Collateralized Debt Networks with Lender Default *

Jin-Wook Chang †

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Abstract

Lehman Brothers’ 2008 bankruptcy spread losses to its counterparties and led to a freezing of lending markets. This contagion occurred even when Lehman was a lender of cash, because collateral for that lending was tied up in the bankruptcy process. I study the implications of such lender default using a general equilibrium model with collateral featuring endogenous leverage, endogenous asset prices, and endogenous network formation. In this model, borrowers diversify their portfolios of lenders due to the possibility of counterparty defaults. However, to diversify counterparties, borrowers have to deal with lenders who lend with lower leverage. This diversification generates positive externalities by reducing systemic risk. Because agents do not fully internalize these externalities, any decentralized equilibrium is constrained inefficient due to under-diversification. The key externalities here, arising from the trade-off between counterparty risk and leverage, are absent in models with exogenous leverage or exogenous networks. I use this framework to analyze the introduction of a central clearing counterparty (CCP). I show that the loss coverage by the CCP reduces diversification incentives and exacerbates the externality problem.

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†Department of Economics, Yale University, Email: jin-wook.chang@yale.edu.
1. Introduction

This paper studies how initial shocks propagate through a network of counterparties in a collateralized debt market, and how counterparties borrow and lend when they account for this contagion channel. This work is motivated in part by the great financial crisis. The financial crisis in 2008 is considered to be a failure of collateralized debt markets, which is the most common form of short-term financing among financial institutions, including repurchase agreement (repo) markets and asset-backed commercial paper (ABCP) markets (Gorton and Metrick, 2012; Copeland et al., 2014). The collapse in prices of subprime mortgages in 2007 had a direct effect on many financial institutions. But the initial shock was exacerbated by the resulting bankruptcy of Lehman Brothers in 2008 which spread the initial losses to Lehman’s counterparties (Copeland et al., 2014; De Haas and Van Horen, 2012; Singh, 2017). I propose a general equilibrium model with collateral featuring endogenous leverage and price, and endogenous network formation to study this problem.

Lender default was important in 2008 as Lehman’s defaults on its lender obligations to return collateral to its borrowers caused a significant loss. Many borrowers from Lehman Brothers, typically hedge funds through prime brokerage accounts, had to over-collateralize their positions to protect the lender in case of borrower default. Right after the bankruptcy of Lehman Brothers, the borrowers did not know when their collateral would be returned to them, nor did they know how much they would recover from the bankruptcy process (Fleming and Sarkar, 2015). This default on lender obligations froze hedge funds’ assets and created sizable losses as a cost throughout the bankruptcy and collateral recovery process. Borrowers reacted by reducing their loans from many counterparties, and this reduction in leverage led to a further plunge in asset prices. Regardless of whether the borrowers stood to recover their assets over the long term, the inability to recover funds in the short term caused problems for certain firms.

A typical form of collateralized debt takes the form of one-on-one interactions between a borrower and a lender because of customization (bespoke) and counterparty-specific contract terms (initial margins). Since every contract is backed by collateral, a collateralized debt network has two transmission channels of shocks – the price channel and the counterparty channel. If the asset price declines, then all the agent’s nominal wealth decreases through the price channel. If an agent defaults and inflicts a cost on its counterparties, then the counterparties’ nominal wealth decreases due to the counterparty channel.

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1 In fact, all of the collateral posted to Lehman was returned to the borrowers.
2 MKM Longboat Capital Advisors closed its $1.5 billion fund partly because of frozen assets, and the chief operating officer of Olivant Ltd. committed suicide, because the fund had $1.4 billion value of assets which was believed to be unlikely to recover from Lehman Brothers (Scott, 2014).
I develop a model to analyze the amplification and feedback between the price channel and the counterparty channel, and how a collateralized debt network is formed endogenously. This paper is the first attempt to endogenize leverage, asset prices, and network formation simultaneously, to the best of my knowledge. When forming a network, borrowing and lending to a specific counterparty, each agent takes into account counterparty solvency. I need to make a number of simplifying assumptions in order to prove theoretical results and make the model computationally solvable. Under the assumptions, I can also perform comparative statics and policy analysis of the introduction of central clearing.

**Model.** The model has six main features. First, agents trade an asset that can be used as collateral in a competitive market. Price changes in the asset market affect each agent’s nominal wealth as a price channel. Second, there is a network of collateralized borrowing and lending. Agents enter one-on-one bilateral contracts specifying who their counterparties are. Third, agents disagree on the fair value of the asset ex ante. Agents buy the asset and use it as collateral to borrow cash because of the belief disagreement. Fourth, the lender of a debt contract holds the collateral and can reuse (rehypothecate) it to borrow money from someone else. An agent can be a lender as well as a borrower at the same time. Fifth, agents are subject to liquidity shocks before paying back their debt. Because of this liquidity shock, agents may have negative nominal wealth and go bankrupt.

Sixth and the most distinct feature is that borrower and lender defaults are treated asymmetrically. Borrowers must put up collateral and failure to repay a promise results in the costless transfer of collateral to the lender. These are no recourse loans. By contrast, when the lender fails to return the collateral, the borrower has to go through a costly process to recover the full collateral from the lender. This lender default cost generates propagation through the counterparty channel.

**Main results.** The first implication of the model is that the lender default cost affects network formation. If there is no lender default cost, then a single intermediation chain is formed endogenously. Since any borrower default propagation is stopped by collateral, due to the non-recourse contracts, lenders are not concerned with the borrower’s balance sheet at

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3The lender default cost is similar to the borrower default cost, which is prevalent in financial network literature. If there is a bankruptcy of one of the counterparties in a contract, then that will incur additional cost in terms of the opportunity cost of time, effort, and litigation costs, which are a deadweight loss to the economy. For example, there were over 100 hedge funds that had prime brokerage accounts or debt obligations under Lehman Brothers, and these accounts were frozen during the bankruptcy of Lehman Brothers. These positions, valued at more than $400 billion, were frozen, which further exacerbated the liquidity shortage of the market [Lleo and Ziemba 2014]. The lender default occurred not only because of rehypothecation but also because of Lehman not holding the collateral in a segregated account [Fleming and Sarkar 2015]. Therefore, the lender default is another significant source of deadweight loss, and the model incorporates this lender default cost as the main channel of propagation in the network.
all. Agents only care about the contracts that are most favorable to them, and in this case, borrowers prefer to maximize their leverage by borrowing from the most favorable lender. The most optimistic agent borrows from the second-most optimistic agent, who borrows from the third-most optimistic agent, and so on.

If there is a lender default cost, then there is a propagation through the counterparty channel. Borrowers diversify their portfolios of lenders due to the possibility of counterparty defaults causing tangible losses. However, there is a trade-off between counterparty risk and leverage as lower counterparty risk comes at the cost of less favorable contracts. Because borrowers have to deal with the lenders who are more pessimistic on the fair value of the asset, they can borrow less than they would have borrowed from an optimistic lender for the same amount of collateral. This endogenous change in the network will create more links than in the case of no lender default cost and reduce the network propagation.

The second implication of the model is that there are positive externalities from diversification. This diversification reduces not only individual counterparty risk but also systemic risk, by limiting the propagation of shocks and resulting price volatility. If a borrower who is also lending to other borrowers becomes safer, then the other borrowers become safer as well so the aggregate counterparty risk becomes smaller. In addition, lower level of debt leads to lower price volatility. Because agents do not fully internalize these externalities, any decentralized equilibrium is constrained inefficient due to under-diversification.

I use this framework to analyze the effect of the introduction of a central clearing counterparty (CCP), which is one of the key elements of the financial system reforms addressed by central banks and financial authorities after the financial crisis in 2008 (Singh, 2010). A CCP novates a contract between two counterparties, i.e., replaces a contract between a borrower and a lender with two different contracts: a contract between the borrower and the CCP, and a contract between the lender and the CCP. Agents still enter debt contracts bilaterally, but the transaction goes through the CCP, and then the CCP covers any losses from counterparty defaults. This novation procedure acts as a pooling of individual counterparty risks into one central agent, as the CCP handles and absorbs any losses from default costs. A CCP can also perform netting of counterparty exposures. If agent A owes $100 to agent B who owes $100 to agent C, then the CCP can net out the obligations. As a result of netting, agent A owes $100 to agent C and agent B has no obligation at all.

The third implication of the model is that the loss coverage by CCP exacerbates the externality problems by eliminating individual agent’s incentive to diversify. The introduction of a CCP eliminates individual agent’s risk concern, since idiosyncratic counterparty risks are covered by the CCP as long as it survives. Because agents are insulated from the counterparty risk, they have no incentive to diversify their counterparties and concentrate
their borrowing with a single most favorable lender. The endogenous response to the new policy will transform the implicit network structure into a single-chain network, which arises in the no default cost case. This situation further exacerbates systemic risk by increasing the riskiness of each agent’s balance sheet and by increasing price volatility due to higher debt and leverage. This reckless borrowing behavior creates huge aggregate default cost and increases systemic risk in the economy. However, netting conducted by the CCP decreases systemic risk even after the change in the network. Therefore, the introduction of a CCP should be performed after comparing the benefits from netting and the costs from the endogenous change in the network.

The fourth implication of the model is that all of the agents hold positive amounts of cash in any equilibrium. Even if there are no lender default costs, agents are still subject to liquidity shocks and the price of the asset may go below the fair value under severe shocks. In the extreme case, all the agents except one might be under huge liquidity shock and have zero cash left. If the surviving agent had small amount cash, then she can buy all the assets in the market with the cash. Thus, every agent holds a positive amount of cash.

**Comparison to exogenous models.** Models with exogenous leverage or exogenous network miss all of the key results in this paper. The positive externalities from diversification, arising from the trade-off between counterparty risk and leverage, are absent in models with exogenous leverage or networks. Exogenous network models do not include endogenous changes in debt networks, and in those models, the CCP only decreases systemic risk via the effect of netting and prevention of second-order bankruptcies. Agents in exogenous leverage models fully diversify their counterparties because the contract terms are fixed and there are no additional benefits of concentrating your counterparties. Introduction of a CCP in those models will have no change even after considering the endogenous response of network formation because agents have no reason to alter their behavior. Thus, the model in this paper captures a novel feature of the endogenous reaction of the market from the combination of the two dimensions.

**Empirical facts.** The model also captures a few empirical facts. First, an increase in counterparty risk leads to an increase in the number of counterparties and a decrease in reuse of collateral and average leverage. Because agents want to diversify more when the counterparty risk increases, the number of linkages increase and the reuse of collateral decreases since the optimists borrow directly from the pessimists. As Singh (2011) documents, the velocity (reuse) of collateral decreased from 3 to 2.4 after the bankruptcy of Lehman Brothers and the average leverage in the Over-the-Counter (OTC) market also went down. Also Craig and Von Peter (2014) shows that the average number of linkages between financial
institutions have increased about 30% over the 4 years after Lehman Bankruptcy. The opposite result happened in unsecured debt markets in which the banks reduced their number of counterparties [Afonso et al. 2011]. This stark comparison shows the role of collateral in network formation.

Second, all of the agents hold positive amounts of cash in any equilibrium. In reality, even the most aggressive investors, such as hedge funds, tend to hold large amounts of daily liquidity that are almost equivalent to cash. In the model, even the most optimistic agents would like to hold some cash to prepare for the case of severe liquidity shocks, which may push down the market price of the asset below the fair price. Furthermore, the amount of cash held by the borrowers can exceed the amount of cash held by lenders. This somewhat counterintuitive result comes from the fact that the potential degree of underpricing is higher for the optimistic borrowers than that of the pessimistic lenders. This property of the equilibrium also matches the empirical facts well. As [Aragon et al. 2017] documented, hedge funds, which are the ultimate borrowers in collateralized lending markets, hold 34% of their assets as daily liquid assets, whereas money market mutual funds, which are the ultimate lenders in collateralized lending markets, hold less than 20% of their assets as daily liquid assets, as documented by [Aftab and Varotto 2017].

Related literature. This paper follows the literature regarding general equilibrium with collateralized debt. The literature started from the seminal paper of [Geanakoplos 1997] and developed through [Geanakoplos 2003], [Geanakoplos 2010a], and [Simsek 2013] which developed models with collateral and how heterogeneous beliefs about the payoff of the asset used as collateral can generate collateralized debt and trade. [Fostel and Geanakoplos 2015] and [Fostel and Geanakoplos 2016] show how endogenous leverage is determined, and the distribution of payoffs affects the dynamics of leverage. [Geerolf 2017] introduces pyramiding, using a contract backed by collateral as collateral, and analyzes its effect on equilibrium. I use a similar idea of heterogeneous beliefs and trades with collateral, although I extend the number of possible reuses of collateral to be any arbitrary number. The literature analyzes how asset prices and leverage are determined in equilibrium but does not answer how the network structure alters the result. The reason for this omission is that in the traditional general equilibrium literature, agents only borrow or lend against the market instead of one specific individual. The model in this paper breaks the anonymity assumption by introducing a network structure.

The financial network literature can be divided into exogenous network literature and endogenous network (network formation) literature. [Eisenberg and Noe 2001] introduced an exogenous network model as a financial network with propagation through the payments, which is extended by [Acemoglu et al. 2015] with a debt financial network without collat-
eral. The payment equilibrium concept employed in such literature is used in this paper as well. Elliott et al. (2014) include discontinuous jumps in the payoffs of agents in the case of bankruptcy, which are also incorporated in this paper. They performed their main analysis via the random graph approach, which is not used in this paper. Di Maggio and Tahbaz-Salehi (2015) incorporates collateral in the network model in the context of the moral hazard problem and a partially endogenous network. This paper is different because of having an endogenous market price for the collateral and free network formation, while not incorporating the moral hazard problem.

Allen and Gale (2000), Chang and Zhang (2016), Elliott and Hazell (2016), Farboodi (2017), and Erol (2017) introduced endogenous network formation in financial networks; they considered the endogenous network structure and possible inefficiencies and systemic risks. Unlike the models in these papers, this paper allows for endogenous contracts in addition to an endogenous asset market and collateral price for both before and after the liquidity shocks. Aymanns et al. (2017) analyzed contagion between two markets of networks. This paper is partially related to this topic because the interaction between the asset market and the collateralized debt market is the main source of price channel contagion in the model. Gai et al. (2011) has analyzed networks with both counterparty channel and price channel as in this paper, but they used quantitative analysis with simulations whereas this paper has theoretical analysis. The literature on financial networks usually focuses on the trade-off between diversification and contagion channel. This trade-off is relatively weak in this paper because the contracts are collateralized, and the explicit asset market operates as the price channel of contagion in the market.

This paper is also related to the literature about central clearing and repo markets. Duffie and Zhu (2011) started the formal discussion about central clearing and its benefits, in addition to possible optimal structures. Biais et al. (2012), Duffie et al. (2015), Arnold (2016) extended the argument both theoretically and empirically to analyze the impact and margin dynamics with a CCP. Biais et al. (2012) uses the search effort cost as the moral hazard problem of clearing members, which is similar to the result of this paper. Because of the pooling of CCP, agents put less effort into searching better counterparties and incur negative externalities to others. This paper, however, has endogenous network changes with a CCP and the externality arises from the leverage decisions rather than search effort decisions. Paddrik and Young (2017) performed systemic risk analysis on the CCP with a network shock transmission model and performed empirical analysis with data. This paper focuses more on analysis with endogenous network change after the introduction of CCP. Gorton and Metrick (2012) documented the important phenomenon of runs on repo markets and how they can cause systemic crises for the financial market. Copeland et al. (2014)
and Krishnamurthy et al. (2014) documented empirical dynamics of bilateral and triparty repo markets and how they reacted differently during the crisis. This paper suggests a possible answer to the puzzle they identified. Gottardi et al. (2017), Infante and Vardoulakis (2018), and Park and Kahn (2018) investigated the lender default problem in collateralized lending and relevant deadweight loss, in addition to contract and intermediation dynamics. This paper incorporates the lender default feature into the endogenous network structure while also having endogenous contract terms and asset prices.

**Layout.** The rest of the paper is organized as the following. Section 2 introduces the model. Section 3 develops results in decentralized OTC market equilibrium. Section 4 illustrates the equilibrium under CCP. Section 5 concludes. The Appendices contain proofs and institutional details on central clearing.

## 2. Model

There are three periods $t = 0, 1, 2$. There are two goods - cash and an asset denoted as $e$ and $h$ respectively. Cash is the only consumption good and storable - one unit of cash at $t$ becomes one unit of cash at $t+1$. The asset yields $s$ amount of cash at $t = 2$ and agents gain no utility from just holding the asset. Each agent has subjective belief on $s$ at $t = 0$. The true $s$ is publicly revealed to everyone at the beginning of $t = 1$ and everyone agrees upon $s$ at $t = 1$. Each agent’s preference is risk-neutral and determined by how much cash she consumes at $t = 2$. Therefore, agents are facing an investment problem. Each agent is endowed with $e_0$ amount of cash and $h_0$ amount of asset.

There are $n$ types of agents and the set of all agents is $N = \{1, 2, \ldots, n\}$. From now on agent $j$ means agent of type $j$. Agent $j$ believes $s = s^j$ with probability 1. This concentrated belief assumption is used in Geerolf (2017) and the assumption is for tractability. Agents are ordered by subjective beliefs on the payoff of the asset as $s^1 > s^2 > \cdots > s^n > 0$. This belief disagreement is the reason why agents trade, borrow, or lend in $t = 0$. The true value of asset payoff $s$ after this information is publicly revealed at the beginning of $t = 1$. However, the asset payoff is realized at $t = 2$ so there is a time gap between uncertainty resolution and payoff realization. The existence of such intermediate period, $t = 1$, plays an important role. Also assume that $ne_0 > nh_0s^1$, so the cash in the market is greater than the equivalent cash value of the supply of the asset even in agent 1’s perspective.

For each agent $j \in N$, there can be negative liquidity shocks $\epsilon_j$ at $t = 1$. This $\epsilon_j$ can be interpreted as senior debt that precedes debt obligations among the agents in the current
economy in the flavor of Diamond and Dybvig (1983). Liquidity shocks are very commonly used in the financial network literature such as in Acemoglu et al. (2015) and Elliott and Hazell (2016), to see how such external shocks propagate through the network. The size of the shock $\epsilon_j$ is independent and identically distributed across $j \in N$ and the common distribution function is denoted as $G_j$ where the support of $G_j$ is $[0, \bar{\epsilon}]$ and differentiable in the support for $j \in N$. Suppose that the upper bound of liquidity shock is large enough that $\bar{\epsilon} > e_0 + h_0 s^1$. The subscript of the distribution just specifies the identity of the agent under that specific shock as the the distributions are iid. The probability of arrival of liquidity shock is $\theta_j$ for any $j \in N$. Even though the distribution of liquidity shocks are the same, the arrival rate of the liquidity shock may differ across agents. Denote $\epsilon_j = 0$ if $j$ did not receive liquidity shock at $t = 1$ (which is also measure zero event even if $j$ received the shock). There are no additional asset endowments at $t = 1$. Without loss of generality, there are no additional endowments of goods at $t = 2$.

2.1. Market Structure

Agents are fully competitive and know each other’s type. Agents are competitive, so every agent believes that she is a price-taker. This assumption is following the tradition of general equilibrium literature and abstracting out from market power and bargaining problem. One way to interpret this assumption is to consider that each agent $j$ consists of a continuum (or hundreds) of homogeneous agents within the same type of $j$ with perfectly correlated uncertainties. Since there is no asymmetric information, the model abstracts out from any adverse selection problem in the market. Also, agents agree to disagree over the payoff $s$ of the asset. The markets for both goods are competitive Walrasian markets. Price of cash is normalized to 1 at any period and price of the asset is $p_t$ for $t = 0, 1, 2$.

At $t = 0$, agents can buy or sell the asset in the competitive market. Also at $t = 0$, agents can borrow cash using an asset as collateral or lend money an taking asset as collateral. All borrowing contracts are 1-period contract between $t = 0$ and $t = 1$. A borrowing contract consists of:

1. the amount of collateral posted,
2. the amount of promised cash per 1 unit of collateral, and
3. the identities of the borrower and the lender.

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4 In fact, even if 1-period contracts between $t = 1$ and $t = 2$ are allowed, agents will only trade borrowing contracts between $t = 0$ and $t = 1$ endogenously. This is because there is no belief disagreement at $t = 1$. 

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Figure 1: Lender’s net payoff from a collateralized debt contract

Denote $y_{ij}$ as promised cash amount in $t = 1$ from $j$ to $i$ per unit of collateral. All borrowing contracts are no recourse so the actual payment from $j$ to $i$ is $x_{ij} = \min\{y_{ij}, \bar{p}_1\}$ per unit of collateral. Because contracts are nonrecourse debt secured by collateral, every borrower with the same promise and collateral makes the same delivery. Shocks to the borrower’s wealth do not get transferred to different deliveries. Therefore, the lenders are insulated from the borrower’s bankruptcy risk. The payoff of the lender from a collateralized debt contract is depicted in figure [1]. Denote $q_{ij}(y_{ij})$ as the amount of cash $i$ lends to $j$ in $t = 0$ per unit of collateral. Subscript of $q_{ij}$ will be omitted from now on. This borrowing amount can be considered as the price of the contract and $q$ is a function of the promise. The gross interest rate is $1 + r(y_{ij}) \equiv \frac{y_{ij}}{q(y_{ij})}$.

Denote $c_{ij}$ as the amount of collateral posted by the borrower $j$ to the lender $i$. This $c_{ij}$ amount of asset is held by the lender until $t = 1$. If the borrower $j$ pays back the full amount of promise $c_{ij}y_{ij}$, then the lender returns the collateral. Otherwise, lender keeps the collateral and the cash value of the collateral is $c_{ij}p_1$. Figure [2] visualizes the flow of cash and collateral. The red solid lines depict collateral flows and the blue dotted lines depict cash flows between the agents. The top left side of the figure visualizes the transaction at $t = 0$ where agent $j$ posts collateral to the lender $i$ in the amount of $c_{ij}$ and $i$ lends cash in the amount of $c_{ij}q(y_{ij})$ to agent $j$. If price of the asset $p_1$ is greater than the promise $y_{ij}$ at $t = 1$, then the borrower $j$ pays the promise and the lender $i$ returns the collateral as seen in the top right side of the figure. The bottom two sides visualize the other case. The bottom left side of the figure has the same transaction at $t = 0$ as in the top case. However, the price of the asset $p_1$ is now lower than the original promise $y_{ij}$ and the borrower does not pay the promise at $t = 1$ as seen in the bottom right side of the figure. The lender $i$ just keeps the collateral for herself when the borrower defaults.
Figure 2: Flow of cash and collateral at $t = 0$ and $t = 1$

Define $C = [c_{ij}]$ and $Y = [y_{ij}]$ as collateral matrix and contract (promise) matrix respectively. A (collateralized) debt network is a double of directed weighted graphs $(C, Y)$ at $t = 0$. That implies a network describes how much each agent borrows or lends from other agents. Let $c_{ii}$ denote the amount of asset agent $i$ holds that has not been used as collateral at $t = 0$. Naturally $y_{ii} \geq s^1$ can be set without loss of generality. Also $C \circ Y$ represents the promise multiplied by its amount traded in the market, where $\circ$ is a Schur (Hadamard) product of matrices. In other words, every $ij$-element of this Schur product matrix $C \circ Y$ is the product of $ij$-elements of $C$ and $Y$, i.e.

$[C \circ Y]_{ij} = c_{ij}y_{ij}$.

2.2. Lender Default

The lenders are obliged to return the collateral when the borrower pays the promise in full. However, some lenders may default on their promise to return collateral to borrowers. This lender default problem is solved by full recourseness of the lender’s obligation towards borrowers. Suppose that the borrowers have full recourse on the lender’s balance sheet when the lender defaults on returning the collateral. If a lender defaults on her promise, then borrowers can take the equivalent amount of cash from the lender’s wealth up to the corresponding value of the assets posted as collateral. Given this full recourseness, the lenders have full incentive to return the collateral. However, if a lender has negative wealth at $t = 1$ because of liquidity shocks, then the lender goes bankrupt and defaults on the contract.
While the borrower default does not incur any deadweight loss to the economy, the lender default involves bankruptcy and incurs deadweight loss towards the borrower.

The financial market has evolved to insulate lenders from borrower counterparty risk over the last few decades. The evolution of securitization has made the cash flows of contracts remote from borrower bankruptcies and significantly eliminated tangible losses that could follow from borrower default (Gorton et al., 2010). However, borrowers are now exposed to the risk of lender default. A good example other than Lehman Brothers, which took almost five years to return all the collateral to the borrowers, is the case of MF Global, a prominent broker-dealer that went bankrupt in 2011. The bankruptcy procedure took nearly five years to resolve all the claims for customers and creditors of MF Global including their borrowers such as hedge funds. By the end of 2015, more than half of the borrowers recovered their collateral in full, but they had to go through the lengthy process of bankruptcy procedure with considerable costs to stay involved and also could not get access to the assets that were used as collateral (SIPC, 2016).

The lender default cost includes the opportunity cost of time and effort caused by involvement into costly and lengthy bankruptcy procedure (similar to bankruptcy and liquidation costs in Elliott et al. (2014) and Acemoglu et al. (2015)), immediate liquidity needs caused by a run of depositors on an agent (bank) with large exposure to the bankrupt agent, a fraction of the collateral assets lost, legal costs for hiring lawyers, reputation cost from the clients of a financial institution and so on. Due to this lender default cost, borrowers face counterparty risk, and they may want to diversify their counterparties.

In the model, the deadweight loss from lender bankruptcy is incorporated as a lender default cash cost \( \zeta(c) \) which is a function of the amount of collateral posted \( c \). If agent \( j \) is borrowing from agent \( i \) and the lender \( i \) goes bankrupt, the borrower \( j \) has to pay \( \zeta(c_{ij}) \) amount of cash as the lender default cost. The lender default cost function is twice-continuously differentiable and \( \zeta(0) = 0, \ \zeta'(0) = 0, \ \zeta'(c) > 0, \ \zeta''(c) > 0, \ \forall c > 0 \). However, I assume that there is no collateral lost during the bankruptcy process, so all of the collateral will be eventually returned to the borrower. This assumption resembles the Lehman bankruptcy case in which all the collateral returned to the original borrowers. In the case of MF Global, some of the borrowers lost a fraction of their collateral, but the majority of the borrowers received all of their collateral. We focus on the case similar to Lehman and abstract from the loss of collateral under the bankruptcy procedure. Also, this assumption is justified by the endogenously arising rehypothecation constraints which will be introduced in the next section.

This cost structure is not only a tractable alternative to assuming risk-aversion of the agents which induces diversification behavior but also a representation of realistic implica-
tions. If a hedge fund posted one treasury bond as collateral to Lehman Brothers, then they might find out where the original collateral went (say Lehman Brothers’ London office which was taken over by Barclays in UK) and retrieve it easily. However, if the hedge fund posted one thousand different bonds as collateral to Lehman Brothers, then this may take much more time and cost to identify and retrieve all the collateral of the hedge fund. For example, Lehman Brothers’ Europe branch had $2.16 billion value of collateral in segregated accounts which are much easily recoverable. In December 2009, a U.K. High Court judge held that clients whose assets should have been segregated but were instead commingled would not receive the same protections as those entities whose asset had actually been segregated. But, in August 2010, an appeals court reversed the decision and ruled that clients whose money should have been segregated would be treated as if their funds had been. The decision slowed the return of assets to clients as it required a longer time of sorting through the bankruptcy procedure (Scott, 2014). As a lot of investment opportunities require large lump-sum cash, this hold of liquidity caused by slow down due to a larger pool of assets to the process would increase the opportunity costs exponentially.

The slope of $\zeta$ can proxy for how risk-averse agents are. For example, risk-averse agents would worry more about lender bankruptcy when their risk-aversion goes up and they would diversify their lenders, or even reduce the amount of the total debt. Similarly, as $\zeta'$ increases faster, the agents are more willing to diversify lenders, or even reduce the amount of the total debt. Therefore, this convexly increasing cost assumption represents the risk-aversion and aligns with the institutional facts of the bankruptcy related costs.

2.3. Rehypothecation

The lender who is holding the collateral can reuse the collateral to borrow cash from someone else. This reuse of collateral is called rehypothecation in the financial market and rehypothecation is prevalent in a wide variety of collateralizable assets including repo contracts of treasury bonds (Singh, 2017). Figure 3 depicts how rehypothecation works where the blue dashed lines are cash flows and the red solid lines are collateral flows. Agent $k$ borrows cash from agent $j$ and posts $c_{jk}$ amount of collateral. Agent $j$ in return lends $c_{jk}q(y_{jk})$ amount of cash to $k$. Now $c_{jk}$ amount of collateral is sitting on $j$’s balance sheet.

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5 As in Infante (2015), there are several difficulties in recovering either the debt payment or collateral from defaulting counterparty which is not due to strategic default. Even though some collateralized lending such as repurchase agreements are bankruptcy remote, there can be frictions in actually enforcing the original agreement when the counterparty is under bankruptcy, especially on the lender side.

6 Note that this cost could have been symmetrically applied to the borrower default as well. The results in this paper mostly hold for the case with the borrower default cost with a similar structure of convexly increasing cost. The only difference it makes is the difference in leverage (contract price) determination.
and she can reuse the collateral to borrow cash from agent $i$. In this contract, $j$ posts $c_{ij}$ amount of collateral and $i$ buys the contract with price $q(y_{ij})$ per collateral and $c_{ij}q_{ij}$ is the total amount of cash lent to $j$. At $t = 1$, the opposite flows of cash and collateral occur. Agent $k$ pays his promised cash amount $c_{jk}y_{jk}$ to $j$ and $j$ pays $c_{ij}y_{ij}$ to $i$. On the other hand, the lenders return their collateral to the borrowers. Agent $i$ returns $c_{ij}$ amount of collateral to $j$ and agent $j$ returns $c_{jk}$ amount of collateral back to $k$. The same collateral can be reused for arbitrary number of times in contrast to other models of rehypothecation such as Geerolf (2017), Gottardi et al. (2017), Infante and Vardoulakis (2018), and Park and Kahn (2018).

In reality, borrowers prefer to allow rehypothecation of their collateral. Even after the fall of Lehman Brothers, most borrowers continued to allow rehypothecation of their collateral (Singh, 2017). The reason for the prevalent use of rehypothecation is that reuse of collateral generates more funding and market liquidity for the borrower themselves. Since the lender can reuse the collateral to borrow money from someone else, the lender can provide even more cash to the borrower for the same collateral and this increases funding liquidity. Furthermore, since the collateral can be used multiple times, the price of the collateral also goes up. This price effect can be thought of as the velocity of capital (Singh, 2010) or the collateral multiplier (Gottardi et al., 2017) which contributes to higher market liquidity of the asset that can be used as collateral.

Figure 3 shows an example of borrowing without rehypothecation and borrowing with rehypothecation. Suppose agents $i, j, k$ all have the same cash endowment of 50 and they have different beliefs as $s^i = 40$, $s^j = 80$, $s^k = 100$. Also suppose that there is no risk in $t = 1$, asset price in $t = 0$ is $p_0 = 100$, and the interest rate is zero. Agent $k$ is the most
optimistic agent and would like to buy the asset as much as possible. Agent $k$ can increase the amount of asset purchase by leveraging more. If agent $k$ wants to borrow from agent $i$, any promise above 40 will not be made by $k$. This is because agent $i$ believes the payoff of the asset is 40 and any promise above 40 will just be the same as 40 due to borrower default under agent $i$’s perspective. Then, the maximum amount of cash $k$ can borrow from $i$ is 40. If agent $k$ wants to borrow from agent $j$, then $k$ will promise up to 80 which provides $k$ a higher leverage than the leverage of borrowing from $i$. However, since agent $j$’s endowment of cash is only 50, $k$ cannot borrow more than 50 from $i$ if there is no rehypothecation allowed. In contrast, if $j$ is allowed to reuse the collateral, then $j$ can borrow 40 from $i$. Now the effective cash available for $j$ becomes $50 + 40 = 90$ and $k$ can borrow 80 from $j$ which is greater than the borrowing amount of 50 under no rehypothecation. The leverage of $k$ with no rehypothecation is $100/(100 - 50) = 2$ while the leverage of $k$ with rehypothecation is $100/(100 - 80) = 5$. Therefore, agent $k$ can increase leverage by 150 percent by allowing rehypothecation and would prefer to do so to increase her return.
2.4. Timeline and Uncertainties

The timeline of the model, which is depicted in figure 5, is the following. Agents are endowed with cash and asset at the beginning of $t = 0$. Agents buy or sell the asset, and simultaneously they form a collateralized debt network at $t = 0$. At the beginning of $t = 1$, asset payoff $s \in S \equiv \{s^1, \ldots, s^n\}$ becomes publicly known. Also at the beginning of $t = 1$, liquidity shocks (negative endowments) of $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_n)$ are realized. Given that some agents may have more cash to be paid to the outside senior debt, (i.e., $\epsilon_j$ is greater than the total cash value of his wealth) some agents may go bankrupt. All the debt is paid back, either by the promise amount or by giving up the collateral, and the collateral is returned to the borrower (if not defaulted). Some borrowers may have to pay additional lender default costs if the counterparties went bankrupt. At the end of $t = 1$, all agents’ final asset holdings are determined. At $t = 2$, the payoff of the asset is realized and agents consume all the cash they have and enjoy utility.

The model has two sources of uncertainty, the revelation of the payoff of the asset $\tilde{s}$ and the realization of negative liquidity shocks for each agent. At the beginning of $t = 1$, both of the uncertainties are resolved. Therefore, there will be no heterogeneous beliefs on the actual asset payoff in $t = 1$ and everyone agrees upon the asset return. However, the actual payoff realization of the asset occurs in $t = 2$ while they still have to pay back the debt they promised for $t = 1$ and towards the liquidity shocks. Without the convex lender default costs or no one going bankrupt, there will be no reason that $p_1$ is different from the commonly known payoff of $s$. However, due to the lack of liquidity (cash) in the market, there may be not enough cash in the market to buy all the assets at the fair price of $s$. Figure 6 is
an example tree that depicts the underlying states and price realizations. There is a finite
number of different $s$ realization and continuum of different $\epsilon$ shocks. Agent 1 believes that
only the first set of states in $t = 1$ occur with positive probability. Agent 2 and 3 believes
that only the second set of states and the third set of states in $t = 1$ occur with positive
probability respectively. The asset price in $t = 1$, $p_1$ depends on the state realization $s$
and liquidity shock realization $\epsilon$. Thus, each agent has their own distribution of prices as
depicted in figure 6. Given the subjective distributions, each agent buys or sells, borrows
or lends for different promises and the equilibrium prices at $t = 0$ for the asset and for all
the promises will be determined. Note that agents agree upon the distribution of liquidity
shocks. Each agent’s subjective belief simply puts different upper bounds on price which is
$s^j$ for agent $j \in N$.

2.5. Optimization Problem and Equilibrium Concept

Now, all the model structure is defined, an agent’s optimization problem can be defined.
Each agent maximizes their expected payoff in $t = 2$ at the beginning of $t = 0$ by choosing
her investment portfolio. Each agent $j \in N$ can

(1) hold cash, amount denoted as $e_j^1$,

(2) can purchase the asset directly and carry it to the next period, in the amount denoted
   as $c_{jj}$,
(3) borrow money from agent \( i \in N \), posting collateral in the amount of \( c_{ij} \) and promise per collateral as \( y_{ij} \),

(4) or lend money to agent \( k \in N \), holding collateral in the amount of \( c_{jk} \) and promise per collateral as \( y_{jk} \).

Note that the portfolio decision does not affect the macro variables such as contract prices \( q(\cdot) \) and asset price \( p_0 \) under agent \( j \)'s perspective because each agent is a price-taker. For a given portfolio, the agent’s expected wealth (cash equivalent of total cash and asset holding of the agent) in \( t = 1 \) is determined. However, these wealth value should be evaluated by the marginal value (utility) of cash for each state. The marginal value of cash could be greater than 1 if the asset price \( p_1 \) is under the fundamental value of the asset \( s \). This underpricing can happen if the economy does not have enough aggregate cash in \( t = 1 \) due to liquidity shocks and bankruptcy-induced lender default costs. If the market is liquidity (cash) constrained, then \( p_1 \) can be lower than \( s \) and the marginal value of cash in such state could be greater than one. Thus, for each realization of liquidity shocks \( \epsilon \), agent \( j \)'s nominal wealth changes but the marginal value of cash also changes as well. Agent \( j \)'s maximization problem becomes as the following,

\[
\max_{e_j^1, \{c_{ij}, y_{ij} \}_{i \in N}, \{c_{jk}, y_{jk} \}_{k \in N}} E_j \left[ \left( e_j^1 - \epsilon_j + c_{jj}p_1 + \sum_{k \in N \setminus \{j\}} c_{jk} \min \{ y_{jk}, p_1 \} \right) - \sum_{i \in N \setminus \{j\}} c_{ij} \min \{ y_{ij}, p_1 \} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) \mathbb{1}[p_1 > y_{ij}] \right] s \frac{p_1}{p_1} \right]^+ 
\]

s.t.

\[
c_{jj} + \sum_{k \in N \setminus \{j\}} c_{jk} \geq \sum_{i \in N \setminus \{j\}} c_{ij},
\]

\[
e_0 + h_0 p_0 = e_j^1 - \sum_{i \in N \setminus \{j\}} c_{ij} q(y_{ij}) + \sum_{k \in N \setminus \{j\}} c_{jk} q(y_{jk}) + c_{jj} p_0
\]

where \( B(\epsilon) \) is the set of bankrupt agents for given liquidity shock realization \( \epsilon \), \( [\cdot]^+ = \max\{\cdot, 0\} \), and \( \mathbb{1}[\cdot] \) is an indicator function. The first constraint is the collateral constraint and the second constraint is the budget constraint. The collateral constraint is the constraint that agent \( j \) should have enough assets, either from direct purchase or from collateral posted by \( k \) who is borrowing from \( j \), to post collateral. The underlying implication of the collateral constraint is the same as Geanakoplos (1997), but this model keeps track of the identity of borrowers and lenders to analyze the network effect and rehypothecation structure.
The equilibrium concept that will be used throughout the paper is a hybrid version of general equilibrium with price functions that are affected by the network structure as the following.

Definition 1. For a given economy \((N, (s^j, \theta_j, e_0, h_0)_{j \in N}, \zeta, G)\), a sextuple \((C, Y, e_1, p_0, \tilde{p}_1, q)\) where \(C, Y \in \mathbb{R}^n_+ \times \mathbb{R}^n_+, e_1 \in \mathbb{R}^n_+, p_0 \in \mathbb{R}_+,\) and functions \(p : \mathbb{R}_+ \to \mathbb{R}_+\) and \(q : \mathbb{R}_+ \to \mathbb{R}_+\) is a network equilibrium if \((C, Y, e_1)\) solves agent maximization problem while satisfying budget and collateral constraints, markets are cleared as \(c_{ij}\) for the solution of agent \(j\) is the same as \(c_{ij}\) for the solution of agent \(i\) for all \(i, j \in N\), asset market clears as \(\sum_{j \in N} c_{jj} = H \equiv \sum_{k \in N} h_0\), and \(p_0\) and \(\tilde{p}_1\) is realized at \(t = 1\) for the given network structure and market clearing in \(t = 1\).

The network dynamics is essentially occurring in \(t = 1\) through repayment and default costs from bankruptcy. This \(t = 1\) network effect also feeds back into \(t = 0\) optimization decisions which lead to network formation. The exact market clearing in \(t = 1\) will be defined in the beginning of the following section.

3. Network Equilibrium

This section characterizes the network equilibrium, the general equilibrium with collateral debt network formation and payment realization. The payment realization in \(t = 1\) shows how the network structure and shocks affect the market price and the final wealth (and equivalently payoffs) of the agents. The endogenous network of collateralized debt contracts in \(t = 0\) is formed based on the consideration of the properties of a network and how agents clear markets of the asset and contracts. This section will solve for the equilibrium backwards - first, analyze the network property in \(t = 1\) and then derive the optimal contract decisions and network formation in \(t = 0\) for the given expected price distribution.

3.1. Payment equilibrium in \(t = 1\)

First, I solve the problem starting from the last period. Since \(t = 2\) is merely the realization of the payoff of the asset and utility, we move to \(t = 1\) and solve for the equilibrium prices and wealth for a given debt network \((C, Y)\), cash holdings \(e_1\), shock realization \(\epsilon\), and payoff revelation of the asset \(s\). Each agent pays back and gets paid according to the payment rule \(x_{ij}\), and after the repayment subtracted by liquidity shock \(\epsilon_j\), the agent’s total nominal wealth (evaluated by cash) could be negative. An agent with negative wealth goes bankrupt, and their wealth does not enter into the demand side of the market. Only agents with positive
post-payment wealth can enter the asset market at \( t = 1 \) and affect the market price. If the asset is underpriced \((p_1 < s)\), then all agents will spend all their wealth to buy the asset and the price that makes the aggregate wealth equal to \( nh_0 p_1 \) will be the market clearing price. This market clearing price and allocation can be defined as payment equilibrium\(^7\) which is an intermediate equilibrium of \( t = 1 \) as the following.

**Definition 2.** For a given period-1 economy of \((N, C, Y, e, \epsilon, s)\), a payment equilibrium is \((M, h^2, p_1)\) where \( M \) is the wealth vector, \( h^2 \) is the asset holding vector, and \( p_1 \) is the price of the asset such that \( M \) satisfies the payment rule, \( h^2 \) is determined after the bankruptcy and default costs, and \( p_1 \) makes the asset market clear.

From the payment rule \( x_{ij} = \min\{y_{ij}, p_1\} \), contracts with promise of \( y_{ij} > p_1 \) will be paid less than the face value that is just the price of the asset, and the contracts with promise of \( y_{kl} \leq p_1 \) will be paid in full for any \( i, j, k, l \in N \). If an agent’s total wealth is negative, then the agent cannot even fulfill its obligations to the senior outside debtors (i.e., the liquidity shock of \( \epsilon_j \)) and the agent will go bankrupt. The model considers any event or cost related to the bankruptcy is outside of the collateral debt network other than the counterparty (lender) default cost\(^8\). As defined before, \( B(\epsilon) \) is the set of agents who go bankrupt under the shock vector \( \epsilon \). The market clearing price will indirectly determine this set because in some cases agent could have survived in high \( p_1 \) but would go bankrupt in low \( p_1 \). Thus, this set might not be well defined as there could be multiple sets that constitute payment equilibria. Among multiple \( B(\epsilon) \)'s, selecting the smallest set of \( B(\epsilon) \) that holds as payment equilibrium implies selecting the maximum price payment equilibrium. This equilibrium selection rule is well-defined which will be shown later in this subsection. Omit the subscript of \( p_1 \) from now on throughout this subsection since this subsection only focuses on \( t = 1 \).

The total nominal wealth of agent \( j \) after all the payment is

\[
m_j(p) = e_j^1 - \epsilon_j + c_{jj}p + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{p, y_{jk}\} - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) 1[p > y_{ij}]\]

\(^7\) In the exogenous debt network literature stemming from [Eisenberg and Noe \(2001\)] and the following papers such as [Acemoglu et al. \(2015\)], their main equilibrium concept is almost the same as the payment equilibrium (the name which I coined from this literature) in this paper. This intermediate step also provides a comparison between the model in this paper and the literature of exogenous financial networks and propagation dynamics. The crucial difference of the model in this paper is that the model here has an additional market for the asset used as collateral which induces the network propagation and the asset price feedback to each other.

\(^8\) In a similar logic, suppose that the agents will fulfill their promises to each other unless they go bankrupt. This structure means the collateralized debt is a contingent contract (ultimately) by the use of collateral and the lenders will try to fulfill their obligations of returning the asset even under the situation when they have to pay the cost of retrieving the collateral from a bankrupt counterparty.
where \( e_j^1 - \epsilon_j \) is the remaining cash you have from \( t = 0 \) subtracted by (possibly zero) liquidity shock \( \epsilon_j \). To consider the wealth that is actually effective in demand when we compute the equilibrium, define the effective nominal wealth of each agent as

\[
[m_j(p)]^+ = \left[ e_j^1 - \epsilon_j + c_{jj}p + \sum_{k \in N\setminus\{j\}} c_{jk} \min\{p, y_{jk}\} - \sum_{i \in N\setminus\{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) 1[p > y_{ij}] \right]^+
\]

If \( m_j(p) < 0 \), then \( j \in B(\epsilon) \) and agent \( j \) will liquidate all of their holdings to pay \( \epsilon_j \). The bankruptcy decision prevents negative asset holding and the equilibrium asset holding \( h^2_j \) is determined by

\[
h^2_j = \frac{[m_j(p)]^+}{p}
\]

when \( p < s \). If \( p = s \), \( h^2_j \leq \frac{[m_j(p)]^+}{p} \) but the asset holding cannot be pinned down and also irrelevant to pin down due to invariance in final utility at \( t = 2 \) between holding the asset by paying the price and holding the equivalent amount of cash.

The aggregate cash value of the supply of the asset should equal to the aggregate cash value of the demand of the asset. As long as \( p \leq s \) there will be an agent who would spend all the excess cash they have to buy the asset. The cash value of the aggregate supply is

\[
\sum_{j \in N} c_{jj}p = nh_0p.
\]

Note that the equality is coming from the market clearing from \( t = 0 \). The cash value of the aggregate demand is a function of the asset price as well. If the price reaches \( s \) and there is enough money to buy up all the supply, then that is an equilibrium. Therefore, the aggregate effective cash value of demand in the market becomes

\[
\sum_{j \in N} \left[ e_j^1 - \epsilon_j + c_{jj}p - \sum_{i \in N\setminus\{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) 1[p > y_{ij}] + \sum_{k \in N\setminus\{j\}} c_{jk} \min\{p, y_{jk}\} \right]^+.
\]

Therefore, the market clearing condition that determines the price is going to be as the
The aggregate effective nominal wealth increases as the price increases and the lender default cost decreases. But, then again there is a feedback from the nominal wealth to the price. Note that the equality should hold unless \( p = s \). We can interpret this as the price is going to be the level that makes the aggregate amount of liquidity that can cover both all available assets and the costs from defaults, cash-in-the-market pricing. Another case to consider is when \( p = 0 \). If there is any extra cash left to the economy, then \( p = 0 \) cannot be true. However, if there is no cash left in the economy after paying out the liquidity shocks and default costs, then \( p = 0 \) can occur, and the market is broken down and all the asset holdings become indeterminate as in the case of \( p = s \). However, in the individual agent’s perspective, the marginal utility of cash is infinity as the limit of marginal utility is \( \lim_{p \to 0} \frac{s}{p} \to \infty \). Because of this property, (which is similar to the Inada conditions) every agent holds a positive amount of cash at \( t = 0 \), but this will be formally proven in the next subsection.

In order to make the collateral is actually returned to the borrower and also available to the lender in the case of default of promise, we have to impose rehypothecation constraints on the network. In fact, these rehypothecation constraints are not binding in the endogenous network formation in \( t = 0 \) which will be showed in the next subsection. However, I assume the result for now as a restriction and then analyze what happens under the class of networks that satisfy rehypothecation constraints. For given level of payment \( \hat{y} \), agent \( j \) should hold enough collateral (either held directly as \( c_{jj} \) or indirectly by lending as \( c_{jk} \)) that promises greater than or equal to \( \hat{y} \) to cover all the debt promised to pay \( \hat{y} \) or greater value. Formally, the rehypothecation constraint for agent \( j \) is

\[
\sum_{i \in N \setminus \{j\}} c_{ij} \leq \sum_{k \in N} c_{jk} \quad \text{for any } \hat{y} \in S. \tag{4}
\]

This rehypothecation constraint is equivalent to only allowing pyramiding of contracts, that is promising a delivery using another contract as a collateral introduced by Geanakoplos (1997). If agent \( j \) uses the contract by agent \( k \) with promise of \( y_{jk} \) as collateral to promise \( y_{ij} \) to agent \( i \), the actual delivery becomes \( \min\{y_{ij}, \min\{p, y_{jk}\}\} = \min\{p, \min\{y_{ij}, y_{jk}\}\} \), so
\( y_{ij} \leq y_{jk} \) to be a non-trivial pyramiding of the contract.

For example, if agent \( k \) promises 20 to \( j \) but \( j \) reuses the collateral and promises 30 to \( i \), then, agent \( j \) violates the rehypothecation constraint. In this case, agent \( j \) might not have enough cash to repay \( i \), if the price is between 20 and 30, so he cannot receive the collateral from \( i \) and then return it to \( k \). The rehypothecation constraint guarantees that if the ultimate borrower (collateral provider) fulfills her promise, then the intermediary (who reuses the collateral) also has enough cash to fulfill his promise to the ultimate lender (cash provider). We call a network satisfies rehypothecation constraints if the above equation (4) holds for all \( j \in N \). Note that rehypothecation constraint implies collateral constraints.

Under the rehypothecation constraints, we can interpret the market clearing condition in a more intuitive way. The negative liquidity shocks \( \epsilon \) destroys the aggregate available cash. The destruction of cash for the demand can be decomposed into three factors:

1. each agent’s liquidity shock \( \epsilon_j \),
2. lender default costs for a non-bankrupt agents from bankrupt lenders \( \sum_{i \in B(\epsilon)} \zeta(c_{ij}) \),
3. second order bankruptcy from the first two effects which amplifies second factor.

The second and third factors create a feedback loop in the market through the price channel and the counterparty channel similar to debt network models without collateral. Note that the first factor is bounded above by the nominal wealth of the agent before the negative liquidity shock, \( \mu_j(p) \equiv m_j(p) + \epsilon_j \), because any excess liquidity shock still makes the same effective nominal wealth of zero. For a given price \( p \), the actual destruction of cash from liquidity shock to \( j \) relevant to the demand of the asset market is \( \min\{\epsilon_j, \mu_j(p)\} \).

For a given price \( p \), the excess cash of the network can be computed as the original cash subtracted by the total destruction of cash. Hence, the remaining cash becomes

\[
RM \equiv \sum_{j \in N} c_j^j - \sum_{j \in N} \min\{\epsilon_j, \mu_j(p)\} - \sum_{i \in B(\epsilon)} \sum_{j \in N} \zeta(c_{ij}) + \sum_{j \in B(\epsilon)} \left( \sum_{k \in N} c_{jk}P - \sum_{i \notin j} c_{ij}P \right)
\]

which is the cash saved from \( t = 0 \) subtracted by the liquidity shocks and lender default costs except for the last term. The last term controls for the over-subtraction from \( \mu_j(p) \)

\( \text{Rearranging the equation results in} \)

\[
RM = \sum_{i \notin B(\epsilon)} (\epsilon_i^1 - \epsilon_i) + \sum_{j \in B(\epsilon)} \sum_{i \notin j} c_{ij}y_{ij} - \sum_{j \in B(\epsilon)} \sum_{k \in N} c_{jk}y_{jk} - \sum_{j \in B(\epsilon)} \sum_{k \notin B(\epsilon)} \sum_{p \geq y_{jk}} \zeta(c_{jk}).
\]
because the cash used to buy the collateral held by the bankrupt agent still remains in the
market. On the other hand, the amount of collateral that are sold in the market is the
amount of collateral from bankrupt agents’ balance sheets, that is the total fire-sales of the
assets denoted as

\[ FS \equiv \sum_{j \in B(\epsilon)} \sum_{k \in N} c_{jk} - \sum_{j \in \bar{B}(\epsilon) \setminus B(\epsilon)} \sum_{i \neq j \setminus y_{ij}} c_{ij}. \]

Suppose that the price \( p \) is neither 0 or \( s \). Then, the market clearing condition, equation (2) becomes

\[ \pi(p) = \frac{RM}{FS} \]  

which is the remaining cash divided by the total fire-sales of the assets that are under
bankrupt agents’ balance sheets.

By rehypothecation constraints, the denominator is always nonnegative. However, if
there are no assets to be bought (i.e. denominator of \( \pi(p) \) is zero) then the price of the asset
will be trivially its fair value price \( s \). If there is no asset to be sold by the bankrupt agent,
then there is no reason to lower the price of the asset. If there are enough cash in the market
to cover the extra supply (fire-sales) with the fair price (i.e. \( \pi(p) > s \)) then the price is also
set as fair value price \( s \). If there are some leftover cash after the payments and costs that is
not sufficient to buy all of the assets in fair price, then the market price will be \( \pi(p) \) which
we define as liquidity constrained (positive) price of the asset.

The post-shock market clearing condition, equations 2 and 3 can be rearranged to obtain
the price equation as below.

\[ p = \begin{cases} 
0 & \text{if } \sum_{j \notin B(\epsilon)} e_{j} < \sum_{i \in D \setminus B(\epsilon)} \epsilon_{i} \text{ for any } p \in [0, s] \\
\pi(p) & \text{if } \pi(p) > s \text{ or } B(0) = \emptyset \\
s & \text{otherwise.} 
\end{cases} \]  

By the nature of the default costs, agents who lent towards the bankrupt agent (who will
be less optimistic agents later in the endogenous network) do not care about this cost. Also,
borrowers who promised greater than the revealed asset payoff \( s \) will default and they would
be irrelevant in network propagation effect. The only case when the counterparty risk is
relevant is that the agent promised less than or equal to \( s \), but the counterparty (lender)
goes bankrupt. The second order bankruptcy, a lender’s counterparty goes bankrupt, is only
relevant if that makes the first order lender go bankrupt as well. For example, consider a simple 3-agent lending chain and all the promises are below \( s \). If \( k \) has borrowed from \( j \) who borrowed from \( i \), \( k \) is affected by \( i \)’s bankruptcy only if \( \zeta(c_{ij}) + \epsilon_j > \epsilon_j^1 + c_{jk}y_{jk} + c_{ij}(p - y_{ij}). \) Then, \( \zeta(c_{jk}) \) is incurred in \( k \)’s balance sheet. Thus, the higher the asset payoff realization \( s \) is, the more the counterparty risk is relevant. The overall price could be still higher but the amount of price decline from the aggregate counterparty bankruptcy is also higher. When it comes to \( t = 0 \) decision of the agents, this property affects their cash holding and investment decisions which will be described later.

Under rehypothecation constraints, I can show that a payment equilibrium always exists and the set of equilibrium prices is a complete lattice.

**Proposition 1 (Existence and Lattice Equilibrium Prices).** For any given collateralized debt network \((N,C,Y,e_1,\epsilon,s)\) with \( C > 0 \) that satisfies rehypothecation constraints, there exists a payment equilibrium \((M,h^2,p_1)\). Furthermore, among the possible equilibria there always exist the maximum equilibrium that is \((M,h^2,p_1)\) where \( p_1 \) is the highest price among all possible equilibrium prices and the minimum equilibrium that is \((M,h^2,p_1)\) where \( p_1 \) is the lowest price among all possible equilibrium prices.

All the proofs are relegated to the appendix. The intuition of the proof is the following. The delivery \( x_{ij} = \min\{y_{ij}, p\} \) towards the lender increases as the price \( p \) increases. By rehypothecation constraint, every borrower or intermediary payoff also increases as \( p \) increases. Thus, every individual nominal wealth increases in \( p \). Since individual nominal wealth \( m_j \) is increasing in \( p \), as shown by lemma \( 4 \) in the proof, the aggregate nominal wealth is increasing in asset price \( p \) and decreasing in lender default cost \( \zeta \). Since increase in wealth also means bankruptcy is less likely, the lender default cost also decreases when \( p \) increases. Therefore, every single variable that is included in the market clearing condition (weakly) increases in price \( p \). Although the equilibrium is determined by the vector of all the wealths, we can summarize each equilibrium by the price level \( p \) and there exists a fixed point price that clears the market.

Although we can show that the payment equilibrium always exists, we cannot guarantee its uniqueness. This multiplicity is mainly due to the jumps in \( m_j(p) \) at the point of bankruptcy of other agents. The actual bankruptcy set may also depend on the market clearing price as \( B(\epsilon|p) \). An agent may have \( m_j(p) > 0 \) for given price \( p \) and bankruptcy set \( B(\epsilon|p) \) but her wealth may be negative at \( p' \) and given bankruptcy set \( B(\epsilon|p') \) so \( m_j(p') < 0 \). Her bankruptcy will generate even more second-order bankruptcy costs and make \( p' \) to be true. The following proposition summarizes this relation between multiplicity (and uniqueness) and bankruptcy.
Proposition 2 (Multiplicity and Bankruptcy). For any given collateralized debt network $(N, C, Y, e_1, \epsilon, s)$ with $C > 0$ that satisfies rehypothecation constraints, there may be multiple equilibria. If $p$ and $p'$ are two distinct prices from the two different payment equilibria, then $B(\epsilon|p) \neq B(\epsilon|p')$.

Figure 7 depicts an example of multiple equilibria. Define the sum of lender default costs coming from agent $l$'s bankruptcy for price $p$ as $\beta_l(p) \equiv \sum_{j \in N} \zeta(c_{lj}) \mathbb{1}_{p > y_{lj}}$. Also denote the nominal wealth substracted by negative liquidity shock as $\mu_l(p) \equiv m_l(p) + \epsilon_l$. There are kinks at prices in which each contract defaults and discontinuous jumps at prices in which each agent goes bankrupt. The first type of kinks occurs for $p < y_{ij}$ which affects both $m_i(p)$ and $m_j(p)$ and the second type of jump occurs at the point where $m_j(p) = 0$. From the second statement of proposition 2 and equation (11) in the proof, the existence of lender default cost plays a significant role in generating multiplicity and also the counterparty contagion effect through the second-order bankruptcy. Due to the multiplicity and the lattice property, we assume $B(\epsilon)$ to be the bankruptcy set from the maximum equilibrium price, i.e. $B(\epsilon) \equiv B(\epsilon, \bar{p})$, from now on. Also, a maximum equilibrium selection rule means choosing the equilibrium with the maximum equilibrium price. We will focus on the results on equilibrium with maximum equilibrium selection rule from now on as in Elliott et al. (2014).

Trivially, if there is no default cost, that is $\zeta(c) = 0$ for any $c$, then the payment equilibrium is unique from the second statement of proposition 2. Also without a default cost, change in counterparty connections does not matter as long as the total borrowing and
lending amount remain the same. The following proposition states this property.

**Proposition 3 (Counterparty Irrelevance).** If there is no lender default cost, that is \( \zeta(c) = 0 \) for all \( c \geq 0 \), then the payment equilibrium is unique for any given network. Furthermore, two networks \((C, X)\) and \((\hat{C}, \hat{X})\) with the same indegrees and outdegrees, that is \( 1(C \circ X) = 1(\hat{C} \circ \hat{X}) \) and \((C \circ X)1 = (\hat{C} \circ \hat{X})1\), will have the same payment equilibrium.

This proposition shows the necessity of assuming the existence of a lender default cost (or any counterparty risk) in order to generate interesting interaction among agents. Figure 8 shows an example of two different networks with the same equilibrium outcome. If all the links have the same weight, that is \( c_{ij} = c \) and \( y_{ij} = y \) for all \( i, j \in N = \{1, 2, 3, 4\} \), then the two networks have the same indegrees and outdegrees. Because of the absence of a default cost, agent’s individual connection does not matter as long as the total borrowing and lending for each agent are the same. The two networks will have the same equilibrium price and allocation.

The result is not so surprising since the main reason for using a collateral is to insulate the lender from the counterparty risk. A collateralized debt network has no counterparty risk as in the anonymous market. This irrelevance result can be extended to a model with default cost caused by borrower’s default. For example, the actual payment when retrieving the collateral from the borrower in case of default can be less than the actual value of the collateral \( p \) which is \( (1 - \phi)p \) with \( \phi > 0 \) due to fire sales cost or collateral seizure cost. The existence of default costs will only scale down the entire values of the collateral for the lender (i.e. less lending in aggregate) but would not change the irrelevance result.

From now on, define *systemic risk* as the expected difference between the fair value of the asset and the actual price of the asset, \( \int (s - p) dG(\epsilon) \) where \( G \) is the convolution of \( G_j \) for all \( j \in N \). This notion of systemic risk is following the definition of *systemic loss* in value defined in Glasserman and Young (2016). Even though the fundamental value of the asset is \( s \), underpricing of the asset, \( s - p \), comes from the liquidity shocks and lender default costs which vary by the network connections. The systemic risk definition here is taking the ex ante expected value of the systemic losses for each subjective belief. The sum
of all the systemic risks for each ex ante beliefs will be closely related to the social welfare computation which we will talk later in the next subsection. By this definition, proposition 4 implies that network with higher aggregate debt has a higher expected systemic risk. The aggregate default costs after the revelation of \( s \) and realization of \( \epsilon \) will determine the difference between the two values and the difference represents how severe the mispricing is due to the total sum of deadweight losses.

Now we briefly describe how to solve the equilibrium in quantitative analysis under maximum equilibrium selection rule.

**Equilibrium Search Algorithm:** Consider the following algorithm of finding the maximum payment equilibrium:

0. Set \( B^{(0)}(\epsilon) = \emptyset \). Start with step 1.

1. For any step \( k \), given \( B^{(k-1)} \), compute \( p^{(k)} \) that satisfies the market clearing condition.

2. For given \( p^{(k)} \), compute \( m_j(p^{(k)}) \) with given \( B^{(k-1)} \) and update \( B^{(k)} \) with the new \( m_j(p^{(k)}) \).

3. If \( B^{(k-1)} = B^{(k)} \), then we have the maximum equilibrium. Otherwise, move to next step \( k + 1 \) and repeat procedures 1 and 2.

This algorithm guarantees to find the maximum payment equilibrium price of the given network. Also, the algorithm finishes within \( n \) steps because the second order bankruptcy (or cascades) could only occur the maximum of \( n - 1 \) times if it happens for one agent by one.

### 3.2. Network Formation in \( t = 0 \)

Given the results from \( t = 1 \), agents form beliefs on the distribution of \( p_1 \) and \( B(\epsilon) \) under shock realizations. As discussed in the model section, agent \( j \) solves the maximization problem:
max \( E_j \left[ \left( e_j^1 - \epsilon_j + c_{jj}p_1 + \sum_{k \in N \setminus \{j\}} c_{jk} \min \{y_{jk}, p_1\} \right) - \sum_{i \in N \setminus \{j\}} c_{ij} \min \{y_{ij}, p_1\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) \frac{1}{p_1} \right] \)

s.t.
\[
\begin{align*}
c_{jj} + \sum_{k \in N \setminus \{j\}} c_{jk} & \geq \sum_{i \in N \setminus \{j\}} c_{ij}, \\
e_0 + h_0p_0 & = e_j^1 - \sum_{i \in N \setminus \{j\}} c_{ij}q(y_{ij}) + \sum_{k \in N \setminus \{j\}} c_{jk}q(y_{jk}) + c_{jj}p_0
\end{align*}
\]

An agent has five different ways to use her budget: holding cash, buying the asset, buying the asset with leverage, lending cash to others, and lending cash with leverage. Agent \( j \)'s return on holding cash is \( E_j \left[ \frac{s^{-}}{p_1} \right] \). From now on, omit the + superscript and denote \( E_j \left[ \cdot \right] \) as nonnegative expected nominal wealth. As any negative wealth will be counted as zero for agent \( j \)'s accounting purposes. Denote \( F_j \) as agent \( j \)'s distribution of prices where the price \( p_1 \) is determined by payment equilibrium with the convolution of density functions \( g = g_1 \ast g_2 \ast \cdots \ast g_n \), where \( g_i \) is a (shifted) density function of shock distribution \( G_j \). Note that individual \( g_i \)'s are still relevant when considering the lender default costs and the correlated bankruptcy of agent \( j \). For example, if agent \( i \) goes bankrupt, then agent \( j \) may also go bankrupt due to the cost of \( \zeta(c_{ij}) \) regardless of the size of the \( \epsilon_j \). In this case, agent \( j \) would not consider the lender default cost to be a problem greater than the size of her nominal wealth in \( t = 1 \) under that shock. Denote this implied subjective lender default probability of \( i \) with \( j \)'s belief as \( \omega_{ij} \equiv E_j \left[ \mathbb{1}_{[p_1 > y_{ij}, i \in B(\epsilon)\}} \right] \). Then, the marginal increase in counterparty risk of borrowing from agent \( i \) for agent \( j \) becomes \( \zeta'(c_{ij}) \omega_{ij} \).

From the market pricing equation \([6] \) in \( t = 1 \), price of the asset in \( t = 1 \) can become \( p_1 = 0 \) if all the agents who are holding cash in the economy at \( t = 0 \) go bankrupt. Even if the probability of liquidity shock \( \theta_j \) is small for everyone, if there is a positive probability of bankruptcy of the agent, then there is a positive probability of \( p_1 \) being zero and the return on cash holding becomes infinity. Therefore, every agent in the equilibrium should hold a positive amount of cash. This pins down all the returns from borrowing and lending to the return of holding cash \( E_j \left[ \frac{s}{p_1} \right] \). The cash return becomes the benchmark return for any other investment decision the agent makes.
Lemma 1 (Positive Cash Holdings). If \( \bar{\epsilon} > ne_0 + h_0s^1 \), then \( e_j > 0 \) for every \( j \in N \).

This lemma implies that the model is distinctive from existing models in general equilibrium with collateral literature when agents have linear utility. Linear utility models in Geanakoplos (2010b), Simsek (2013), and Geerolf (2017) all have borrowers only holding the assets and zero amount of cash. In reality, the ultimate borrowers such as hedge funds usually hold a significant proportion of their portfolio as cash equivalent assets. The network model here replicates this observed phenomenon by adding this temporary liquidity shock and the possibility of liquidity constrained-price below the fair value of the asset.

From now on we assume the condition for lemma 1 holds. The return on lending depends on how much you lend but does not depend on which agent you lend to. This irrelevance comes from the fact that lenders do not have counterparty risk due to collateralization and no recourse contracts. Therefore, the contract price \( q \) does not depend on the identity of the borrower.

Suppose \( j \) is lending positive amount of cash without leverage, i.e., \( j \) is a pure lender. The return equation of lending for \( j \) becomes as below.

\[
\frac{1}{q(y)} E_j \left[ \min \left\{ s, y \frac{s}{p_1} \right\} \right] = E_j \left[ \frac{s}{p_1} \right]
\]

The return of lending should equal the return of cash for no arbitrage (indifference). Note that the cash return goes up as \( j \) holds less cash because there would be even more underpricing if all other agents go bankrupt. This equation also represents how price of a contract (or interest rate) is determined if agent \( j \) does not leverage.

\[
q(y) = \frac{E_j \left[ \min \left\{ s, y \frac{s}{p_1} \right\} \right]}{E_j \left[ \frac{s}{p_1} \right]} = \frac{E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} \right]}{E_j \left[ \frac{1}{p_1} \right]}
\]

Where \( y \) is in the range of contracts that \( j \) is the marginal lender in the equilibrium. As we have seen in the payment equilibrium part in \( t = 1 \), as the realization of the asset payoff increases, the asset is more likely to be underpriced than its fundamental value due to more exposure to liquidity shortage and lender default costs. Thus, if the nominal wealth of the agents are identical, the order of return of holding cash also follows the order of optimism over asset payoffs i.e. \( E_j \left[ \frac{s_j}{p_1} \right] > E_k \left[ \frac{s_k}{p_1} \right] \) for any \( j < k \). This ordering implies that interest rates of the same contract increases over agent’s optimism, i.e., optimistic agents demand a higher interest rate than pessimistic agents do. On the other hand, the optimist have more possible underpricing in their subjective beliefs. For example, agent \( n \) believes that
the price is \( s^n \) for any cases with \( \sum [m_j]^+ \geq \sum h_0 s^n \) where as agent 1 believes that \( \sum [m_j]^+ \) can determine prices in the interval \([s^n, s^i]\). Thus,

\[
E_i \left[ \frac{1}{p_1} \right] < E_j \left[ \frac{1}{p_1} \right]
\]

holds if \( i < j \) and \( e^i_1 \leq e^j_1 \).

Return on buying the asset without leverage is \( E_j \left[ \frac{s}{p_0} \right] \) where \( p_0 \) is asset price determined at \( t = 0 \). Since the return does not depend on \( p_1 \), this return is not (directly) influenced by counterparty risk. Hence, this return on asset is ordered directly by the agent’s optimism i.e. \( E_j \left[ \frac{s}{p_0} \right] > E_k \left[ \frac{s}{p_0} \right] \) for all \( j < k \). Return on asset purchase with leverage is

\[
\frac{s^j}{p_0 - q(y)} E_j \left[ \left[ 1 - \frac{y}{p_1} \right]^+ - \frac{\zeta'(c_{ij})}{p_1} \mathbf{1} \left[ 1 > \frac{y}{p_1} \right] \mathbf{1} \{ i \in B(\epsilon) \} \right]
\]

where agent \( j \) is borrowing cash from agent \( i \) with \( c_{ij} \) amount and promises \( y \). Similarly, return on lending with leverage is

\[
\frac{s^j}{q(y') - q(y)} E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \mathbf{1} \left[ 1 > \frac{y}{p_1} \right] \mathbf{1} \{ i \in B(\epsilon) \} \right]
\]  \hspace{1cm} (7)

where \( j \) buys (lends money) a contract with promise \( y' \). From the return comparisons and pure lender’s no arbitrage condition, agent’s individual leverage decision could be derived and the following lemma summarizes the result of leverage maximization.

**Lemma 2 (Maximum Leverage).** Suppose that agent \( j \) lends positive amount of money to an agent (or buys the asset i.e. lends money to herself) and borrows positive amount of money from agent \( i \) in a network equilibrium. Then, the following are true:

1. Agent \( j \) maximizes her leverage by borrowing the maximum amount of money she can borrow from agent \( i \) which is \( s^i \).

2. If \( j \) is borrowing the same amount from agent \( i \) and \( k \) who have the same probability of bankruptcy with \( i < k \), then \( j \) marginally prefer to borrow from \( i \).

The intuition of the proof is the following. If borrower \( j \) and lender \( i \) agree on the distribution of prices below \( s^i \) which only depends on liquidity shocks that both agents agree upon, then \( j \) and \( i \) agree upon the expected delivery. Since \( j \) has higher marginal utility of
cash in \( t = 0 \) than \( i \), agent \( j \) would like to increase borrowing at any point below \( s^i \). At the point of \( s^i \), agents disagree with the promised delivery above \( s^i \). Agent \( j \) believes the price of the asset \( p_1 \) can be greater than \( s^i \) if the aggregate liquidity shock is not large enough, but \( i \) believes the price is bounded above by \( s^i \) even if there is zero liquidity shock. Therefore, the endogenous leverage is determined by the promise of \( y = s^i \) and its price \( q(s^i) \). The logic can be considered as a generalization of the three state case in Geanakoplos (2003). With this lemma, we can pin down \( q(y) \) for agents who both borrow and lend. If agent \( j \) borrows \( y \) from agent \( i \) and lends \( y' \) to some other agent (or herself if she buys the asset directly) then her no-arbitrage contract price becomes as below.

\[
q(y') = q(y) + E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \mathbb{1} \left[ 1 > \frac{y}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right] \frac{1}{p_1}
\]

By maximum leverage lemma we only need to focus on kink points for borrowing. Hence, any agent who is willing to borrow from agent \( j \) will face the willingness to pay as

\[
q(y) = q(s^i) + E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \mathbb{1} \left[ 1 > \frac{s^i}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right] \frac{1}{p_1}
\]

(8)

The following proposition and figure 9 describe the relationship between interest rate and loan-to-value ratio.

**Proposition 4** (Concave Credit Surface). In any network equilibrium, the contract price function \( q(y) \) is piece-wise concave in the amount of promise \( y \) and has kinks and jumps at each payoff points \( s^1, s^2, \ldots, s^{n-1}, s^n \). Furthermore, the credit surface of the equilibrium (the graph between leverage \( q(y)/p_0 \) and interest rate \( y/q(y) \)) is piece-wise concave and continuous in the amount of leverage \( q(y) \) and has kinks at each corresponding payoff points \( q(s^1), q(s^2), \ldots, q(s^{n-1}), q(s^n) \) and right derivative of each kink point is greater than the left derivative. Also, the interest rate goes to infinity at the point \( q(s^1) \).

Now the remaining parts of the equilibrium are actual amount of cash holdings and the amount of each contract. From the return equation of leverage, equation 7 and the 

\[\text{This intuition also brings light to how complicated the model would be if concentrated beliefs are not assumed. For example, if agent’s optimism is ordered by first-order stochastic dominance, then endogenous leverage depends not only on the relative hazard ratio, but also on the difference in marginal utilities of cash which also changes endogenously and is extremely intractable to pin down. Moreover, they differ with the distribution of liquidity shocks.} \]
convexly increasing lender default cost $\zeta$, borrowers would diversify their borrowings across different lenders. Even if agent $j$ can borrow more from $i$ as $q(s^i)$, higher $\zeta'(c_{ij})$ would make $j$ to borrow from $i+1$ with lower leverage $q(s^{i+1})$ because of lower default cost $\zeta'(c_{(i+1)j})$. Hence, there is a trade-off between leverage and counterparty risk. An agent wants to maximize leverage to maximize her return but she has to face higher counterparty risk from the concentration. If the agent wants to diversify her lenders, then she has to deal with more pessimistic agents who only provides low leverage which means low return for the borrower. Thus, the network becomes a multi-layered chain network instead of a single-chain when $\zeta$ becomes large and $\theta_i$ is non-negligible. Thus, the equilibrium cash holdings depend on the gap between beliefs on asset payoffs and the slope of the contract price function.

3.3. Results on Decentralized OTC Market

Now with the given contract prices, the asset price can be analyzed. The first result is about who is buying the asset in the equilibrium. Not surprisingly, the agent with the most optimistic belief on the asset payoff, agent 1 who believes the asset payoff will be $s^1$, always buys the asset.

**Lemma 3** (Natural Buyers). *In a network equilibrium with maximum payment equilibrium selection rule, the most optimists, agent 1, buys the asset with positive quantity, thus $c_{11} > 0$. Similarly, agent $i$ borrows from agent $i+1$ with positive amount, $c_{i+1,i} > 0$ for any $i \in N, i \neq n$.***
The intuition of the lemma is that if any agent \( j > 1 \) is buying the asset, then agent 1 will have even higher return than \( j \) by using the same leverage decisions as \( j \) unless agent 1’s cash holding is huge enough to make her required return low. However, when 1’s cash holding is large then \( j \)’s return of cash is enormous in case of agent 1 goes bankrupt and agent \( j \) should either increase his cash holding or increase the return on asset purchase, i.e. \( p_0 \) goes down. But either of them should make agent 1’s perceived return on asset purchase (with leverage) increase even faster due to \( \frac{s_1}{p_0} > \frac{s_j}{p_0} \). Thus, agent 1 should be a natural buyer of the asset (but not necessarily the only buyer). Similar logic can be applied to any subsequent contracts \( \min\{p_1, s^i\} \) and by induction, we can show that natural buyers of any contracts with a promise of \( s^i \) is agent \( i \). Note that agents other than agent 1 can also hold some amount of assets because it is possible to have

\[
E_j \left[ \frac{s^j}{p_1} \right] = E_j \left[ s^j - \min \left\{ s^i \frac{s^j}{p_1}, s^j \right\} - \frac{\zeta^i(c_{ij})s^j}{p_1} \mathbb{I} \left[ 1 > \frac{s^i}{p_1} \right] \mathbb{I} \{ i \in B(\epsilon) \} \right]
\]

for multiple \( j \in N \). In this case, agent 1 holds more cash than agent \( j \) so that the possible underpricing coming from larger support for agent 1 is mitigated by being less vulnerable to liquidity shocks to others including agent \( j \). Thus, \( e_1 > e_j \) in such cases.

This property of optimists holding more cash than pessimists can be formalized for certain parameter region. Belief disagreements are harmonically dispersed if \( s^j s^{j+2} \leq (s^{j+1})^2 \) for any \( j < n - 2 \). Harmonically dispersed belief disagreements imply that the belief of one agent among three consecutive agents are not too radically skewed. For example, belief disagreements are not harmonically dispersed if agent 2 and 3 believes \( s \) to be 20 and 10 respectively but agent 1 believes \( s \) to be 100. Agent 1’s belief should be less than or equal to 40 in order to be harmonically dispersed.

**Proposition 5.** Suppose the network equilibrium is a single chain network i.e. \( c_{i+1,i} = c_{i+2,i+1} = c > 0 \) for \( i < n - 2 \) and \( c_{ij} = 0 \) for any \( ij \) not in the path between agent 1 and \( n \) and \( i \neq j \). Also suppose that the belief disagreements are harmonically dispersed. Then, agents hold cash as \( e_1 > e_2 > \cdots > e_n \), that is the order of amount of cash holdings is the same as the order of optimism on the asset payoff.

**Corollary 1.** If there is no lender default cost, i.e. \( \zeta(c) = 0 \) for any \( c \in \mathbb{R}^+ \) and the belief disagreements are harmonically dispersed, then agents hold cash as \( e_1 > e_2 > \cdots > e_n \), that is the order of amount of cash holdings is the same as the order of optimism on the asset payoff.

This result is in contrast to standard results in general equilibrium with collateral litera-
ture such as Geanakoplos (1997), Fostel and Geanakoplos (2015), Simsek (2013), and Geerolf (2017) in which optimists spend more, if not all, cash to purchase assets and pessimists hold more cash and sell assets. Although agent 1 values the asset the most, they also have the highest marginal utility of cash in \( t = 1 \). Because the asset value is so high, the price of the asset is also vulnerable to liquidity shortage in the market. Under agent 1’s perspective, the market should have \( nh_0s^1 \) amount of cash to clear the market with the asset’s fundamental value. On the contrary, agent \( n \) believes the market can be cleared in fair value in \( t = 1 \) even with \( nh_0s^n \) amount of cash and underpricing only happens when the economy is under severe liquidity shocks. Holding more cash is possible because of the possibility of leveraging through the lending chain. The down payment (cash paid for the levered purchase) for agent 1, \( q(s^1) - q(s^2) \), can be less than the down payment for agent \( n - 1 \), \( q(s^{n-1}) - q(s^n) \). Also, the cash holding dispersion will be even more severe if leverage increases.

This cash holding result may seem unintuitive. However, the empirical facts support this result. On average, 34\% of hedge fund’s assets can be liquidated within one day (without fire sale discounting) according to Aragon et al. (2017) using Form Periodical Filings over 2013-2015. This proportion is well above the proportion of money market mutual funds (MMFs) SEC reformed regulation by 10 percentage point. Before the regulation, daily liquid assets for MMF is on average less than 20\%, and even after the regulation the daily liquidity in the portfolio is still below 31\% (Aftab and Varotto, 2017). Because hedge funds are the ultimate asset buyers (as agent 1) in a collateralized debt market and money market mutual funds are pure lenders in the market (as agent \( n \)), the empirical findings are consistent with the result of the proposition.

Given all the tools from \( t = 1 \) payment equilibrium and \( t = 0 \) borrowing and lending behavior, we can prove existence of a network equilibrium as well as the properties of network equilibria.

**Theorem 1** (Existence and Characterization of Network Equilibrium).
For a given economy \((N, (s^j, \theta_j, e_0, h_0)_{j \in N}, \zeta, G)\) and maximum equilibrium selection rule, there exists a network equilibrium \((C, Y, e_1, p_0, \tilde{p}_1, q)\) and any network equilibrium is characterized as the following:

1. For any \( y \in [s^{j+1}, s^j] \)

\[
q(y) = q(s^{j+1}) + E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^{j+1}}{p_1} \right\} - \frac{\zeta'(c_{(j+1)j})}{p_1} 1 \left( 1 > \frac{s^{j+1}}{p_1} \right) \ 1 \left\{ j + 1 \in B(\epsilon) \right\} \right]
\]

where we set \( q(s^{n+1}) = s^{n+1} = 0 \) and \( \max_j E_j [1 \{ n + 1 \in B(\epsilon) \}] = 0 \).
2. For any $i, j \in N, i \neq j$, $y_{ij} = s^j$.

3. For any counterparties $i, k$ of $j$ with $c_{ij} > 0$, $c_{kj} > 0$,

$$
\frac{s^j}{q(s^j) - q(s^i)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \zeta'(c_{ij}) \frac{1}{p_1} \left[ 1 > \frac{s^i}{p_1} \right] \{ i \in B(\epsilon) \} \right] \\
= \frac{s^j}{q(s^j) - q(s^k)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^k}{p_1} \right\} - \zeta'(c_{kj}) \frac{1}{p_1} \left[ 1 > \frac{s^k}{p_1} \right] \{ k \in B(\epsilon) \} \right]
$$

4. For any $j, i \in N$ and $j < i$, $c_{ji} = 0$.

5. Cash holdings of each agent is determined by

$$
e^j_1 = e^j_0 + b^j_0 p_0 + \sum_{i \in N \setminus \{ j \}} c_{ij} q(s^i) - \sum_{k \in N \setminus \{ j \}} c_{jk} q(s^j) - c_{jj} p_0
$$

6. The price of the asset at $t = 0$ is determined by

$$
p_0 = q(s^1).
$$

7. The price of the asset at $t = 1$, $\tilde{p}_1$ is determined by payment equilibrium for the given network.

Also the set of network equilibria forms a complete lattice and there exists a maximum leverage network equilibrium that is the equilibrium with the collateral matrix which has the highest aggregate debt among all other equilibria.

The theorem contains several implications. First of all, the theorem suggests the network structure change for the given economy. Any equilibrium collateral matrix should be an acyclical network as agents only borrow from more pessimistic agents, and contract matrix follows rank order due to lemma 2. For negligible default cost (small $\zeta$ and $\theta_j$), a single-chain network is formed that is agent $j$ borrows from agent $j + 1$ only for all $j < n - 1$. This is because even if $c_{j+1,j} = \sum_{k \in N} c_{jk}$ the return from borrowing from $j + 1$ is still greater than return from borrowing from $l > j + 1$ as the counterparty risk increase is small. Figure [10] is an example of such a network. Agents are not concerned about diversifying their counterparty and choose the most profitable counterparty, that is the most optimistic agent after herself, and concentrates all the borrowing. On the other hand, if the default cost $\zeta$ is non-negligible, then a multi-chain network is formed in equilibrium. Figure [11] is an example of such a multi-chain network. Agent $j$ borrows not only from $j + 1$ but also from $j + 2$ as well. Agents
would rather diversify their counterparties and would like to link with several levels down of optimism.

The second implication of theorem 1 is that individual agent’s diversification behavior generates positive externalities through amplification and feedback effects in both asset price channel and network channel with default cost in $t = 1$. If agent $j$ diversifies more and lowers her own return due to counterparty risk concerns, then it will lower the leverage through $q(y)$ and also decrease price volatility in $t = 1$. Furthermore, this risk-reducing behavior makes agent $j$’s balance sheet $m_j$ safer and decreases the probability of $j$’s bankruptcy. Thus, second-order bankruptcy contagion decreases even further. This result is shown in the next proposition.

Before stating the proposition, we have to define directions of lowering the aggregate debt level. First, for a given collateral matrix $C$, a collateral matrix $C^*$ is uniformly less indebted if $c_{ij} \geq c_{ij}^*$ for any $i, j$ and $c_{ij} > c_{ij}^*$ for at least one pair $ij$. The second direction comes from diversification. Define $L_j$ as the largest holder of $j$’s collateral, thus, $\max_{i \in N \setminus \{j\}} c_{ij} = c_{L_jj}$. For a given network equilibrium and its collateral matrix $C$, $C^*$ is a diversification of agent $j$ from $C$, if

1. $c_{L_jj} > c_{L_jj}^* \geq \max_{i \in N \setminus \{j\}} c_{ij}^*, c_{ij} \leq c_{ij}^*$ for all $i > L_j$,
2. $\zeta(c_{L_jj}^*) \omega_{L_jj} \geq \zeta(c_{ij}^*) \omega_{ij}$ for any $i > L_j$,
3. $\sum_{i \in N \setminus \{j\}} c_{ij} \geq \sum_{i \in N \setminus \{j\}} c_{ij}^*$,
4. $c_{ik} \geq c_{ik}^*$ for all $i, k \in N$ with $k \neq j$,
5. and $(C^*, Y)$ satisfies rehypothecation constraints.

This diversification of agent $j$ from an equilibrium collateral matrix implies that agent $j$ has her counterparties more diversified than the original network in either intensive or extensive margins while still maintaining the perceived counterparty risk not exceeding the original largest holder of collateral.

**Proposition 6** (Diversification Externality). Suppose that $(C, Y, e_1)$ is a network equilibrium. Suppose there is a collateral matrix $C^*$ and either of the two holds:

1. $C^*$ is uniformly less indebted than $C$
2. $C^*$ is a diversification of agent $j$ from $C$ for $j < n$. 

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Then, ex ante expected equilibrium price $p_1$ under $C^*$ is greater than that under $C$ and ex ante expected volatility of $p_1$ under $C^*$ is lower than that under $C$ for each subjective belief of $j \in N$.

This proposition implies that the higher the debt level is, either uniformly more indebted or under the direction of less diversification, the more the underpricing occurs both in terms of likelihood and intensity. The intuition is that the increase in lender default cost as $\zeta(c)$ increases convexly in $c$ and also the contagion intensifies through both second-order bankruptcy of counterparty channel and asset price channel. If a borrower is more indebted, the expected sum of lender default costs is higher. Also if a borrower is less diversified, the expected sum of lender default costs is higher because of convexity of $\zeta$. The second-order bankruptcy contagion only makes it even worse in expected sense because that only increases the probability of bankruptcy even more. Thus, diversification generates benefits to the overall distribution and all of the agents.

However, these positive externalities from diversification are not included in individual agent $j$’s concern. Therefore, the degree of diversification is always less than the optimal degree in the economy, and the equilibrium is constrained inefficient. Define the social welfare of the economy as the sum of ex ante expected utilities of all the agents as

$$\sum_{j \in N} E_j \left[ m_j(\epsilon) \frac{s_j}{p_1(\epsilon)} \right].$$

An equilibrium is constrained inefficient if a social planner can generate higher social welfare by adjusting the allocation while the resource constraints and collateral constraints are satisfied and leaving the $t=1$ market interaction decentralized.

**Theorem 2.** Any network equilibrium under OTC market is constrained inefficient due to under-diversification if $\zeta$ is non-negligible.

The third implication of theorem 1 is leverage stacking through the lending chain. Increase in $q(s^n)$ increases all the subsequent contract prices which imply that the lending amount increases. Therefore, lending or leverage at any point in the lending chain has a multiplier effect on the economy. This leverage multiplier effect due to reuse of collateral has been mentioned in Gottardi et al. (2017) as well. A distinct feature of the model in this paper is that different level in the lending chain has different multiplier effects. An increase in $s^n$ will have much more impact than an increase in $s^2$ as agent $n$’s lending stacks $n-1$ times through the lending chain through equation 8.

The fourth implication is the dispersion of gains of trade. Unlike the result in one link of borrowing and lending in Simsek (2013) and Geerolf (2017), where the gains of trade are fully
concentrated to the borrower (agent 1), the gains of trade are dispersed across all agents through competition across different agents and also varying degree of liquidity shortage. The literature regarding the principal-agent problem in lending contract usually focuses on the special case in which borrowers have all the bargaining power (Gale and Hellwig, 1985; Holmstrom and Tirole, 1997) but the result of the network equilibrium shows that even if the market is competitive and each individual agent is a price-taker, there can be some surplus distributed to either side. In particular, even if we retract the model to a single contract case as \( n = 2 \), the dispersion of bargaining power still occurs. This is partly due to positive cash holdings for either side which is coming from liquidity shock and differential marginal utility of cash. Also in the network or lending chain literature context, this feature implies bargaining power, division of gains of trade, between borrowers and lenders is determined endogenously in contrast with the papers such as Farboodi (2017) and Hugonnier et al. (2018) where they assume exogenous bargaining power as some constant. The more cash you are holding, the less power you have in terms of bargaining power as your outside option becomes less profitable and cannot charge a higher interest rate.

The fifth implication of theorem 1 is the endogenous market reaction to the change in counterparty risk. From theorem 1 and proposition 6 the connection between degree
centrality and contract prices (interest rates) can be deduced. As the debt of the network increases, the equilibrium contract prices become lower. This is due to the second term of equation 8. The denominator increases while the numerator does not increase as much due to the boundedness of contract returns. The intuition for this result is the following. Since the network has a higher amount of debt, the market in \( t = 1 \) can suffer more from liquidity shocks and further propagation in case of bankruptcy. Agents prefer to hold cash in case of huge liquidity shocks and also willing to lend less for the same promise as a lender. Similar comparative statics can be done for the equilibrium contract prices. For example, if all of the agent’s liquidity shock arrival rate \( \theta_j \) increases, then contract price for an agent who borrows cash would decrease as the return from the leverage decreases. Also, change in the asset payoff belief \( s^j \) would affect both the amount of debt as well as contract prices. The comparative statics results above are summarized as the next proposition.

Before stating the proposition, define the velocity\(^{11}\) of collateral in a network \( C \) as the volume of total collateral posted divided by the stock of source collateral\(^{12}\)

\[
Velocity(C) \equiv \frac{\sum_{i \in N} \sum_{j \neq i} c_{ij}}{\sum_{j \in N} c_{jj}}.
\]

This velocity of collateral represents volume of the reuse of collateral within the network. For example, if the network \( C \) is a single chain network using all of the source collateral which is all held by agent 1 repeatedly, then the velocity of \( C \) is \( n - 1 \) because \( c_{21} = c_{32} = \cdots = c_{n,n-1} = c_{11} \) and \( Velocity(C) = \frac{c_{21} + c_{32} + \cdots + c_{n,n-1}}{c_{11}} = n - 1 \). The velocity of collateral is also an approximate measure of the average length of the lending chain in the network as argued in Singh (2017).

**Proposition 7** (Comparative Statics on Borrowing Pattern). For a given network equilibrium with maximum equilibrium selection rule, the following statements are true.

1. If \( s^j \) increases (decreases) by the same amount for every \( j \in N \), then the equilibrium contract prices and leverage for each agent increases (decreases). Also there must be weakly less (more) links between agents and the velocity of collateral increases (decreases).

2. If \( \theta_j \) increases (decreases) by the same amount for every \( j \in N \), then the equilibrium contract prices and leverage for each agent decreases (increases). Also there must be

\(^{11}\)Since this model is not dynamic, the “velocity” here means how much a collateral moves around in the market.

\(^{12}\)This definition is similar to the definition of the velocity of collateral in Singh (2017) that is the volume of secured transactions divided by the stock of source collateral.
weakly more (less) links between agents and the velocity of collateral decreases (increases).

The results above can be summarized as the following theorem.

**Theorem 3** (Network Change under Crisis). *If the economy is under financial distress and the counterparty risks become greater, then agents diversify more, the velocity of collateral decreases, and the average number of counterparties (weakly) increases.*

The results of theorem 3 are consistent with the empirical facts. As Singh (2017) documented, the velocity (reuse) of collateral decreased from 3 to 2.4 right after the bankruptcy of Lehman Brothers and the average leverage in the OTC market also went down. Also Craig and Von Peter (2014) shows that the average number of linkages between financial institutions have increased about 30% over the 4 years after Lehman bankruptcy. The dynamics of theorem 3 has occurred even before the Lehman bankruptcy. In the wake of Bear Stearns’ demise, hedge funds had increasingly used multiple prime brokers to mitigate counterparty risk. In fact, despite the traditionally concentrated structure of the prime brokerage business, as far back as 2006, about 75% of hedge funds with at least $1 billion in assets under management relied on the services of more than one prime broker (Scott, 2014). On the contrary, the opposite result happened in unsecured debt markets. Afonso et al. (2011) finds that the banks in the federal funds market reduced their number of counterparties after Lehman bankruptcy. This stark comparison shows the importance of collateral in network formaion.

Results of propositions 4 and 7 also provide insights on the puzzles in repo markets mentioned in Copeland et al. (2014) and Krishnamurthy et al. (2014). In particular, concavely increasing credit surface might help to explain the difference in market dynamics between bilateral and triparty repo markets. As the bilateral market is more dispersed in counterparty exposure already, the general reaction to bad news about the asset payoff will be dispersed across all agents and they would generally increase the margin and lower leverage. On the contrary, triparty repo market is already full of contracts with maximum leverage and focused counterparty risk exposure. Under the bad news, one of the early channels may break down and the total money injection to the market suddenly decreases by a huge amount and the entire market dynamics is rather a huge decrease in volume of lending relative to stable margins.

Numerical examples in figure 12 and table 1 show the comparative statics in theorem 3. Figure 12 represents the collateral flow of promises of no risk, moderate risk, and significant

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13The velocity went further down to 1.8 as of 2015. Singh (2017) argues that the collateral landscape has changed further due to central banks’ quantitative-easing policies and new regulations which are beyond the scope of this paper.
Figure 12: Network Comparative Statics – no risk, moderate risk, and significant risk

<table>
<thead>
<tr>
<th></th>
<th>No risk</th>
<th>Moderate risk</th>
<th>Significant risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pr(Bankruptcy)</td>
<td>0%</td>
<td>9.6%</td>
<td>25.4%</td>
</tr>
<tr>
<td>Leverage</td>
<td>10</td>
<td>2.0766</td>
<td>1.7411</td>
</tr>
<tr>
<td>Velocity</td>
<td>3</td>
<td>1.6870</td>
<td>1.4149</td>
</tr>
<tr>
<td>$ Volume</td>
<td>2400</td>
<td>756</td>
<td>431</td>
</tr>
<tr>
<td># of links</td>
<td>3</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 1: Network Comparative Statics

risk cases respectively. Each numbered node represents the agent and the arrowed link represents the direction and the size of the promise. As the risk increases, equilibrium network changes from a single chain network with a large size of collateral flows to a multi-chain network with a smaller size of collateral flows. The no risk case has $\theta_j = \theta = 0$ for all $j \in N$, the moderate risk case has $\theta_j = \theta = 0.4$ for all $j \in N$, and significant risk case has $\theta_j = \theta = 0.8$ for all $j \in N$. As the liquidity shock and counterparty risk become more relevant, the probability of bankruptcy increases. The leverage of a natural buyer, agent 1, decreases by a huge margin and the velocity of collateral decreases as agents diversify their counterparties which reduces the reuse of collateral. The number of links increases due to diversification and the total nominal cash volume of promises decreases.

4. Central Clearing

As discussed in the introduction, central clearing and the introduction of central clearing counterparty (CCP) is one of the major issues in market structure regulations. In this section, I define a theoretical way of introducing CCP and perform counterfactual analysis on the impact of introducing CCP to a decentralized OTC market.

CCP novates one contract between a borrower and a lender into two contracts – a contract between the borrower and the CCP and a contract between the lender and the CCP. This
implies the CCP can be considered as a new agent, defined as agent 0, and the CCP simply duplicates the already existing debt network \( C, Y \) into its balance sheet. This can be done by first adding all the columns of \( C \) and each column sum will be \( c_{0i} \) for all \( i \in N \) and then adding all the rows of \( C \) and each row sum will be \( c_{i0} \) for all \( i \in N \). The contract matrix \( Y \) can also be modified by adding the new row and column for 0 with all the relevant promises of \( s^j \) for each \( j - 1 \) row and column. CCP also does pooling which is buffering the counterparty risk with its own balance sheet. The CCP’s cash holdings \( e_0 \) can be considered as cash buffer as CCP guarantee funds that are coming from \( n \) client agents with \( g \) amount of contribution, so \( e_0 = ng \). This structure of participation fee is in fact, how the actual CCP manages its guarantee funds in the CCP’s “default waterfall.” Define the new network with CCP as \( (C_{ccp}, Y_{ccp}) \).

CCP also nets out obligations between two counterparties. We can consider netting of borrower obligations as a transformation of the debt matrix \( C \circ Y \) that is \( \hat{C} \circ \hat{Y} \) s.t.

\[
\hat{c}_{ij} \hat{y}_{ij} = [c_{ij}y_{ij} - c_{ji}y_{ji}]^+ 
\]

for all \( i, j \in N \). This can be considered by transformation of matrix as \( [C \circ Y - C' \circ Y']^+ \) If this netting procedure is done for the original debt network, then this is a *bilateral netting procedure*. If we run the bilateral netting procedure after the inclusion of CCP i.e. \( [C_{ccp} \circ Y_{ccp} - C'_{ccp} \circ Y'_{ccp}]^+ \), then it becomes the *multilateral netting* \( \hat{C}_{ccp} \circ \hat{Y}_{ccp} \) which is relatively straightforward operation equivalent to the double summation operation in Duffie and Zhu (2011).

The netting should be considered more carefully when it comes to lender obligations since the lender obligation may not be relevant under certain prices when the borrower defaults on their promises. The netting procedure works as the following:

1. For the given price \( p_1 \), compute the entry-by-entry indicator matrix \( \Gamma \equiv 1(Y = X) \).

2. Compute the effective collateral matrix \( C' \equiv C \circ \Gamma \).

3. Perform the CCP netting procedure above to derive \( \hat{C}'_{ccp} \).

4. Redistribute the relevant collateral obligations from the updated \( \hat{C}'_{ccp} \) to corresponding columns and rows for \( C \).

This redistribution is done by whoever is the final holder of the asset. Under acyclical networks which arises endogenously in the model as seen in theorem there is no indeterminacy of redistribution. Also any left over wealth of the CCP is equally distributed to the surviving
agents. Thus, the CCP’s nominal wealth after payments becomes

\[ m_0(\epsilon|p_1) = n g - \sum_{j \in N} \sum_{k \in N} \zeta(c_{jk}) I \{ j \in B(\epsilon), p_1 \geq y_{jk} \} \]

and CCP goes bankrupt when \( m_0(\epsilon|p_1) = 0 \).

There are many important properties of a CCP in reality, such as enhanced transparency and collateral management, that are abstracted out from the model. Other than the pooling and netting of the contracts, I assume that the CCP is exactly the same as the other agents in the economy. The CCP still has to pay the same \( \zeta \) cost for bankrupt lenders and does not have any additional benefits on collateral management or efficiency in margin settings. Also when the CCP goes bankrupt (if the nominal wealth of the CCP becomes negative), then all the agents suffer \( \zeta \) cost as the OTC market case. Obviously, these assumptions are strong. For example, the vast majority of Lehman’s clients who went through CCPs obtained access to their accounts within weeks of Lehman’s bankruptcy (Fleming and Sarkar, 2015). This implies the \( \zeta \) when the CCP is the borrower might be lower than \( \zeta \) of the other agents. But, the cost of retrieving collateral after the CCP went bankrupt might be much higher than the lender default costs from the OTC markets. Also CCPs often do not allow rehypothecation or sometimes themselves are restricted in rehypothecating the assets. But, then this restriction comes with a cost of worse flow of collateral and liquidity as the velocity of collateral decreases (Singh, 2017). The main point of this analysis is rather to focus on the understudied property of endogenous reaction of the market, in terms of change in network formation. Any other properties are abstracted out from the model and are subject to further studies.

4.1. CCP without Netting

First, consider the effect of novation and pooling only. Since agents are protected from direct counterparty risk when the CCP survives, agent \( j \)’s optimization problem becomes as below.
From proposition 6 and theorems 1 and 2, the following proposition holds.

**Proposition 8.** For a given network equilibrium with maximum equilibrium selection rule under OTC market with collateral matrix $C$, suppose that a CCP without netting is introduced to the market.

1. If the CCP never goes bankrupt by (implicit) guarantee by the government, then the new network with collateral matrix $C_{ccp}$ maximizes the systemic risk.

2. If agents have to contribute $g$ to the CCP and the size of the contribution is large enough to cover any lender default costs, then the new network with collateral matrix $C_{ccp}$ maximizes the systemic risk.

3. If agents have to contribute $g$ to the CCP and the CCP can go bankrupt in some states, then the new network with collateral matrix $C_{ccp}$ has higher systemic risk than the original network with collateral matrix $C$.

The CCP’s pooling feature eliminates direct counterparty risk concern from agents. They connect for the most favorable contracts with the most concentrated counterparties. This elimination of positive externalities from diversification amplifies the cost even more through increase in debt. The intuition can be explained as a metaphor for fire insurance. If an agent joins the fire insurance, her individual fire risk can be fully covered. However, since she does not care about her own fire risk anymore, the moral hazard problem occurs. She does not care for fire safety which incurs individual effort cost and her probability of fire increases. Thus, the aggregate fire risk rather increases when the economy-wide fire insurance is introduced. In addition, her individual fire safety might have also prevented some spillover fire to other...
people. Thus, the amplification of aggregate fire risk occurs even further. If $g$ is high, then some agents, say $j + 1, j + 2, \ldots, n$, may not participate in the market, if they had the choice, since their return from borrowing or lending in the market does not justify paying $g$. However, the individual incentives of the participants are still the same since marginal incentives are the same. Even though the lending chain leverage may decrease, the network they have is going to maximize the systemic risk for the given component of the network.

The graphical dynamics of the above result is described in figure 13. The top graph is the decentralized OTC network where each agent diversifies their counterparties. The bottom graph is the new network after introducing a CCP in the middle. The notional link in the new network looks like the black links which are only the contracts between the CCP and the other agents. However, the actual contract flows are the single chain network in red links which is different from the OTC network in the top graph. If the endogenous change in the network, from a multi-chain network to a single chain network, is not taken into account, then the impact of introducing a CCP on systemic risk could be under-evaluated.

4.2. CCP with Netting

A CCP indeed provides positive benefits in reducing systemic risk through the netting. Bilateral netting does not reduce systemic risk at all because there is no cycle in an endogenously formed network. However, multi-lateral netting does reduce counterparty exposure.
Figure 14: Single Leverage Complete Bi-partite Network

**Proposition 9.** Bilateral netting does not affect systemic risk. Multi-lateral netting always decreases systemic risk.

Multi-lateral netting can reduce risk even if there is no cycle. For example, if agent 1 is borrowing from 2 who is borrowing from 3 and agent 2 goes bankrupt, then agent 1 suffers from default cost. However, if CCP nets out the contracts, then agent 1 can pay 3 to retrieve her collateral and not suffer from default cost because of going through the additional chain of agent 2. Hence, the introduction of a CCP has the cost of systemic risk caused by the network structure (from higher leverage and concentrated counterparty risk) because of pooling and the benefit of reducing net counterparty exposure by multilateral netting.

Exogenous leverage models completely miss all these cost and benefit features. If there is an exogenously given leverage that is fixed as $y$ and its market clearing price is fixed as $q(y)$, then agents will be divided into two groups, buyers (borrowers) and sellers (lenders) of the asset. Then, there is no trade-off between leverage and counterparty risk since there is only one contract. Agents will fully diversify their counterparties, even for an infinitesimally small default cost. Thus, a complete bi-partite network as in figure 14 is the equilibrium network under exogenous leverage. Since agents are already diversifying fully, pooling has zero effect on network formation. On the other hand, since all the paths in the network have length of 1 and there is no cycle, netting has zero effect as well.

**Proposition 10** (Irrelevance of CCP). If there is only one contract $y$ that is available in the market, then the decentralized OTC equilibrium network is a complete bi-partite network. Introduction of a CCP (with or without netting) to such market has no impact on leverage and endogenous network formation.
4.3. Numerical Examples

In this subsection, I perform a quantitative analysis of the model to provide for numerical examples. The number of agents is four, with endowments of 5000 cash and 25 assets for each agent, where \( \zeta(c) = c^3 \), and \( S = \{10, 9, 8, 7\} \). The common shock distribution is a log-normal distribution with the mean of 5 and the standard deviation of 5. For 500 samples of this distribution and the given seed of random number generation, the average shock size is 2406957 and median shock size is 347.1644. The equilibrium selection rule is maximum equilibrium selection rule. The algorithm is the following:

**Quantitative Algorithm.**

1. Guess the initial equilibrium collateral matrix \( C_0 \).

2. Compute the payment equilibrium prices \( \tilde{p}_1 \) and bankruptcy sets \( B(\epsilon) \) for each simulated state \( \epsilon \) out of \( k \) different states and for each subject beliefs \( s^j \) of agents. (total \( n \times k \) matrix of prices and \( n \times n \times k \) array of bankruptcy indicators)

3. Compute each agent’s expected returns on each investment decision in \( t = 0 \).

4. Compute the market prices of the asset \( p_0 \) and contracts \( q(y) \).

5. Derive agent’s optimal portfolio decisions starting from agent 1 to agent \( n \). By acyclicity and rehypothecation constraints and lemma [1], this procedure satisfies agents’ optimality and market clearing conditions. Set the new collateral matrix as \( C_1 \).

6. Compare \( C_0 \) and \( C_1 \). If the difference is above the tolerance level, then update \( C_0 = C_1 \) and go back to step 2. If the difference is smaller than the tolerance level, then set \( C_1 \) as the equilibrium network and compute the rest of the variables of the equilibrium.

First, suppose that the CCP never defaults as the government guarantees the solvency of CCP by tax payer’s cash. Under this case, we compare three different cases of the market structure: decentralized OTC market, CCP without netting, CCP with netting. For each market structure, we change the values of \( \theta \), which is the common arrival rate of liquidity shock, and compare the three cases for each \( \theta \) value. In the graphs, the blue solid lines represent the numbers from decentralized OTC market, the red dashed lines represent the numbers from market under a CCP without netting, and the black dotted lines represent the numbers from market under a CCP with netting.

As in the top left graph in figure [15], the leverage of three cases starts with 10. In the OTC market, leverage drops around 2 and stays low as the increase in counterparty risk concern
reduces the leverage. On the other hand, two cases with CCP has almost the maximum leverage as agents are not concerned with lender default costs, which is fully covered by the CCP. The top right graph in figure 15 shows the sum of ex ante social welfare for each case. All of the cases have lower social welfare as the arrival rate of shock increases. However, the OTC market has the highest social welfare compared to the two CCP cases. This is due to agents’ diversification in the OTC market, which is absent from the CCP markets. Also netting has an important impact as it limits the duplication of lender default costs from bankruptcies which makes a noticeable difference between the two CCP cases. However, the probability of bankruptcy is still the highest in the OTC market as can be seen in bottom left of figure 15. The reason is that there exists a contagion channel in the OTC market which is nonexistent in CCP cases because the counterparty channel is insulated by the CCP. As predicted by the theory, the velocity of collateral in the network for the OTC market goes down as $\theta$ increases while the velocity remains the same for two CCP cases.

Now, suppose that the CCP does not have the government guarantee and only covers its losses by the member contribution for the default guarantee fund $g$. The size of $g$ is set as 1000. Under this case, the CCP can actually go bankrupt if the sum of the lender default costs is too large. The leverage graph in the top left of figure 16 shows an interesting shape. In the market with CCP without netting, the leverage rather increases almost to 30 and then start to revert back to 10, which is still much larger than the OTC market case. These dynamics come from the interaction between the counterparty channel and the price channel through the leverage. As $\theta = 0.2$ is still a small number, agents are willing to borrow and lend still very aggressively, however, when the CCP goes bankrupt with the low probability then it will make a huge crash in this case. Agents are gambling for the CCP to survive which is very costly for the agents. Also, since the CCP failure implies total market failure, agents are much less concerned about the event of market failure because that implies the agents themselves are also out of the market as well. In the meantime, they can have large return from cash holdings if they survive. All of these features contribute to the enormous leverage. This colossal leverage also results in lower social welfare as can be seen in the top right of figure 16. The leverage for the case of CCP with netting is much lower than the case without netting. The first reason is, of course, the reduction of counterparty exposure due to netting and much lower likelihood of market breakdown. The agents do not expect the total market break down and they do care about having more cash in case of a CCP failure, but they still survive. Another reason for the moderate leverage is the diversification behavior of agent 1. As the netting cancels out all the exposures between the intermediaries, agent 1 is still exposed to agent $n$’s counterparty risk even after the netting. Therefore, agent 1 wants to diversify and reduces leverage. Since agents are internalizing some of the
lender default costs and the netting reduces the total expected lender default costs for a given network, the social welfare under CCP with netting is greater than the social welfare under the OTC market. The bottom left of figure 16 also shows the similar pattern for bankruptcy probabilities. Because agents are recklessly borrowing and lending under CCP without netting, the probability of bankruptcy is very high. The OTC market case is much lower due to diversification but still the CCP with netting has the lowest bankruptcy rate. The velocity of collateral also follows a similar pattern.

I also test the effect of a CCP when the network is exogenously fixed as the decentralized OTC market equilibrium. Suppose that even after the introduction of a CCP, agents still maintain the same links as before. Figure 17 plots social welfare of the three cases – OTC market, CCP without netting, and CCP with netting. Numerical results imply that CCP always increases social welfare if the network remains the same. Since netting reduces
counterparty exposure, social welfare under CCP with netting is the highest as seen from the previous results. Figure 17 shows that the reversal of social welfare between the OTC market and the market under a CCP without netting in figure 15 and 16 comes from the endogenous change in network formation.

4.4. Policy Implications

Having multiple CCPs can decrease expected aggregate default cost in spite of reduced netting opportunities (Duffie and Zhu, 2011) due to diversified counterparty costs. Multilateral netting across different classes of contracts is crucial in effective netting. Therefore, multilateral netting with a maximum number of classes of assets and contracts which is divided by multiple CCPs could be the optimal CCP structure. Existing CCPs such as Eurex and LCH do have such a structure in reality. However, introducing a CCP should be
done after the cost and benefit analysis from pooling and netting. For example, CDS market is already highly centralized, and the cost of centralizing such market with a CCP could be less than cost from other more diversified markets.

Another more direct regulation to solve for diversification externality problem could be introducing a relevant leverage ratio restriction. In Basel III, there is Supplementary Leverage Ratio (SLR), which is effectively a tax on intermediation activity that is proportional to the size of an intermediary’s balance sheet, defined as below.

\[
\frac{\text{Tier 1 Capital}}{\text{Total Leverage Exposure}} \geq 3\%
\]

A slight modification of this ratio, Network Supplementary Leverage Ratio, can be used and risk externality is included as weights of degree centrality in the denominator. Such restrictions provide marginal incentives to diversify and internalize second order default and maintain borrower or lender discipline of agents.

5. Conclusion

I constructed a general equilibrium model with collateral featuring endogenous leverage, endogenous price, and endogenous network formation. The model bridges the theory of financial networks and the theory of general equilibrium with collateral. Collateral generates an additional channel of contagion through asset price risk, the price channel, on top of the balance sheet risk through the debt network, the counterparty channel. Borrowers diversify
Table 2: Product types cleared by CCPs in 2014 (Source: Bank of International Settlements)

<table>
<thead>
<tr>
<th></th>
<th>Securities</th>
<th>Derivatives</th>
<th>Repos</th>
<th>All three</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advanced economies</td>
<td>63.6</td>
<td>72.7</td>
<td>31.8</td>
<td>27.3</td>
</tr>
<tr>
<td>Europe</td>
<td>60.0</td>
<td>80.0</td>
<td>40.0</td>
<td>40.0</td>
</tr>
<tr>
<td>North America</td>
<td>50.0</td>
<td>50.0</td>
<td>25.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Asia-Pacific</td>
<td>75.0</td>
<td>75.0</td>
<td>25.0</td>
<td>25.0</td>
</tr>
<tr>
<td>Emerging economies</td>
<td>84.6</td>
<td>76.9</td>
<td>61.5</td>
<td>53.8</td>
</tr>
<tr>
<td>Asia</td>
<td>100.0</td>
<td>87.5</td>
<td>87.5</td>
<td>25.0</td>
</tr>
<tr>
<td>Latin America</td>
<td>60.0</td>
<td>60.0</td>
<td>20.0</td>
<td>87.5</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>71.4</strong></td>
<td><strong>74.3</strong></td>
<td><strong>42.9</strong></td>
<td><strong>37.2</strong></td>
</tr>
</tbody>
</table>

their portfolios of lenders due to the possibility of lender defaults. However, lower counterparty risk comes at the cost of lower leverage. There are positive externalities from diversification because it reduces not only the individual counterparty risk but also the systemic risk, by limiting the propagation of shocks and resulting price volatility. Because agents do not internalize these externalities, any decentralized equilibrium is only constrained inefficient. The key externalities here, arising from the trade-off between counterparty risk and leverage, are absent in models with exogenous leverage or exogenous networks. The model also replicates the change in network structure during the financial crisis. Greater counterparty risk induces agents to diversify more which lowers leverage and the velocity of collateral and increases the number of links. I performed a counterfactual analysis on the introduction of a CCP with this model. The loss coverage by CCP exacerbates the externality problems by eliminating individual agents incentives to diversify. Thus, the endogenous network change after the introduction of a CCP creates additional systemic risk that exogenous leverage or exogenous network models do not capture.

Appendix

A. Institutional Details on Central Clearing

In the aftermath of Lehman Brother’s collapse, G20 reform of the over-the-counter (OTC) derivatives market mandated the clearing of standardized derivative contracts by CCPs. One of the principal risks in the financial system which CCPs seek to address is the counterparty credit risk. CCPs enable multilateral netting, and an empirical analysis in Cecchetti et al. (2009) shows that CCP can reduce gross notional exposures by approximately 90 percent. There are also benefits of transparency from CCPs. In a bilateral market, parties know their cross exposures to counterparties but they do not know their counterparties’ exposures to third parties. The lack of transparency could significantly increase the systemic risk.
both by misaligned risk management (Tirole, 2011), for example, excessive sales of CDS without proper collateral, and the increase in uncertainty. If we consider the risk from derivative contracts sold to unregulated counterparties such as foreign financial institutions as counterparties, it creates further risk exposures (Cecchetti et al., 2009). Both private and public sector responses to failures of large financial institutions could be even more complicated. In addition CCP may decrease transaction costs significantly (Case et al., 2013). Multilateral netting also reduces collateral requirements as shown in Duffie et al. (2015). Standardized products by CCPs makes it easier to adjust for appropriate margin calls and allows supervisors to monitor the solvency of CCPs better. With the concentrated counterparty risk to CCPs, the central clearing market usually pools the “default fund” from all the clearing members or more sophisticated loss-absorbing predetermined contingent equity resources termed “rights of assessment” which mutualizes the aggregate shock to all the clearing members. Furthermore, CCPs require initial margins as collateral from clearing members to recover idiosyncratic counterparty shock, which is usually the operating cost from novating the contract to another potential buyer or seller of the contract. In summary, the reduction of counterparty risks through netting, pooling (mutualization), and orderly distribution of losses are the key differences between trades that are centrally cleared compared to non-cleared transactions.

However failure of a large CCP could act as a channel of contagion, CCPs actions may have ‘procyclical’ effects by adjusting initial margin demands, and strict requirements upon its members cause limited access to the market to members with adequate financial and technical resources. Historically, there have been few incidences of CCPs failing, but when this has happened, the impacts on financial markets have been significant. In 1974, the Caisse de Liquidation failed due to trades put forward by members without the consent of their clients and high volatility in Paris White Sugar Market, leading to large margin calls that participants were unable to meet. More recently, the Kuala Lumpur Commodity Clearing House failed in 1983 after massive defaults on palm oil contracts following a market squeeze. The Hong Kong Futures Guarantee Corporation failed in the aftermath of the stock market crash of 1987 which led to the closure of stock and futures exchanges in Hong Kong for four days (Rehlon and Nixon, 2013).

Through intermediation of OTC counterparties, CCPs face a great amount of concentrated counterparty risk which makes it CCPs as systemically important financial institutions. It might even be the case that a CCP, while solvent, cannot meet immediate demands for the return of clearing member collateral as well (Singh, 2010). Thus, CCPs demand collateral (initial margin) from their counterparties. CCPs may decrease the systemic risk by reducing the impact of a default of bilateral clearing while they may increase the sys-
temic risk by increasing margin requirements during financial turmoil which exacerbates procyclicality. CCPs may be considered as risk pooling and sharing mechanism through the mutualization of the default funds. Hence, central clearing may reduce the overall margin requirements which are required in bilateral trades (Duffie et al., 2015).

B. Proofs

The following lemma is useful for the proofs of the next two results.

**Lemma 4.** For a given financial network that satisfies collateral constraints, the effective demand \([m_j(p)]^+\) is increasing in \(p\) for any \(j \in N\).

**Proof of Lemma 4.** It is enough to show that

\[
m_j(p) = c_1^j - \epsilon_j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) \mathbb{1}[p > y_{ij}] + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\}
\]

is increasing in \(p\). Since \(\min\{y_{ij}, p\} \leq p\), both \(\min\{p, y_{ij}\}\) and \(\min\{y_{jk}, p\}\) are increasing in \(p\) as can be seen in figure 1. For any value of promise \(\hat{y}\),

\[
\sum_{i \in N \setminus \{j\}} c_{ij} \min\{y_{ij}, \hat{y}\} \leq \sum_{k \in N} c_{jk} \min\{y_{jk}, p\}
\]

by collateral constraints. Therefore, the sum of the payments from other agents will always exceed the sum of payments that \(j\) has to pay to others.\(^{14}\) For fixed \(B(\epsilon)\), each \(m_j(p)\) is increasing in \(p\). Therefore, for any \(p' < p\), \(B(\epsilon) \subseteq B(\epsilon')\) and the indicator function for the bankruptcy cost is decreasing in \(p\). ■

**Proof of Proposition 1.** If \(p = s\), then we automatically have an equilibrium that satisfies inequality (3) or otherwise \(p\) cannot be \(s\). Now suppose \(p < s\). The equilibrium equation can be represented as

\[
p = \frac{\sum_{i \in N}[m_i(p)]^+}{\sum_{j \in N} c_{jj}}.
\]

Since the denominator is constant and \(h_i(p)\) and \([m_i(p)]^+\) are increasing in \(p\) by lemma 4, the equilibrium condition is a order-preserving function of \(p\) while \(p\) is in a subset of \(\mathbb{R}^+\) that

\(^{14}\)This is, in fact, the reason why there is a collateral constraints. It guarantees the agent to have non-negative amount of cash from all the payments netted out so that they can actually pay the debt at any price level of the given collateral.
is a complete lattice and the range is also a complete lattice. Then, by Knaster-Tarski’s
fixed point theorem, there exists a fixed point \( p \) and the set of \( p \) that satisfies the equilibrium
condition is also a complete lattice.
Now suppose that the supremum fixed point price \( \bar{p} \) is greater than \( s \), and we will show
that then either there exists a price \( 0 < p \leq s \) that is also a fixed point or \( p = s \) satisfies
equilibrium condition (3). If equation (2) is true when \( p = 0 \), then we already have a fixed
point with \( p \leq s \). If equation (2) is not true when \( p = 0 \), then that implies at least some
\( m_j(0) \) is positive for \( j \in N \) after subtracting the counterparty bankruptcy costs. Therefore,
\[
\sum_{i \in N} m_i(p) + \sum_{j \in N} c_{jj} \geq 0
\]
This implies that as \( p \) increases, the difference between the \( p \) and
\[
\sum_{i \in N} m_i(p) + \sum_{j \in N} c_{jj}
\]
will be eventually closed out at \( \bar{p} \) by intermediate value theorem. Therefore,
the two functions either meet for some \( p \leq s \) or the gap between them does not close out
even when \( p = s \) so equation (3) holds.

Proof of Proposition 2. For the proof, suppress the \( \epsilon \) term in bankruptcy set. If no
agent is going to bankrupt at any price \( p \in [0,s] \) then the equilibrium price is trivially and
uniquely determined as \( p = s \). Now suppose some agents go bankrupt at a low price \( \tilde{p} \) i.e.
\( B(\tilde{p}) \neq \emptyset \). Denote \( V_l \) as the set of agents such that there is a path between \( l \) and \( i \) for any
\( i \in V_l \). Suppose that \( l \notin B(\tilde{p}) \) and \( V_l \cap B(\tilde{p}) \neq \emptyset \). Thus, at least at some price close to (or
equal to) \( \tilde{p} \) the agent \( l \) will bear some bankruptcy cost and may go bankrupt. If there is no
agent \( l \) that satisfies \( z^l < \sum_{i \in V_l \cap B(\tilde{p})} \zeta(c_{il})1[\tilde{p} > y_{il}] \) from \( \tilde{p} = 0 \) to all the way up to \( s \), then
\( B(p) = B(p') \) for any \( p, p' \in [0,s] \) and in fact there is unique equilibrium.

Now suppose that for some price \( \tilde{p} \) and some agent \( l \), \( z^l < \sum_{i \in V_l \cap B(\tilde{p})} \zeta(c_{il})1[\tilde{p} > y_{il}] \) is
satisfied. Then, there exists \( p^* \) less than \( p \) (due to monotonicity of \( m_i(p) \)) such that \( \forall p' < p^* \),
\( m_i(p') < 0 \) and let \( l \) be the only one who goes bankrupt due to the price decline from \( p \) to
\( p < p^* \) without loss of generality. The left-hand-side of the market clearing condition, the
sum of effective money, can be decomposed as
\[
\sum_{j \in N} m_j(p)^+ = \sum_{j \in N} e_j^1 + \sum_{j \in N} c_{jj}p - \sum_{j \in N} \sum_{i \in B(p)} \zeta(c_{ij})1[p > y_{ij}]
- \sum_{j \in N} \min \left\{ \epsilon_j, e_j^1 - \sum_{i \in N \setminus \{j\}} c_{ij} \min \{p, y_{ij}\} - \sum_{i \in B(p)} \zeta(c_{ij})1[p > y_{ij}] + \sum_{k \in N} c_{jk} \min \{p, y_{jk}\} \right\}
\]
where the second term on the right hand side is coming from
\[ (p - y_{ij})^+ + \min\{p, y_{ij}\} = p. \]

Since the term is the same as the supply side of the equation, price is determined by the remaining cash from \( t = 0 \) and the amount of aggregate liquidity shock to the demand, bounded by its entire position, and the counterparty default costs. We can rewrite the market clearing condition into loss-coverage with remaining cash equality as
\[
\sum_{j \in N} e_j^i = \sum_{i \in B(p)} \sum_{j \in N} \zeta(c_{ij}) \mathbb{1}[p > y_{ij}]
\]
\[
+ \sum_{j \in N} \min \left\{ \epsilon_j, e_j^i - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(p)} \zeta(c_{ij}) \mathbb{1}[p > y_{ij}] + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\} \right\}
\]
(10)

Then, there can be a price \( \hat{p} \) such that the additional jump in bankruptcy cost \( \beta_l(p) \equiv \sum_{j \in N} \zeta(c_{ij}) \mathbb{1}[p > y_{ij}] \) coincides with the amount of decrease in losses from bankrupt agent’s endowments and counterparty costs i.e.
\[
\beta_l(\hat{p}) = \epsilon_l + \sum_{j \in B(p)} \left[ \sum_{i \neq j} (c_{ij} - \mathbb{1}_{i \in B(p)} \zeta(c_{ij})) (\mathbb{1} \{ p > \hat{p} \geq y_{ij} \} (p - \hat{p}) + \mathbb{1} \{ p \geq y_{ij} > \hat{p} \} (p - y_{ij}))
\]
\[
+ \zeta(c_{ij}) (\hat{p} - y_{ij})^+ + \sum_{k \in N} c_{jk} (\mathbb{1} \{ y_{jk} > p \geq \hat{p} \} (p - \hat{p}) + \mathbb{1} \{ p \geq y_{jk} > \hat{p} \} (y_{jk} - \hat{p})))
\]
\[
- \left[ e_1^i - \sum_{i \neq l} c_{il} \min\{\hat{p}, y_{il}\} - \sum_{i \in B(p)} \zeta(c_{il}) \mathbb{1}[\hat{p} > y_{il}] + \sum_{k \in N} c_{lk} \min\{\hat{p}, y_{lk}\} \right]
\]
(11)

and therefore, \( \hat{p} \) is also an equilibrium price. □

Proof of Proposition 3. For fair price, there exists unique equilibrium price no matter what happens in shocks and bankruptcies. Now focus on liquidity constrained prices. When \( \zeta(c) = 0 \) for any \( c \geq 0 \), equation (10), the market clearing condition with loss-coverage, becomes
\[
\sum_{j \in N} e_j^i = \sum_{j \in N} \min \left\{ \epsilon_j, e_j^i - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\} \right\}
\]
and by rehypothecation constraints, the right hand side is increasing in \( p \). Also the right hand side is bounded below by \( \sum_{j \in N} \min\{ \epsilon_j, e^j \} \) when \( p = 0 \). By intermediate value theorem, there exists a unique equilibrium price \( p \) between \([0, s]\) that satisfies the market clearing condition above.

For the second statement of the proposition, first note that the nominal wealth with no lender default cost is

\[
m_j(p) = e^j - \epsilon_j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\}.
\]

The sum of nonnegative nominal wealth is

\[
\sum_{j \in N} [m_j(p)]^+ = \sum_{j \in N} e^j + \sum_{j \in N} c_{jj} p - \sum_{j \in N} \min\{ \epsilon_j, e^j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\}\}
\]

which can be re-written as the sum of indegrees and outdegrees as below.

\[
\sum_{j \in N} [m_j(p)]^+ = \sum_{j \in N} e^j + nh_0 p - \sum_{j \in N} \min\{ \epsilon_j, e^j - \sum_{i \in N \setminus \{j\}} c_{ij} x_{ij} + \sum_{k \in N} c_{jk} x_{jk}\}
\]

which will have the same value with a network with

\[
\sum_{i \in N \setminus \{j\}} c_{ij} x_{ij} = \sum_{i \in N \setminus \{j\}} \hat{c}_{ij} \hat{x}_{ij}
\]

\[
\sum_{k \in N} c_{jk} x_{jk} = \sum_{k \in N} \hat{c}_{jk} \hat{x}_{jk}
\]

so networks \((C, X)\) and \((\hat{C}, \hat{X})\) have the same equilibrium price and final asset holdings. 

**Proof of Lemma**: For each agent \( i \in N \), the maximum cash he can hold for \( t = 1 \) is by saving all the cash while not lending any cash because borrowing requires collateral and no arbitrage condition will prevent anyone from making positive cash from borrowing. The price of the asset at \( t = 0 \) cannot exceed the most optimistic agent’s fair value since there is always a possibility of liquidity constrained under-pricing in \( t = 1 \). Thus, \( e_0 + h_0 s^1 \) is always the upper bound of the maximum amount of cash each agent can hold by selling all the asset endowments and not borrowing from or lending to anyone. Since \( G_j \) is differentiable with full support of \([0, \tilde{c}]\), any agent can go bankrupt regardless of how much cash they hold in \( t = 0 \) since \( G_j([e_0 + h_0 s^1, \tilde{c}]) \) is positive. Now suppose that agent \( j \) has zero cash holdings i.e.
$e^j_1 = 0$. Agent $j$’s nominal wealth depends on asset price $p_1$ which becomes zero if $p_1 = 0$. By equation \[10\] this implies that if every other agent $i \neq j$ goes bankrupt because of liquidity shocks, which happens with probability greater than $[G_j([e_0 + h_0 s^j, c])]^{n-1}$, while agent $j$ is not, which happens with positive conditional probability, the price of the asset becomes zero while agent $j$ is not bankrupt. Marginal utility of cash in such state becomes $\lim_{p_1 \to 0} \frac{s^j}{p_1}$ which is infinity. Hence, expected marginal utility of holding cash in $t = 0$ becomes infinity as well and agent $j$ would like to hold positive amount of cash for any $j \in N$. If $e^j_1 > 0$, then the only state that with infinite marginal utility of cash is when $\epsilon^j = e^j_1$ which happens with zero probability by differentiability of $G_j$. Thus, for any $j \in N e^j_1 > 0$ but $e^j_1$ is not necessarily all the cash available for $j$. ■

The following lemma shows that whenever leveraging is profitable for certain investment, the same leverage makes other investment more profitable than not leveraging.

**Lemma 5.** Suppose $\frac{a - p}{b - q} = \pi = \frac{c - p}{d - q}$ and $\frac{a}{b} < \frac{a - p}{b - q}$ for $a, b, c, d, p, q, \pi > 0$. Then, $\frac{c}{d} \leq \frac{c - p}{d - q}$.

**Proof of Lemma 5** Since $\frac{a - p}{b - q} = \pi$, $a - p = b\pi - q\pi$. By $\frac{a}{b} < \frac{a - p}{b - q}$, we obtain $a < b\pi$. By combining the previous equation and inequality, we have $p < q\pi$. Now suppose that $\frac{c}{d} > \frac{c - p}{d - q}$. Then, $\frac{c - p}{d - q} = \pi$ implies $c > d\pi$. Combining this with $p < q\pi$, we get $\frac{c - p}{d - q} > \pi$ which is a contradiction. Therefore, $\frac{c}{d} \leq \frac{c - p}{d - q}$.

**Proof of Lemma 2** From the return equation \[7\] we immediately get $y' > y$ should hold for agent optimality. First order derivative of leveraged return with respect to $y$ is

$$
\frac{s^j}{(q(y') - q(y))^2} \left( - E_j \left[ \frac{1}{p_1} \right] p_1 \geq y \right) \Pr_j(p_1 \geq y) - \frac{\zeta'(c_{ij})}{y} \mathbb{1} \{ i \in B(\epsilon) \} f_j(p_1) \left( q(y') - q(y) \right) + q'(y) E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} \right] - \min \left\{ 1, \frac{y}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \mathbb{1} \left[ 1 > \frac{y}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right)
$$

where $F_j$ and $f_j$ are distribution and density functions for $p_1$ with $j$’s belief. From the first order derivative of leveraged return, all we need to show is that

$$
\left( E_j \left[ \frac{1}{p_1} \right] p_1 \geq y \right) - \frac{\zeta'(c_{ij})}{y} \mathbb{1} \{ i \in B(\epsilon) \} f_j(p_1) \Pr_j(p_1 \geq y) \left( q(y') - q(y) \right) < q'(y) E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} \right] - \min \left\{ 1, \frac{y}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \mathbb{1} \left[ 1 > \frac{y}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right)
$$

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and then the return is increasing in $y$ and $j$’s optimal promise will be the corner solution.

From the positive cash holding and optimality we know that $q(y) = \frac{E_i \left[ \min \left\{ \frac{y}{p_1}, 1 \right\} \right]}{E_i \left[ \frac{1}{p_1} \right]}$.

By optimality of leveraged purchase of agent $j$, we have

$$E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} \right] \leq \frac{E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \left[ 1 \geq 1 \frac{y}{p_1} \right] \{ i \in B(\epsilon) \} \right]}{q(y') - q(y)}.$$ 

Since leveraging is profitable,

$$E_j \left[ \frac{\min \left\{ 1, \frac{y}{p_1} \right\}}{q(y)} \right] + \frac{\zeta'(c_{ij})}{p_1} \{ 1 \geq 1 \frac{y}{p_1} \} \{ i \in B(\epsilon) \} \leq \frac{E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} \right]}{q(y')}.$$ 

Thus, we get

$$(q(y') - q(y)) E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \left[ 1 \geq 1 \frac{y}{p_1} \right] \{ i \in B(\epsilon) \} \right] < q(y) E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \left[ 1 \geq 1 \frac{y}{p_1} \right] \{ i \in B(\epsilon) \} \right]$$

By dividing both sides with $y$, and since $\zeta'(c_{ij}) > 0$, we obtain

$$(q(y') - q(y)) E_j \left[ \min \left\{ 1, \frac{1}{y' p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1 y} \{ i \in B(\epsilon), p_1 \geq y \} \right] < \frac{q(y)}{y} E_j \left[ \min \left\{ 1, \frac{1}{y p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \left[ 1 \geq 1 \frac{y}{p_1} \right] \{ i \in B(\epsilon) \} \right].$$

By dividing both sides with $y$, and since $\zeta'(c_{ij}) > 0$, we obtain

$$(q(y') - q(y)) \frac{\min \left\{ 1, \frac{y}{p_1} \right\}}{E_i \left[ \frac{1}{p_1} \right]} < \frac{\zeta'(c_{ij})}{p_1 y} \frac{\min \left\{ y, \frac{1}{p_1} \right\}}{E_i \left[ \frac{1}{p_1} \right]} \{ i \in B(\epsilon), p_1 \geq y \}.$$ 

Thus, we get

$$q(y) = \frac{E_i \left[ \min \left\{ \frac{y}{p_1}, 1 \right\} \right]}{E_i \left[ \frac{1}{p_1} \right]} > \frac{E_j \left[ \min \left\{ \frac{y}{p_1}, 1 \right\} \right]}{E_j \left[ \frac{1}{p_1} \right]}.$$ 

This is true because of the following. Agents have same probability distribution over shock distribution $G$ and its density $g$. The shock distribution and corresponding market price distribution is a one-to-one mapping which has different upper bound for each $F_j$. Hence we have $E_j \left[ \min \left\{ \frac{y}{p_1}, 1 \right\} \right] = E_j \left[ \frac{y}{p_1} \right] Pr(p_1 \geq y) + Pr(p_1 < y)$ and $E_j \left[ \frac{1}{p_1} \right] = \frac{1}{p_1}$. 

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Agent $i$ and agent $j$ agree on the distribution of prices below $s_i$. Also they agree on the probabilities, but they do not agree on price distribution at prices greater than or equal to $s_i$. Thus, only thing that matters is the term $\xi_j \equiv E_j \left[ \frac{1}{p_1} \right] \text{Pr}(p_1 \geq y)$. Note that $\xi_j$ is increasing in $j$ because there will be more chances of higher prices for $p_1 \geq y$ which decreases $1/p_1$. Also $\gamma \equiv E_j \left[ \frac{1}{p_1} \right] \text{Pr}(p_1 < y)$ is the same for $i$ and $j$ for $y \leq s_i$. Thus, 

$$E_j \left[ \min \left\{ \frac{y}{p_1}, 1 \right\} \right] \frac{y\xi_j + \text{Pr}(p_1 < y)}{\xi_j + \gamma}.$$ 

Differentiating the return ratio with respect to $\xi_j$ gives 

$$\frac{d}{d\xi_j} \left( \frac{y\xi_j + \text{Pr}(p_1 < y)}{\xi_j + \gamma} \right) = \frac{y(\xi_j + \gamma) - y\xi_j - \text{Pr}(p_1 < y)}{(\xi_j + \gamma)^2}$$ 

which is positive since $y\gamma = E_j \left[ \frac{y}{p_1} \right] \text{Pr}(p_1 < y) > \text{Pr}(p_1 < y)$. Thus, return ratio is increasing in $\xi_j$ so return ratio of $i$ is greater than return ratio of $j$. Given that substituting the term derives 

$$q(y) = \frac{E_j \left[ \min \left\{ \frac{1}{p_1}, \frac{1}{y} \right\} \right]}{\frac{y}{p_1}} > \frac{E_j \left[ \frac{1}{p_1} \right] \text{Pr}(p_1 \geq y) + \frac{1}{y} \text{Pr}(p_1 < y)}{E_j \left[ \frac{1}{p_1} \right]}.$$ 

For $y$ less than $s_i$, 

$$q'(y) = \frac{E_i \left[ \frac{1}{p_1} \right] \text{Pr}(p_1 \geq y)}{E_i \left[ \frac{1}{p_1} \right]} > \frac{E_j \left[ \frac{1}{p_1} \right] \text{Pr}(p_1 \geq y)}{E_j \left[ \frac{1}{p_1} \right]}.$$
We can decompose $E_j \left[ \min \left\{ \frac{1}{y}, \frac{1}{p_1} \right\} - \zeta'(c_{ij}) \frac{1}{p_1 y} \mathbb{1} \{ i \in B(\epsilon), p_1 \geq y \} \right]$ into

$$E_j \left[ \frac{1}{p_1} - \zeta'(c_{ij}) \frac{1}{p_1 y} \mathbb{1} \{ i \in B(\epsilon) \} \big| p_1 \geq y \right] \Pr(p_1 \geq y) + \frac{1}{y} \Pr(p_1 < y).$$

Therefore, equation \[12\] becomes

$$(q(y') - q(y)) \left( E_j \left[ \frac{1}{p_1} - \zeta'(c_{ij}) \frac{1}{p_1 y} \mathbb{1} \{ i \in B(\epsilon) \} \big| p_1 \geq y \right] \Pr(p_1 \geq y) + \frac{1}{y} \Pr(p_1 < y) \right)$$

$$< \frac{E_j \left[ \frac{1}{p_1} \right] \mathbb{1} \{ p_1 \geq y \} \Pr(p_1 \geq y)}{E_j \left[ \frac{1}{p_1} \right]}$$

$$\times E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \mathbb{1} \left[ 1 > \frac{y}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right]$$

and by changing the coefficients

$$(q(y') - q(y)) \left( E_j \left[ \frac{1}{p_1} \mathbb{1} \{ p_1 \geq y \} \Pr_j(p_1 \geq y) - \frac{\zeta'(c_{ij})}{y} \mathbb{1} \{ i \in B(\epsilon) \} \mathbb{1} \{ p_1 = y \} f_j(p_1) \right)$$

$$< (q(y') - q(y)) E_j \left[ \frac{1}{p_1} - \zeta'(c_{ij}) \frac{1}{p_1 y} \mathbb{1} \{ i \in B(\epsilon) \} \big| p_1 \geq y \right] \Pr(p_1 \geq y)$$

$$< q'(y) E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ij}) \left[ 1 - \frac{y}{p_1} \right]^+ \mathbb{1} \{ i \in B(\epsilon) \} \right]$$

and thus the derivative is positive. Hence, agent $j$ will borrow as much as possible which is $s^i$ and maximizes her leverage. The same procedure can be done in the case where $i$ is also leveraging by borrowing from some agent $k > i$. Thus, every agent in the economy will promise the lender’s belief on asset payoff if they are gonna borrow positive amount. Also, the leverage is profitable for any $y$ by lemma \[5\].

In the second part of the lemma, we apply the result from the first part of the lemma and fix the contracts with promises of expected asset payoffs of the lenders. Suppose agent $j$ is borrowing both from $i$ and $k$ with the same probability of bankruptcy and $i < k$, for the same amount of contracts i.e. $c \equiv c_{ij} = c_{ik}$. The returns from both leveraged positions for $j$
are
\[ R_j^i \equiv \frac{s^j}{q(s^j) - q(s^i)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} 1 \left[ 1 > \frac{s^i}{p_1} \right] 1 \{ i \in B(\epsilon) \} \right] \]
\[ R_j^k \equiv \frac{s^j}{q(s^j) - q(s^k)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^k}{p_1} \right\} - \frac{\zeta'(c)}{p_1} 1 \left[ 1 > \frac{s^k}{p_1} \right] 1 \{ k \in B(\epsilon) \} \right] \]

Since \( q' \) is increasing at the right limit of \( s^k \) and \( j \) maximizes over \((s^k, s^i)\) at \( s^i \), relative increase in amount of borrowing (or decrease in down payment) should exceed the relative decrease in expected payoff at \( t = 1 \). Therefore, agent \( j \) prefers to borrow more from \( i \) over \( k \). ■

**Proof of Proposition 4.**

By lemmas 2 and 3, agent 1 borrows from 2, who borrows from 3, who borrows from 4, and so on. There will be no missing chain because of lemma 2 and the property of lender cost function \( \zeta \), that is at least some positive amount of borrowing occurs through the lending chain linking the agents in the order of optimism. Hence, all the contract prices are determined by the subsequent lender. In other words, contract prices for \( y \in [s^{j+1}, s^j] \) is determined by \( j \). From equation (8), we have \( j \)'s contract pricing formula as below.

\[
q(y) = q(s^{j+1}) + E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^{j+1}}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} 1 \left[ 1 > \frac{s^{j+1}}{p_1} \right] 1 \{ j + 1 \in B(\epsilon) \} \right] \frac{1}{p_1}.
\]

Since \( q(s^{j+1}) \) is determined by the perspective of \( j + 1 \), the only relevant factor is the second term. As \( y \) increases, the relevant lower bound of price for borrower default increases. Obviously, \( s^j \) is the maximum price in \( j \)'s perspective and \( q'(y) = 0 \) at \( y = s^j \). On the other hand, \( y = s^{j+1} \) provides no additional value and simply becomes \( q(y) = q(s^{j+1}) \). By Leibniz integral rule, for any \( y \in [s^{j+1}, s^j] \),

\[
q'(y) = \frac{1}{E_j \left[ \frac{1}{p_1} \right]} \int_y^{s^j} \frac{1}{p_1} dF(p_1) > 0
\]
\[
q''(y) = -\frac{1}{E_j \left[ \frac{1}{p_1} \right]} \frac{f(y)}{y} < 0
\]

where \( F \) is the distribution function of the asset price in \( t = 1 \) which comes from convolution of shock distributions and \( f \) is the density function of \( F \). Thus, \( q(y) \) is concavely increasing in
Denote $\gamma$ as the inverse function of $q(y)$ which is well defined in the domain of $y \in [s^{j+1}, s^j]$ since $q'(y) > 0$ in the domain and $q'(s^j) = 0$. By inverse function theorem of first and second order derivatives, for any $q(y)$ in the range of original function, we obtain

$$
\gamma'(q(y)) = \frac{1}{q'(y)} > 0
$$
$$
\gamma''(q(y)) = -\frac{q''(y)}{(q'(y))^3} > 0
$$

Now denote the gross interest rate function as $\delta(q) \equiv \frac{\gamma(q)}{q}$ where $q$ is in the range of $q(y)$. The first derivative of the gross interest rate function becomes

$$
\delta'(q) = \frac{\gamma'(q)q - \gamma(q)}{q^2} = \frac{q(y)}{q'(y)} - \frac{y}{q(y)^2}
$$

where $\gamma(q) = y$. The numerator of the term can be rearranged as $q(y) - yq'(y)$ and this is positive because

$$
q(y) = q(s^{j+1}) + \left[ E_j \left[ \min \left\{1, \frac{y}{p_1} \right\} - \min \left\{1, \frac{s^{j+1}}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \mathbb{1} \left[ 1 > \frac{s^{j+1}}{p_1} \right] \mathbb{1} \{j + 1 \in B(\epsilon)\} \right] E_j \left\{ \frac{1}{p_1} \right\} \right]
$$

$$
> \frac{E_j \left[ \frac{y}{p_1} \mathbb{1} \{p_1 > y\} \right] Pr_j(p_1 > y)}{E_j \left\{ \frac{1}{p_1} \right\}}
$$

where the last inequality is positive because the leveraged return should be greater than the return from non-leveraged purchase of contract of $y$ subtracted by the value of keeping the collateral in case of borrower default. Therefore, the gross interest rate is increasing in $y$. The second derivative of the gross interest rate function becomes

$$
\delta''(q) = \frac{1}{q^4} \left[ q^2 \left( \gamma''(q)q + \gamma'(q) - \gamma'(q) \right) - 2q \left( \gamma'(q)q - \gamma(q) \right) \right]
$$
and the numerator is
\[
\gamma''(q)q^3 - 2q^2\gamma'(q) + 2q\gamma(q) = -q''(y) + 2q(y) [y - q(y)\gamma'(q(y))] \\
= \frac{f_j(y)/y}{E_j \left[ \frac{1}{p_1} \right]} - 2q(y) \left[ \frac{q(y)/q'(y) - y}{E_j \left[ \frac{1}{p_1} \right]} \right] \\
= \frac{f_j(y)/y}{E_j \left[ \frac{1}{p_1} \right]} - 2q(y) \left[ q(y) \int_{y}^{s_j} \frac{1}{p_1} dF_j(p_1) \right] - y
\]

which is negative because \(q(y) > yq'(y)\) as above and also \(q(y)/q'(y) - y > 1\) implies the inequality to be trivial and \(q(y)/q'(y) - y \leq 1\) also means the first term to be negligible compared to the integral in \(q(y)\) of the second term. Thus, \(y/q(y)\) is concavely increasing in the interval of \(q(y) \in [q(s_j^{i+1}), q(s_j)]\). Now we need to check for the kink points and the whole graph. Because \(q'(s_j) = 0\), \(\delta'(q)\) goes to infinity and that is why \(q'(s_j)\) is infinity. A unique property of the pricing of equation (8) is that \(y\) close to \(s_j^{i+1}\) will make \(q(y) < q(s_j^{i+1})\) coming from the left limit of \(q(s_j^{i+1})\). Therefore, there are intersections around each point of \(s_j\) for \(j \in N\) as can be seen in the figure 18. Since the borrowers would rather prefer to borrow from low \(y\) for higher \(q(y)\), the market price function for \(q(y)\) will take the upper envelope of the functions \(q\)'s defined for each interval \((s_j^{i+1}, s_j]\) for \(j = 1, 2, \ldots, n - 1\). Hence, the inverse function of \(q\), \(\gamma\) will have jumps at each point of \(q(s_j)\) for \(j \neq 1, n\) and the right derivative is greater than the left derivative of each point. Finally, since the upper envelope of \(q\)'s are continuous because above \(s_j\) there is a point that borrowers prefer to simply borrow from \(j\) at constant price rate up to the point that \(j - 1\) becomes the preferred lender when \(q(y)\) is greater than or equal to \(q(s_j)\). Therefore, both the upper envelope function of market price \(q(y)\) is continuous and interest rate function is also continuous. \(\blacksquare\)

**Proof of Lemma 3.** Suppose agent \(j > 1\) is buying the asset while agent 1 is not buying, then agent 1 will have even larger amount of cash holdings. If agent 1’s cash holdings \(e_1^1\) is large, then \(j\)'s return of cash is large. Return from the asset purchase for agent \(j\) is \(s_j/p_0\). By lemma 1, agent \(j\) should equate the returns from cash and asset as
\[
\frac{s_j}{p_0} = E_j \left[ \frac{s_j}{p_1} \right].
\]
But then, \(\frac{s_j}{p_0} < \frac{s_1}{p_0} < E_1 \left[ \frac{s_1}{p_1} \right]\). Agent \(j\) can sell the asset lower than the market price to agent 1 and accumulate more cash because of the gap between the two returns. This implies
agent \( j \) would rather sell her asset to agent 1 and both make profitable trades. The same inference can be done with levered purchases as both agents can do the same borrowing from the same set of lenders and simply change the price as the down payment such as \( p_0 - q(s^i) \). The second statement holds with the similar argument as the problem becomes isomorphic by substituting the asset with the promise of \( s^2 \) (which is coming from lemma 2) from agent 1 and so forth. ■

**Proof of Proposition 5.** By lemma 2 fix the equilibrium contract matrix \( Y \) as \( y_{ij} = s^i \) for any \( i > j \). From the contract pricing equation from equation (8),

\[
q(s^j) - q(s^{j+1}) = E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^{j+1}}{p_1} \right\} - \frac{\zeta'(c_{j+1,j})}{p_1} \mathbb{1} \left[ 1 > \frac{s^{j+1}}{p_1} \right] \mathbb{1} \{ j + 1 \in B(\epsilon) \} \right]
\]

\[
= E_j \left[ 1 - \min \left\{ 1, \frac{s^{j+1}}{p_1} \right\} - \frac{\zeta'(c_{j+1,j})}{p_1} \mathbb{1} \left[ 1 > \frac{s^{j+1}}{p_1} \right] \mathbb{1} \{ j + 1 \in B(\epsilon) \} \right]
\]

Agent \( j \) makes positive return out of this margin purchase only if \( p_1 > s^{j+1} \). The denominator of the equation is

\[
E_j \left[ \frac{1}{p_1} \right] = \int \frac{1}{p_1} dG(\epsilon)
\]

Figure 18: Graph of Contract Prices
while the numerator without the counterparty risk becomes

\[ E_j \left[ 1 - \min \left\{ 1, \frac{s_j^{j+1}}{p_1} \right\} \right] = \int_{p_1 > s_j^{j+1}} \frac{p_1 - s_j^{j+1}}{p_1} dG(\epsilon). \]

As \( j \) decreases, i.e. becomes more optimistic agent, the probability of \( \frac{s^j - s_j^{j+1}}{s^j} \) becomes smaller as agents agree upon the distribution of liquidity shocks and underpricing. Also the maximum return from leveraged purchase \( \frac{s^j - s_j^{j+1}}{s^j} \) is (weakly) increasing with \( j \) as well because the belief is harmonically dispersed and

\[
\begin{align*}
s^j s^{j+2} & \leq (s^{j+1})^2 \\
s^j s^{j+1} + s^j s^{j+2} & \leq s^j s^{j+1} + (s^{j+1})^2 \\
s^j s^{j+1} - (s^{j+1})^2 & \leq s^j s^{j+1} - s^j s^{j+2} \\
\frac{s^j - s^{j+1}}{s^j} & \leq \frac{s^{j+1} - s^{j+2}}{s^{j+1}}.
\end{align*}
\]

Each agent’s cash holding becomes

\[ e^j_1 = e_0 + h_0 q(s^1) - (q(s^j) - q(s^{j+1})) c \]

for all \( j \in N \) where \( q(s^{n+1}) = 0 \). Difference of cash holdings between agent \( j \) and \( j+1 \) is

\[ e^j_1 - e^{j+1}_1 = (q(s^{j+1}) - q(s^{j+2})) - (q(s^j) - q(s^{j+1})) > 0 \]

for any \( j < n \), so \( e^1_1 > e^2_1 > \cdots > e^n_1 \). ■

**Proof of Theorem 1.** The first and second properties come directly from propositions \[4\] and lemmas \[2\] and \[3\]. The third property comes from the indifference equation for the borrower \( j \) who has to be indifferent between borrowing cash from \( i \) and \( k \) if \( j \) is borrowing from the two in positive amount. The fourth property is simply from the budget constraint and contract prices and the fifth property is again derived from lemma \[3\].

Now we show that an equilibrium that satisfies those properties exists. Denote \( \hat{C} \) as \( C \) with all diagonal elements \( c_{ii} \) for \( i \in N \) set as zero. Define \( Z \equiv \hat{C} \circ Y \). Consider a class of networks \( Z \) such that \( Z \in Z \) satisfies rehypothecation constraint for fixed \( Y \) s.t. \( y_{ij} = s^i \) for any \( i, j \in N \). Now use the matrix ordering to compare the total amount of promises as \( Z > Z' \) implies \( Z_{ij} \geq Z'_{ij} \) for all \( i, j \in N \) and at least one element has strict inequality. Similarly, \( Z \geq Z' \) can be defined allowing equality for every entry. Note that this ordering is only a partial ordering among \( Z \). There can be networks \( Z, Z' \in Z \) with neither \( Z \geq Z' \) nor
\( Z' \geq Z \) is true. However, \( (Z, \geq) \) forms a complete lattice because for any subset \( Z' \subseteq Z \), the least upper bound \( \overline{Z} \) with \( \overline{Z}_{ij} = \max_{\overline{Z}_{ij} \in Z'} \overline{Z}_{ij} \) and the greatest upper bound \( \overline{Z} \) with \( \overline{Z}_{ij} = \min_{\overline{Z}_{ij} \in Z'} \overline{Z}_{ij} \) exist because each element is from a subset of Euclidean space.

Let \( V : Z \rightarrow Z \) be a function from network to network that is given the price and counterparty risk distribution of the first network in \( t = 1 \), what are the agents’ optimal network formation decisions as best responses (which is unique under the characterization of the theorem because of linear utility under maximum payment equilibrium selection rule).

Now I show that \( V \) is monotonous in \( Z \). Let \( Z \) be the network with \( \|Z\| = 0 \) i.e. no risk of bankruptcy and dispersion of cash holdings. Under \( Z \), return from cash holding is minimized by proposition \( 6 \). By rehypothecation constraint, \( V(Z) \leq \overline{Z} \) where \( \overline{Z} \) denotes the maximum leverage network i.e. the single chain network with full borrowing. Similarly, \( V(Z) \geq Z \) due to lower bound. Therefore, range of \( V \) is compact. Since the network satisfies rehypothecation constraints, lemma 2 and proposition 6 imply large \( Z \) implies the existing network has low degree of diversification and large default costs. Thus, \( p_1 \) is more volatile under higher \( Z \). All of these make \( E[1/p_1] \) greater, i.e. return of cash holding is greater and agent’s return on leverage goes down. Thus, any increase in \( Z \) will make the optimal response to the given distribution of \( Z \) to be lowering \( \|Z\| \). Increase in \( Z \) has two effects to the return calculation. First, it increases counterparty exposure \( \omega_{ij}(Z) \) and default cost which implies \( E_j[1 \{i \in B(\epsilon)\}] \) and \( \beta_i(p_1) \) increase. Second, the state in which the liquidity is constrained is exactly the state in which the optimists are under liquidity shock i.e. when they would have really wanted to have additional liquidity. The marginal value of cash in such state is even greater. Thus, large \( Z \) implies greater price and counterparty risks which make agents to diversify or reduce borrowing/lending in general. The equilibrium portfolio decision holds as

\[
E_j[1/p_1] = \frac{s^j}{q(s^j) - q(s^k)} E_j \left[ 1 - \min \left\{ 1, \frac{s^j}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} 1 \{ 1 > \frac{s^j}{p_1} \} 1 \{ i \in B(\epsilon) \} \right]
\]

\[
= \frac{s^j}{q(s^j) - q(s^k)} E_j \left[ 1 - \min \left\{ 1, \frac{s^k}{p_1} \right\} - \frac{\zeta'(c_{kj})}{p_1} 1 \{ 1 > \frac{s^k}{p_1} \} 1 \{ k \in B(\epsilon) \} \right].
\]

The critical feature of the optimization problem is that higher leverage in the economy implies that there are multiple ways of contagion of bankruptcy compared to well diversified lower leverage case. Whoever you are borrowing from has increased probability of bankruptcy and eventually you end up paying the cost no matter what. Thus, \( V(Z) \) decreases as \( Z \) increases. Then, \( V \) is a monotonic function on a complete lattice and there exists a fixed point network \( Z^* \) such that \( Z^* = V(Z^*) \). Therefore, there exists a network equilibrium.

The following lemma characterizes the properties of the inverse of equilibrium price,
especially with respect to indegree of the bankrupt agents. It will be used to prove proposition 4.

**Lemma 6 (Convexity of Inverse Price).** Consider a class of collateral debt networks \((N, C, Y, z, s)\) with \(C > 0\) that satisfies rehypothecation constraints. Suppose that \(j \in B(\epsilon)\). Then, the inverse of the price of the asset \(\frac{1}{p}\) is convexly decreasing in \(c_{ij}\) and convexly increasing in \(c_{jk}\) for any \(i\) and \(k\) in \(N\). The convexity of inverse of the price with respect to \(c_{ij}\) and \(c_{jk}\) is strict up to the point \(p = y_{ij}\) and \(p = y_{jk}\) respectively.

**Proof of Lemma 6.**

For prices \(p = s\) and \(p = 0\), the result is trivially true. Now consider the intermediate case of \(p = \pi(p)\). Recall that

\[
\frac{1}{p} = \sum_{i \notin B(\epsilon)} z_i^2 + \sum_{j \in B(\epsilon)} \sum_{p < y_{ij}} c_{jk} - \sum_{j \in B(\epsilon)} \sum_{i \neq j}^{p < y_{ij}} c_{ij} + \sum_{j \in B(\epsilon)} \sum_{k \in N} \sum_{p < y_{jk}} c_{jk} y_{jk} - \sum_{j \in B(\epsilon)} \sum_{k \notin B(\epsilon)} \sum_{p < y_{jk}} \zeta(c_{jk}).
\]

Denote \(\frac{1}{p} = \left(\frac{\text{num}}{\text{den}}\right)\). Suppose \(j \in B(\epsilon)\) and we differentiate the inverse price with respect to \(c_{ij}\) which will become

\[
\frac{\partial(1/p)}{\partial c_{ij}} = \begin{cases} 
-\frac{1}{\text{den}} < 0, & \text{if } p < y_{ij} \text{ and } i \notin B(\epsilon) \\
-\frac{(\text{num})y_{ij}}{\text{den}^2} < 0, & \text{if } p \geq y_{ij} \text{ and } i \notin B(\epsilon) \\
0, & \text{if } i \in B(\epsilon)
\end{cases}
\]

and differentiating with respect to \(c_{ij}\) once more gives

\[
\frac{\partial^2(1/p)}{\partial c_{ij}^2} = \begin{cases} 
0, & \text{if } p < y_{ij} \text{ or } i \in B(\epsilon) \\
2\frac{(\text{num})^2y_{ij}^2}{\text{den}^3} > 0, & \text{if } p \geq y_{ij} \text{ and } i \notin B(\epsilon).
\end{cases}
\]

Thus, \(\frac{1}{p}\) is convexly decreasing in \(c_{ij}\) with strict convexity up to the point \(p = y_{ij}\). Now
differentiate inverse price with respect to lending of bankrupt agent \( j \), \( c_{jk} \).

\[
\frac{\partial(1/p)}{\partial c_{jk}} = \begin{cases} 
\frac{1}{(\text{den})(y_jk + \zeta'(c_{jk}))} > 0 , & \text{if } p < y_jk \text{ and } i \notin B(\epsilon) \setminus \{j\} \\
\frac{(\text{den})^2}{(\text{num})(y_jk + \zeta'(c_{jk}))} > 0 , & \text{if } p \geq y_jk \text{ and } i \notin B(\epsilon) \setminus \{j\} \\
0 , & \text{if } i \in B(\epsilon) \setminus \{j\}
\end{cases}
\]

The second derivative becomes zero for the case of \( p < y_jk \) and \( i \in B(\epsilon) \setminus \{j\} \). In the case of \( p \geq y_jk \) and \( i \notin B(\epsilon) \setminus \{j\} \), the numerator of the second derivative becomes

\[
(den)^2(num)\zeta''(c_{jk}) + 2(den)(y_jk + \zeta'(c_{jk}))^2
\]

which is again positive. Therefore, inverse of price is convexly increasing in indegree and strict convexity holds up to the point \( p = y_jk \). \bull

The following lemma is also used in proving proposition 6.

**Lemma 7** (Counterparty Risk Order). For any network equilibrium and any agent \( j \in N \), \( \zeta(c_{ij})\omega_{ij} \geq \zeta(c_{kj})\omega_{kj} \) for any \( j < i < k \).

**Proof of Lemma 7**. Consider the return equations. For \( c_{ij} = c_{kj} = c \), \( R_j^i > R_j^k \) as shown in lemma 2 where

\[
R_j^i \equiv \frac{s^j}{q(s^i) - q(s^i)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \frac{\zeta'(c)}{p_1} \left[ 1 > \frac{s^i}{p_1} \right] \left\{ i \in B(\epsilon) \right\} \right]
\]

\[
R_j^k \equiv \frac{s^j}{q(s^j) - q(s^k)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^k}{p_1} \right\} - \frac{\zeta'(c)}{p_1} \left[ 1 > \frac{s^k}{p_1} \right] \left\{ k \in B(\epsilon) \right\} \right]
\]

Since agent \( j \) has the highest return when she from borrow from the more optimistic lender, agent \( k \) should have a lower counterparty risk in the perspective of agent \( j \) in order to make the indifference condition \( R_j^i = R_j^k \) hold. If agent \( j \) is posting zero amount of collateral to \( k \), then the condition still trivially holds as \( \zeta(c_{kj})\omega_{kj} = 0 \) while \( \zeta(c_{ij})\omega_{ij} > 0 \). \bull

**Proof of Proposition 6**. Since every belief is bounded above by \( s^i \), increase in volatility implies decrease in expectation of \( p_1 \) and also increase in expected sum of default costs. Suppose that \( (s, \epsilon) \) is realized and \( j \in B(\epsilon) \), which happens with positive probability because of the distribution of \( \epsilon_j \). By lemma 6, \( c_{ij} \) convexly decreases the inverse price and \( c_{jk} \) convexly increases the inverse price for \( i, k \in N \). By rehypothecation constraints and \( Y \) following rank order, \( c_{jk} \geq c_{ij} \) and \( y_{jk} > y_{ij} \). Also from the proof of lemma 6, the slope from \( c_{jk} \) dominates the slope from \( c_{ij} \). Thus, any inverse of price \( p_1 \geq y_{ij} \) will be convexly increasing in \( c_{jk} \).
Suppose that $C^*$ is uniformly less indebted than $C$. Holding the bankruptcy state realizations the same, this decrease in debt decreases volatility directly from the above argument of lemma 6 as for each realization of $(s, \epsilon)$ involving bankruptcy will have smaller impact on increase in inverse price and price decline. Also the decrease will generate less states of bankruptcy as every agent becomes less susceptible to price as in the wealth equation

$$m_j(p) = e^j_1 - \epsilon_j - \sum_{i \in N\setminus\{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) \mathbb{1}[p > y_{ij}] + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\}$$

has smaller coefficients on prices and also the bankruptcy of lenders have smaller impact and less second-order bankruptcy will occur for the same state realizations.

Now suppose that $C^*$ is a diversification of $j$ from $C$. From the application of lemma 6 in the beginning, the direct price effect from deconcentration is always positive and the states that incur bankruptcy are fewer. In order to consider the effect from the counterparty channel, consider the simplest case of three agents 1, 2, and 3 in a network. Suppose agent 1 is borrowing more from agent 2 than from agent 3 i.e., $c_{21} > c_{31}$. By diversification of agent 1, $\zeta(c_{21}^*) + \zeta(c_{31}^*) < \zeta(c_{21}) + \zeta(c_{31})$ by convexity of $\zeta$. Also, agent 2 has less collateral from agent 1 to reuse. Lower collateral makes agent 2’s borrowing from agent 3 less so $c_{32} \geq c_{32}^*$ due to rehypothecation constraint. Even though agent 1’s promise becomes smaller by $y_{21} > y_{31}$ which implies that it is more susceptible to lender bankruptcy, but the reduction of rehypothecation means the susceptibility is only replaced by the identity of the agent, from 2 to 1.

The only case left is that diversification happens and it does not affect any change in intermediation i.e., rehypothecation constraint is not binding. Now the conditions are

$$\zeta(c_{21}) > \zeta(c_{31})$$
$$\zeta(c_{21}^*) > \zeta(c_{31}^*)$$
$$\zeta(c_{21})\omega_{21} > \zeta(c_{31})\omega_{31}$$
$$\zeta(c_{21}^*)\omega_{21} > \zeta(c_{31}^*)\omega_{31}$$
$$\zeta(c_{21}) + \zeta(c_{21}^*) > \zeta(c_{31}) + \zeta(c_{31}^*)$$
$$\zeta(c_{21}) > \zeta(c_{21}^*)$$
$$\zeta(c_{31}) < \zeta(c_{31}^*)$$

where the third and fourth equalities come from lemma 7 and the condition of diversification.
By rearranging the inequalities we obtain
\[ \zeta(c_{21})\omega_{21} + \zeta(c_{31})\omega_{31} > \zeta(c^*_{21})\omega_{21} + \zeta(c^*_{31})\omega_{31} \]
thus, the expected default cost is lower under diversification. Also, even the bankruptcy probability change goes into the same direction. By the distributional assumption on \( G \) and because of the second order bankruptcy of agent 2 is now even more likely when agent 3 is bankrupt, \( \omega_{21|C^*} - \omega_{21|C} > \omega_{31|C^*} - \omega_{31|C} \). Thus,
\[ \zeta(c_{21})\omega_{21|C} + \zeta(c_{31})\omega_{31|C} > \zeta(c^*_{21})\omega_{21|C^*} + \zeta(c^*_{31})\omega_{31|C^*} \]
and the increased case of greater default cost from 3 is dominated by decrease of default cost from more likely event of agent 2’s bankruptcy. Thus, the counterparty channel also decreases the aggregate expected deadweight loss and increases expected price. Therefore, diversification in this case decreases aggregate expected deadweight loss and increases expected price and decreases volatility.

Finally, we can extend this argument of three agents to any general number of agents. For any \( j \in N \), \( c_{L,j} > c^*_{L,j} \) while keeping \( \sum_{i \in N \setminus \{j\}} c_{ij} = \sum_{i \in N \setminus \{j\}} c^*_{ij} \) implies there is an agent \( i > L \) such that \( c_{ij} < c^*_{ij} \). Using the same argument for agent 1, 2, and 3 on agent \( j \), \( L_j \), and \( i \) will provide for the same result. If agent \( j \) is diversifying even further, then that will divide \( c_{L,j} \) into even further diversification and convexity will make it even lower aggregate expected default cost. Thus, any deconcentration increases expected price and decreases aggregate expected default cost and volatility.

**Proof of Theorem 2.** As discussed in the description
\[ \frac{\partial}{\partial c_{ik}} \sum_{j \in N} E_j \left[ m_j(\epsilon) \frac{s^j}{p_1(\epsilon)} \right] \neq \frac{\partial}{\partial c_{ik}} E_j \left[ m_j(\epsilon) \frac{s^j}{p_1(\epsilon)} \right] \]
and due to counterparty externality and price externality coming from the arguments in proposition 6, the direction of inefficiency is coming from under-diversification.

**Proof of Proposition 7.**

1. Suppose that \( s^i \) increases to \( s^i + \eta \) for every \( i \in N \). As shown in proposition 4, \( q(y) \) is increasing in \( y \) for any \( y \in [s^i, s^i + \eta] \) and \( q'(y) < 1 \) by the lower bound of \( y \) in the
numerator. By equation (8) the function for contract price becomes

\[ q(y) = q(s^{i+1}) + \frac{E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^{i+1}}{p_1} \right\} - \frac{\zeta'(c_{j+1,j})}{p_1} \mathbb{1} \left[ 1 > \frac{s^{i+1}}{p_1} \right] \mathbb{1} \{ j + 1 \in B(\epsilon) \} \right]}{E_j \left[ \frac{1}{p_1} \right]} \]

Any change in the terms related to \( q(s^j) \) has a direct effect of increase in \( q(s^i) \) in linear terms for any \( i < j \) by the recursive equation

\[ q(s^i) = q(s^j) + \sum_{k \in \{ i+1, i+2, \ldots, j-1 \}} E_k \left[ 1 - \min \left\{ 1, \frac{s^{k+1}}{p_1} \right\} - \frac{\zeta'(c_{k+1,k})}{p_1} \mathbb{1} \left[ 1 > \frac{s^{k+1}}{p_1} \right] \mathbb{1} \{ k + 1 \in B(\epsilon) \} \right] \]

As in the argument in the proof of proposition 4, for any agent \( k < j \), prices relevant to cashflow of the leveraged contracts are bounded below by the subject belief of the lender \( k + 1 \), \( s^{k+1} \) as in \( s^k E_k \left[ 1 - \min \left\{ 1, \frac{s^{k+1}}{p_1} \right\} - \frac{\zeta'(c_{k+1,k})}{p_1} \mathbb{1} \left[ 1 > \frac{s^{k+1}}{p_1} \right] \mathbb{1} \{ k + 1 \in B(\epsilon) \} \right] \).

However, the return from cash holdings, \( s^k E_k \left[ \frac{1}{p_1} \right] \) is not bounded by any prices. The ratio between the changes of the two terms is increasing in \( k \) as the lower bound of the price distribution becomes smaller, that is

\[ \frac{\Delta E_k \left[ 1 - \min \left\{ 1, \frac{s^{k+1}}{p_1} \right\} - \frac{\zeta'(c_{k+1,k})}{p_1} \mathbb{1} \left[ 1 > \frac{s^{k+1}}{p_1} \right] \mathbb{1} \{ k + 1 \in B(\epsilon) \} \right]}{\Delta E_{k+1} \left[ 1 - \min \left\{ 1, \frac{s^{k+2}}{p_1} \right\} - \frac{\zeta'(c_{k+2,k+1})}{p_1} \mathbb{1} \left[ 1 > \frac{s^{k+2}}{p_1} \right] \mathbb{1} \{ k + 2 \in B(\epsilon) \} \right]} \]

Thus, a direct increase in \( s^i \) dominates the changes in the denominator and in the expectations of the return equation

\[ R_i^{i+1} = \frac{s^i}{q(s^i) - q(s^{i+1})} E_{i+1} \left[ \min \left\{ 1, \frac{s^i}{p_1} \right\} - \min \left\{ 1, \frac{s^{i+1}}{p_1} \right\} - \frac{\zeta'(c)}{p_1} \mathbb{1} \left[ 1 > \frac{s^{i+1}}{p_1} \right] \mathbb{1} \{ i + 1 \in B(\epsilon) \} \right] \]

Thus, higher counterparty risk can be justified as the leverage return for agent \( i \) increases. Agent \( i \) will increase \( c_{i+1,i} \) more which implies less links (intensively and extensively) if \( i \) was diversifying and the velocity of collateral (weakly) increases by
the increase in \( c_{i+1,i} \) and also relaxing collateral constraints for the subsequent agents \( i + 1, i + 2, \ldots, n \).

Also changes in \( q(s^j) \) has indirect effects by the induced borrowing pattern changing the relative distribution of prices \( F_i \) for given liquidity shocks \( \epsilon \) and return on cash holdings \( E_i \left[ \frac{S^j}{p_1} \right] \) and also changing the probability of bankruptcy of the lenders. First, there will be a change in price distribution of \( \tilde{p}_1 \) which influences both denominator and numerator of equation (8). The increase in agents’ debts will increase the price volatility by proposition 6. The effect from indirect increase in bankruptcy probability is confined by the increase in \( E_r \left[ \min\{1, \frac{\eta}{p_1}\} \right] / E_r \left[ \frac{1}{p_1} \right] \) always dominates the indirect effect. Hence, \( q(s^j) \) and leverage increases and \( R^j_r \) increases for all \( i < j \) which implies the velocity of collateral increases.

The last thing to check is that whether the change will affect the agents below agent \( i \). Note that the increase in \( c_{k,l} \) for any \( k, l \leq j \) does not affect expected sum of lender default costs of each agent in \( \{ j + 1, j + 2, \ldots, n \} \) because any promise between agents \( k, l \leq j \) is going to be defaulted no matter what in their perspective of the upper bound of the asset price \( s^{j+1} > s^{j+2} > \cdots > s^n \). Thus, the debt amount or even the change in price distribution is irrelevant to these pessimistic agents. The only change for them comes from the increase in asset price \( p_0 = q(s^1) \) which increases their nominal value of endowments which incentivizes them to increase borrowing and increase the reuse of collateral i.e. the velocity of collateral.

2. Suppose \( \theta_j \) decreases by \( \eta \) for all \( j \in N \). Then again from \( R^{i+1}_r \) increases due to lower probability of default costs and \( c_{i+1,i} \) increases. The rest of the argument goes the same as in the previous case. In this case, it is even more simpler because there is a reduction of counterparty risk in every link that even offsets the indirect change.

\[ \square \]

**Proof of Proposition 8.** From equation (9) an individual agent does not care about the terms of \( g \) and \( \sum_{i \in N} \mathbb{1}_{\{i \notin B(\epsilon)\}} \cdot m_0(\epsilon|p_1) \) since they are determined by the macro variables and the agent consider herself as a price-taker. Under the case of 1 and 2, the term \( E_j \left[ \mathbb{1}_{[p_1 > y_{ij}, i \in B(\epsilon)]} \right] \) equals to zero. Therefore, each agent does not have any incentive
to diversify and lower leverage and will maximize their leverage. The equilibrium network under CCP has a collateral matrix $C_{ccp}$ which has a greater debt than the debt of decentralized equilibrium network $C$, by being more indebted (the opposite of less indebted) and less deconcentrated (the opposite of deconcentration) minimizing the deconcentration of a network. By proposition 6, this equilibrium network maximizes the systemic risk by maximizing the sum of expected default costs. Even if $g$ is not large, and CCP can go bankrupt in some states, agent $j$’s perceived risk of borrowing from agent $i$, $E_j [1[p_1 > y_{ij}, 0, i \in B(\epsilon)]]$, is always smaller than $\omega_{ij} = E_j [1[p_1 > y_{ij}, i \in B(\epsilon)]]$ under decentralized equilibrium and the debt of the network becomes larger either by more indebted or less deconcentration. As in the argument in the proof of theorem 2, the positive externality becomes even less incorporated into agent’s individual decision-making and the systemic risk is always greater under $C_{ccp}$ than the systemic risk under $C$.

**Proof of Proposition 10.** Suppose only one contract $y$ is available in the market. As in lemma 3, agent 1 will buy the asset and borrow cash from agents who has $s^j \geq y$ with equal weights as diversification. If agent 1’s endowment $e_0$ is not enough to purchase all the assets with the downpayment, then agent 2 also joins the buyer side and borrows from other pool of lenders. This can be repetitively done for agent 3, 4 and so forth. Similarly, if the demand for cash is too high, then the price of the contract $q(y)$ will decrease and even agents with $s^j < y$ can become a lender similar to the argument in lemma 3. Since the maximization problem and the budget constraints with down payments are all monotone, there is always an equilibrium. The resulting network becomes a complete bi-partite network for the given component of market participants. Since agents have no trade-off between choice of counterparties and choice of leverage, they have no incentives to change their network formation behavior even after eliminating the counterparty risk concerns $E_j [1[p_1 > y_{ij}, i \in B(\epsilon)]]$. Since all the walks in the network have length of 1, there will be no effect from netting as well.

**References**


