde{\Pi}_{1,t} = \left(\frac{P_{1,1,t}}{P_{1,t}}\right)\tilde{c}_{1,1,t} + \left(\frac{q_t P_{2,1,t}}{P_{2,t}}\right)\tilde{c}_{2,1,t}\Omega_{t-1} - \tilde{w}_{1,t}\ell_{1,t} \tag{A.92}$$

$$\tilde{\Pi}_{2,t} = \left(\frac{P_{2,2,t}}{P_{2,t}}\right)\tilde{c}_{2,2,t} + \left(\frac{P_{1,2,t}}{q_t P_{1,t}}\right)\frac{\tilde{c}_{1,2,t}}{\Omega_{t-1}} - \tilde{w}_{2,t}\ell_{2,t} \tag{A.93}$$

[23] $c_{1,1}^{\frac{\sigma-1}{\sigma}} = \int c_{11}(f)^{\frac{\sigma-1}{\sigma}} df = \int \left(d\left(\frac{p_{1,1}(f)}{P_{1,1}}\right)^{-\sigma}\left(\frac{P_{1,1}}{P_1}\right)^{-\mu} c_1\right)^{\frac{\sigma-1}{\sigma}} df = \left(d\left(\frac{P_{1,1}}{P_1}\right)^{-\mu} c_1\right)^{\frac{\sigma-1}{\sigma}} \underbrace{\left(\frac{1}{P_{1,1}}\right)^{1-\sigma}\int p_{1,1,}(f)^{(1-\sigma)} df}_{1}.$

Hence, $c_{1,1} = d\left(\frac{P_{1,1}}{P_1}\right)^{-\mu} c_1$

Price dynamics:

$$1 = \alpha_c \left(\frac{P_{1,1,t-1}}{P_{1,1,t}}\right)^{1-\sigma} + (1-\alpha_c)\left(\frac{p^*_{1,1,t}}{P_{1,1,t}}\right)^{1-\sigma} \tag{A.94}$$

$$1 = \alpha_c \left(\frac{P_{1,2,t-1}}{P_{1,2,t}}\right)^{1-\sigma} + (1-\alpha_c)\left(\frac{p^*_{1,2,t}}{P_{1,2,t}}\right)^{1-\sigma} \tag{A.95}$$

$$1 = \alpha_c \left(\frac{P_{2,2,t-1}}{P_{2,2,t}}\right)^{1-\sigma} + (1-\alpha_c)\left(\frac{p^*_{2,2,t}}{P_{2,2,t}}\right)^{1-\sigma} \tag{A.96}$$

$$1 = \alpha_c \left(\frac{P_{2,1,t-1}}{P_{2,1,t}}\right)^{1-\sigma} + (1-\alpha_c)\left(\frac{p^*_{2,1,t}}{P_{2,1,t}}\right)^{1-\sigma} \tag{A.97}$$

A.7 Steady State

In the steady state, the reset price is equal to the sub-category price index, $p^*_{i,j} = P_{i,j}$, consumption is equal across countries, $\tilde{c}_1 = \tilde{c}_2 = \tilde{c}$, and labor allocations are equal across activities $\ell_1 = \ell_2 = \ell$. In addition, $\Omega = 1, G = 1, Q = 1, P_{1,1}/P_1 = P_{1,2}/P_1 = P_{2,2}/P_2 = P_{2,1}/P_2 = 1$.The household Euler equation in the steady state is identical for both countries,

$$w = \eta \tilde{c} \ell^{\chi}. \tag{A.98}$$

Steady state demand is,

$$\tilde{c}_{1,1} = d\tilde{c} \tag{A.99}$$

$$\tilde{c}_{1,2} = (1-d)\,\tilde{c} \tag{A.100}$$

$$\tilde{c}_{2,2} = d\tilde{c} \tag{A.101}$$

$$\tilde{c}_{2,1} = (1-d)\,\tilde{c} \tag{A.102}$$

Labor input,

$$\ell_1 = \tilde{c}_{1,1} + \tilde{c}_{2,1} \tag{A.103}$$

$$\ell_2 = \tilde{c}_{2,2} + \tilde{c}_{1,2} \tag{A.104}$$

Imposing $\ell_1 = \ell_2 = \ell$ in (A.103) and (A.104) tells us $\tilde{c}_{1,1} = \tilde{c}_{2,2} = d\ell, \tilde{c}_{2,1} = \tilde{c}_{1,2} = (1-d)\,\ell$. Take the steady state values of the **price reset**s and substitute out the wage $w = \eta c \ell^{\chi}$,to get

$$1 = \frac{\sigma\eta}{(\sigma-1)} c \ell^{\chi} \tag{A.105}$$

We also have,

$$\tilde{y}_1 = \tilde{c}_{1,1} + \tilde{c}_{2,1} = \tilde{c} = \tilde{y}_2 = \tilde{c}_{2,2} + \tilde{c}_{1,2.} \tag{A.106}$$

For price setting

$$z_{1,1}^{n} = \frac{d\tilde{W}_1}{(1-\alpha_c\beta)\,P_1} \tag{A.107}$$

$$z_{1,1}^{d} = \frac{d}{1-\alpha_c\beta} \tag{A.108}$$

$$1 = \frac{\sigma}{(\sigma-1)}\frac{z_t^n}{z_t^d} \tag{A.109}$$

$$1 = \frac{\sigma}{(\sigma-1)}\frac{z_{2,1,t}^n}{z_{2,1,t}^d} \tag{A.110}$$

$$z_{2,1}^{n} = \frac{(1-d)\,\tilde{W}_1}{(1-\alpha_c\beta)\,P_1} \tag{A.111}$$

$$z_{2,1}^{d} = \frac{(1-d)}{1-\alpha_c\beta} \tag{A.112}$$

$$1 = \frac{\sigma}{(\sigma-1)}\frac{z_{2,2}^n}{z_{2,2}^d} \tag{A.113}$$

$$z_{2,2,t}^{n} = \frac{d\tilde{W}_2}{(1-\alpha_c\beta)\,P_2} \tag{A.114}$$

$$z_{2,2}^{d} = \frac{d}{1-\alpha_c\beta} \tag{A.115}$$

$$1 = \frac{\sigma}{(\sigma-1)}\frac{z_{1,2}^n}{z_{1,2}^d} \tag{A.116}$$

$$z_{1,2,t}^{n} = \frac{(1-d)\,\tilde{W}_{2,t}}{(1-\alpha_c\beta)\,P_2} \tag{A.117}$$

$$z_{1,2,t}^{d} = \frac{(1-d)}{1-\alpha_c\beta} \tag{A.118}$$

B Equations of interest

Equations of interest are taken from the text above.

Value function

$$\tilde{V}_t = \left\{(1-\beta)\left[\ln(\tilde{c}_t) - \eta\frac{\ell_t^{1+\chi}}{1+\chi}\right] - \frac{\beta}{\alpha}\ln\left[E_t e^{-\alpha\tilde{V}_{i,,t+1}}\right]\right\} + \beta\ln(G_t) \tag{B.1}$$

Auxilary Variable for Value Function

$$U_t = e^{\left(-\alpha\tilde{V}_{t+1}\right)} \tag{B.2}$$

Discount term in Price Setting Equations

$$\Lambda_t = \frac{e^{\left(-\alpha \tilde{V}_t\right)}}{U_{t-1}} \tag{B.3}$$

Real SDF

$$\beta M_{t,t+1} = \beta \left(\frac{\tilde{c}_t}{\tilde{c}_{t+1}}\right) \left(\frac{e^{-\alpha \tilde{V}_{t+1}}}{E_t\left(e^{-\alpha \tilde{V}_{t+1}}\right)}\right) \tag{B.4}$$

Nominal SDF

$$\beta N_{t,t+1} = \beta M_{t,t+1} e^{-\pi_{t+1}} \tag{B.5}$$

Labor supply

$$\eta \tilde{c}_t \ell_t^{\chi} = \tilde{w}_t \tag{B.6}$$

Incomplete Markets: Bonds Consumption Euler Equations

$$\frac{1}{1+i_{1,t}} = \left(\frac{1}{G_{1,t}}\right) E_t N_{1,t+1} \tag{B.7}$$

$$\frac{1}{1+i_{2,t}} = \left(\frac{1}{G_{2,t}}\right) E_t N_{2,t+1} \tag{B.8}$$

$$\left(\frac{1}{1+i_{2,t}}\right)\left(1+\tau_{1,2}\left(\frac{Q_t \tilde{B}_{1,2,t}}{P_{2,t}}\right)\right) = \frac{\beta}{G_{1,t}} E_t M_{1,t+1}\left(\frac{Q_{t+1}}{Q_t}\right)\left(\frac{P_{2,t}}{P_{2,t+1}}\right) \tag{B.9}$$

$$\left(\frac{1}{1+i_{1,t}}\right)\left(1+\tau_{2,1}\left(\frac{\tilde{B}_{2,1,t}}{Q_t P_{1,t}}\right)\right) = \frac{\beta}{G_{2,t}} E_t M_{2,t+1}\left(\frac{Q_t}{Q_{t+1}}\right)\left(\frac{P_{1,t}}{P_{1,t+1}}\right) \tag{B.10}$$

Real Interest Rate

$$r_t = i_t - E_t \pi_{t+1} \tag{B.11}$$

Budget Constraint (Incomplete Markets)

$$\tilde{c}_{1,t} + \frac{\tilde{B}_{1,1,t}}{P_{1,t}} + \frac{Q_t \tilde{B}_{1,2,t}}{P_{2,t}} = \tilde{w}_{1,t}\ell_{1,t} + \frac{\Pi_{1,t}}{P_{1,t}} + \frac{\left(1+i_{1,t-1}\right)\tilde{B}_{1,1,t-1}}{P_{1,t}G_{1,t-1}} \tag{B.12}$$

$$+ \frac{\left(1+i_{2,t-1}\right) Q_t \tilde{B}_{1,2,t-1}}{P_{2,t}G_{1,t-1}} - \frac{\tau_{1,2}}{2}\left(\frac{Q_t\left(\tilde{B}_{1,2,t}\right)}{P_{2,t}}\right)^2 \tag{B.13}$$

$$\tilde{c}_{2,t} + \frac{\tilde{B}_{2,2,t}}{P_{2,t}} + \frac{\tilde{B}_{2,1,t}}{Q_t P_{1,t}} = \tilde{w}_{2,t}\ell_{2,t} + \frac{\Pi_{2,t}}{P_{2,t}} + \frac{\left(1+i_{2,t-1}\right)\tilde{B}_{2,2,t-1}}{P_{2,t}G_{2,t-1}}$$

$$+ \frac{\left(1+i_{1,t-1}\right)\tilde{B}_{2,1,t-1}}{P_{1,t}Q_t G_{2,t-1}} - \frac{\tau_{2,1}}{2}\left(\frac{\left(\tilde{B}_{2,1,t}\right)}{Q_t P_{1,t}}\right)^2$$

Net Zero International Bonds (incomplete markets)

$$0 = \tilde{B}_{1,1,t} + \tilde{B}_{2,1,t}\Omega_{t-1} \tag{B.14}$$

$$0 = \frac{\tilde{B}_{1,2,t}}{\Omega_{t-1}} + \tilde{B}_{2,2,t} \tag{B.15}$$

Complete Markets: Consumption Euler Equations and Real Exchange Rate

$$\frac{1}{1+r_t} = \frac{E_t M_{t,t+1}}{G_t} \tag{B.16}$$

$$\frac{Q_{t+1}}{Q_t} = \frac{M_{2,t,t+1}}{M_{1,t,t+1}}$$

Nominal exchange rate depreciation

$$\frac{S_t}{S_{t-1}} = \frac{Q_t}{Q_{t-1}}\left(\frac{1+\pi_{1,t}}{1+\pi_{2,t}}\right) \tag{B.17}$$

Profits

$$\tilde{\Pi}_{1,t} = \left(\frac{P_{1,1,t}}{P_{1,t}}\right)\tilde{c}_{1,1,t} + \left(\frac{Q_t P_{2,1,t}}{P_{2,t}}\right)\tilde{c}_{2,1,t}\Omega_{t-1} - \tilde{w}_{1,t}\ell_{1,t} \tag{B.18}$$

$$\tilde{\Pi}_{2,t} = \left(\frac{P_{2,2,t}}{P_{2,t}}\right)\tilde{c}_{2,2,t} + \left(\frac{P_{1,2,t}}{Q_t P_{1,t}}\right)\frac{\tilde{c}_{1,2,t}}{\Omega_{t-1}} - \tilde{w}_{2,t}\ell_{2,t} \tag{B.19}$$

Demand functions

$$\tilde{c}_{1,1} = \left(\frac{P_{1,1}}{P_1}\right)^{-\mu} d\tilde{c}_1 \tag{B.20}$$

$$\tilde{c}_{1,2} = \left(\frac{P_{1,2}}{P_1}\right)^{-\mu} (1-d)\,\tilde{c}_1 \tag{B.21}$$

$$\tilde{c}_{2,2} = \left(\frac{P_{2,2}}{P_2}\right)^{-\mu} d\tilde{c}_2 \tag{B.22}$$

$$\tilde{c}_{2,1} = \left(\frac{P_{2,1}}{P_2}\right)^{-\mu} (1-d)\,\tilde{c}_2 \tag{B.23}$$

Equilibrium conditions

$$\tilde{y}_{1,t} = \tilde{c}_{1,1,t} + \tilde{c}_{2,1,t}\Omega_{t-1} \tag{B.24}$$

$$\tilde{y}_{2,t} = \tilde{c}_{2,2,t} + \frac{\tilde{c}_{1,2,t}}{\Omega_{t-1}} \tag{B.25}$$

$$G_{1,t}\ell_{1,t} = \tilde{c}_{1,1,t}v^p_{1,1,t} + \tilde{c}_{2,1,t}v^p_{2,1,t} \tag{B.26}$$

$$G_{2,t}\ell_{2,t} = \tilde{c}_{2,2,t}v^p_{2,2,t} + \tilde{c}_{1,2,t}v^p_{1,2,t} \tag{B.27}$$

$$v^p_{1,1,t} = (1-\alpha_c)\left(\frac{p^*_{1,1,t}}{P_{1,1,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{1,1,t-1}}{P_{1,1,t}}\right)^{-\sigma} v^p_{1,1,t-1} = \int_0^1 \left(\frac{p_{1,1,t}(j)}{P_{1,1,t}}\right)^{-\sigma} dj \tag{B.28}$$

$$v^p_{2,1,t} = (1-\alpha_c)\left(\frac{p^*_{2,1,t}}{P_{2,1,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{2,1,t-1}}{P_{2,1,t}}\right)^{-\sigma} v^p_{2,1,t-1} = \int_0^1 \left(\frac{p_{2,1,t}(j)}{P_{2,1,t}}\right)^{-\sigma} dj \tag{B.29}$$

$$v^p_{2,2,t} = (1-\alpha_c)\left(\frac{p^*_{2,2,t}}{P_{2,2,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{2,2,t-1}}{P_{2,2,t}}\right)^{-\sigma} v^p_{2,2,t-1} = \int_0^1 \left(\frac{p_{2,2,t}(j)}{P_{2,2,t}}\right)^{-\sigma} dj \tag{B.30}$$

$$v^p_{1,2,t} = (1-\alpha_c)\left(\frac{p^*_{1,2,t}}{P_{1,2,t}}\right)^{-\sigma} + \alpha_c\left(\frac{P_{1,2,t-1}}{P_{1,2,t}}\right)^{-\sigma} v^p_{1,2,t-1} = \int_0^1 \left(\frac{p_{1,2,t}(j)}{P_{1,2,t}}\right)^{-\sigma} dj \tag{B.31}$$

Evolution of Prices

$$1 = \alpha\left(\frac{P_{1,1,t-1}}{P_{1,1,t}}\right)^{1-\sigma} + (1-\alpha)\left(\frac{p^*_{1,1,t}}{P_{1,1,t}}\right)^{1-\sigma} \tag{B.32}$$

$$1 = \alpha\left(\frac{P_{1,2,t-1}}{P_{1,2,t}}\right)^{1-\sigma} + (1-\alpha)\left(\frac{p^*_{1,2,t}}{P_{1,2,t}}\right)^{1-\sigma} \tag{B.33}$$

$$1 = \alpha\left(\frac{P_{2,2,t-1}}{P_{2,2,t}}\right)^{1-\sigma} + (1-\alpha)\left(\frac{p^*_{2,2,t}}{P_{2,2,t}}\right)^{1-\sigma} \tag{B.34}$$

$$1 = \alpha\left(\frac{P_{2,1,t-1}}{P_{2,1,t}}\right)^{1-\sigma} + (1-\alpha)\left(\frac{p^*_{2,1,t}}{P_{2,1,t}}\right)^{1-\sigma}$$

Price Indices

$$P_{1,t} = \left[dP^{1-\mu}_{1,1,t} + (1-d)\,P^{1-\mu}_{1,2,t}\right]^{\frac{1}{1-\mu}} \tag{B.35}$$

$$P_{2,t} = \left[dP^{1-\mu}_{2,2,t} + (1-d)\,P^{1-\mu}_{2,1,t}\right]^{\frac{1}{1-\mu}} \tag{B.36}$$

Price setting equations

$$\left(\frac{p_{1,1,t}(f)}{P_{1,1,t}}\right)\left(\frac{P_{1,1,t}}{P_{1,t}}\right) = \frac{\sigma}{(\sigma-1)}\frac{z^n_t}{z^d_t} \tag{B.37}$$

$$z^n_{1,1,t} = \frac{d\tilde{w}_{1,t}}{G_{1,t}} + \alpha_c\beta E_t\Lambda_{1,t,t+1}\left(\frac{P_{1,1,t}}{P_{1,1,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{1,t}}{P_{1,t+1}}\right)^{-\mu} z^n_{1,1,t+1} \tag{B.38}$$

$$z^d_{1,1,t} = d + \alpha_c\beta E_t\Lambda_{1,t+1}\left(\frac{P_{1,1,t}}{P_{1,1,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{1,t}}{P_{1,t+1}}\right)^{1-\mu} z^d_{1,1,t+1} \tag{B.39}$$

$$\left(\frac{p_{2,1}\left(f\right)}{P_{2,1,t}}\right)\left(\frac{P_{2,1,t}}{P_{2,t}}\right)=\frac{\sigma}{\left(\sigma-1\right)}\frac{z^{n}_{2,1,t}}{z^{d}_{2,1,t}} \tag{B.40}$$

$$z^{n}_{2,1,t}=\left(1-d\right)\tilde{w}_{1,t}\frac{\Omega_{t-1}}{G_{1,t}}+\alpha_{c}\beta E_{t}\Lambda_{1,t+1}\left(\frac{P_{2,1,t}}{P_{2,1,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{2,t}}{P_{2,t+1}}\right)^{-\mu}z^{n}_{2,1,t+1} \tag{B.41}$$

$$z^{d}_{2,1,t}=\left(1-d\right)\Omega_{t-1}Q_{t}+\alpha_{c}\beta E_{t}\Lambda_{1,t+1}\left(\frac{P_{2,1,t}}{P_{2,1,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{2,t}}{P_{2,t+1}}\right)^{1-\mu}z^{d}_{2,1,t+1} \tag{B.42}$$

$$\left(\frac{p_{2,2}\left(f\right)}{P_{2,2,t}}\right)\left(\frac{P_{2,2,t}}{P_{2,t}}\right)=\frac{\sigma}{\left(\sigma-1\right)}\frac{z^{n}_{2,2t}}{z^{d}_{2,2t}} \tag{B.43}$$

$$z^{n}_{2,2,t}=\frac{d\tilde{w}_{2,t}}{G_{2,t}}+\alpha_{c}\beta E_{t}\Lambda_{2,t,t+1}\left(\frac{P_{2,2,t}}{P_{2,2,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{2,t}}{P_{2,t+1}}\right)^{-\mu}z^{n}_{2,2,t+1} \tag{B.44}$$

$$z^{d}_{2,2,t}=d+\alpha_{c}\beta E_{t}\Lambda_{2,t+1}\left(\frac{P_{2,2,t}}{P_{2,2,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{2,t}}{P_{2,t+1}}\right)^{1-\mu}z^{d}_{2,2,t+1} \tag{B.45}$$

$$\left(\frac{p_{1,2,t}\left(f\right)}{P_{1,2,t}}\right)\left(\frac{P_{1,2,t}}{P_{1,t}}\right)=\frac{\sigma}{\left(\sigma-1\right)}\frac{z^{n}_{1,2,t}}{z^{d}_{1,2,t}} \tag{B.46}$$

$$z^{n}_{1,2,t}=\left(1-d\right)\frac{\tilde{w}_{2,t}}{\Omega_{t-1}G_{2,t}}+\alpha_{c}\beta E_{t}\Lambda_{2,t+1}\left(\frac{P_{1,2,t}}{P_{1,2,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{1,t}}{P_{1,t+1}}\right)^{-\mu}z^{n}_{1,2,t+1} \tag{B.47}$$

$$z^{d}_{1,2,t}=\frac{\left(1-d\right)}{\Omega_{t-1}Q_{t}}+\alpha_{c}\beta E_{t}\Lambda_{2,t+1}\left(\frac{P_{1,2,t}}{P_{1,2,t+1}}\right)^{\mu-\sigma}\left(\frac{P_{1,t}}{P_{1,t+1}}\right)^{1-\mu}z^{d}_{1,2,t+1} \tag{B.48}$$