The identification of beliefs

with

Felix Kübler

April 4, 2019
The demand for assets as prices vary identify beliefs and preferences towards risk.
An individual trades in financial assets to maximize his stationary and time-separable (subjective) expected utility over two dates.

We observe how his initial date demand for consumption and assets varies with prices and wealth, while the investor’s beliefs over stochastic asset payoffs and endowments remain fixed and have finite support.

We investigate conditions under which one can identify the investor’s beliefs and attitudes towards risk.
Cardinal utility can always be identified (locally) from the demand for date 0 consumption.

Our main result is that identification is possible if the indirect marginal utilities for wealth, across realizations of uncertainty, are linearly independent.

This condition can be verified in a wide variety of situations:

If the payoffs of risky assets separate uncertainty, beliefs can be identified whenever

a) the individual’s endowments are known or

b) no derivative of cardinal utility is the product of the exponential function and some periodic function.
Without any assumptions on the risky assets, in the presence of a risk-free asset, beliefs over endowments and payoffs of assets can be identified if, for any any distinct \((e_k)_{k=1}^K\), the functions \((u'(e_k + x))_{k=1}^K\) are linearly independent.

We characterize classes of utility functions with this property.

If cardinal utility is analytic, in the presence of a risky asset that separates uncertainty, one can drop the assumption that utility is stationary.

The analysis extends to the case in which only aggregate demand or the equilibrium correspondence are observable; and we illustrate how the analysis can be expanded to tackle models with more than two dates.
The identification of fundamentals is of intrinsic theoretical interest; also, it serves to formulate policy.

In order to investigate the extent to which asset prices are determined by fundamentals or the beliefs of investors, it is necessary to identify these beliefs from market data.

Our methods can potentially be applied to data obtained from laboratory experiments or, with modifications, to market data obtained from auctions.

As an extension to our main results, we show that the identification that we derive here is necessary for the convergence of preferences and beliefs constructed with a finite set of observations as the number of observations increases.
The identification of fundamentals from observable data can be posed, most simply, in the context of certainty.

Aggregate demand or the equilibrium correspondence, as endowments vary, also allow for identification.

Identification becomes problematic, and more interesting, when the set of observations is restricted. Under uncertainty, this arises when the asset market is incomplete and the payoffs to investors are restricted to a subspace of possible payoffs.

Identification is possible as long as the utility function has an expected utility representation with a state-independent cardinal utility index, and the distribution of asset payoffs is known. The argument extends to the joint identification of tastes and beliefs; but, it relies on the presence of a risk-free asset and, more importantly, does not allow uncertainty due to future endowments.
It is interesting to note that the identification of preferences from the excess demand for commodities, that corresponds to the demand for elementary securities in a complete asset market, is, in general, not possible.

Here, restrictions on preferences, additive separability and stationarity or state-independence, allow for identification even in an asset market that is incomplete.
A strand of literature in finance focuses on supporting prices and observations for a single realization of the path of endowments or equivalently on equilibrium in an economy with a representative investor.

In models with heterogeneous agents and incomplete financial markets an individual’s consumption will never be Markovian and therefore this approach cannot be extended to make any statements about individuals’ beliefs.
Identification

Dates are $t = 0, 1$, and, at each date-event, there is a single perishable good.

At date 0, assets, $a = 1, \ldots, A$, are traded and they pay off at $t = 1$.

An individual has subjective beliefs over the joint distribution of asset-payoffs and his endowments at $t = 1$ that, we assume, has finite support, $S$.

Consumption at date 0 is $x_0$, and it is $x_s$ at state of the world $s = 1, \ldots, S$ at date 1.
The individual maximizes time-separable expected utility

\[ U(x_0, \ldots, x_s, \ldots) = u(x_0) + \beta \sum_{s=1}^{S} \pi_s u(x_s), \]

with the cardinal utility index, \( u : (0, \infty) \rightarrow \mathbb{R} \), twice continuously differentiable, concave and strictly monotonically increasing, \( \beta \in (0, \infty) \) and \( \pi \) a probability measure.

Payoffs of an asset are \( r_a = (r_{a,1}, \ldots, r_{a,s}, \ldots, r_{a,S})^\top \), and payoffs of assets at a state of the world are \( R_s = (r_{1,s}, \ldots, r_{a,s}, \ldots, r_{A,s}) \). Holdings of assets are \( y = (\ldots, y_a, \ldots)^\top \).

At date 0, the endowment of the individual is \( e_0 \), consumption is numéraire and prices of assets are \( q = (\ldots, q_a, \ldots) \); at state of the world \( s = 1, \ldots, S \), at date 1, consumption is, again, numéraire, and the endowment is \( e_s \); across states of the world, \( e = (e_1, \ldots, e_S) \).
The optimization problem of the individual is

$$
\max_{x \geq 0, y} \quad u(x_0) + \beta \sum_{s=1}^{S} \pi_s u(x_s)
$$

s.t. \quad x_0 + qy \leq e_0

$$
x_s - R_s y \leq e_s, \quad s = 1, ..., S.
$$

The demand function for consumption and assets is \((x_0, y)(q, e_0)\); it defines the inverse demand function \((q, e_0)(x_0, y)\).

Given any \((\bar{q}, \bar{e}_0)\) we suppose that \((x_0, y)(q, e_0)\) is observable and solves the individual’s maximization problem, with \((x_0, ..., x_s, ...) \gg 0\) on an open neighborhood of \((\bar{q}, \bar{e}_0)\).

This gives an open set \(Y \subset \mathbb{R}^J\) of observed asset holdings and \(X_0\), the open interval of observed date zero consumptions.
Unobservable characteristics of an individual are the cardinal utility index, $u : (0, \infty) \to \mathbb{R}$, the discount factor, $\beta > 0$ and beliefs over the distribution of future endowments and payoffs of assets, $S \in \mathbb{N}$, $(\pi, R, e) \in \mathbb{R}_+^S \times \mathbb{R}^{AS} \times \mathbb{R}_+^S$, with $\pi = (\ldots, \pi_s, \ldots)$ a probability measure.

Does the demand function identify the unobservable characteristics of the individual? This is the question we address in this paper. We are mainly concerned with the identification of beliefs.
The following result establishes that the cardinal utility index can be identified over the range of observable first date consumption.

**Proposition 1.**

*The demand function for consumption and assets identifies the first date cardinal utility index* \( u : X_0 \to \mathbb{R} \).

**Proof.**

The demand for consumption and assets is defined by the first order conditions

\[
\sum_{s=1}^{S} \pi_s u'(e_s + R_s y) = u'(x_0)q(x_0, y).
\]
Differentiation with respect to $x_0$ gives that, for any $a = 1, \ldots, A$,

$$0 = u''(x_0)q_a(x_0, y) + u'(x_0)\frac{\partial q_a(x_0, y)}{\partial x_0}.$$ 

Since, for any asset $a$ with non-zero price

$$\frac{u''(x_0)}{u'(x_0)} = -\frac{\partial q_a(x_0, y)/\partial x_0}{q_a},$$

and since inverse demand is observable $u''(x)/u'(x)$ is observable over $X_0$, which identifies $u$ up to an affine transformation. □
Remark.

Note that if $u$ is assumed to be analytic on $(0, \infty)$ observing demand on any open $\mathcal{X}_0 \subset (0, \infty)$ identifies the cardinal utility on all of $(0, \infty)$. For this case, we can therefore take $\mathcal{X}_0$ to be equal to $(0, \infty)$ in the results that follow.
With $u$ given, the unknown characteristics are $\xi = (S, \beta, \pi, R, e)$.

The question of identification is whether, given some $\xi$ that generates the observed demand function, there is a different $\tilde{\xi}$ that would generate the same demand for assets on a specified neighborhood of prices and wealth.

a) $r_{1s} > 0, s = 1, \ldots, S$;

b) there are no $s \neq s'$ with $(e_s, R_s) = (e_{s'}, R_{s'})$;

c) probabilities are strictly positive: $\pi \gg 0$. 
Identification is possible if any two distinct characteristics in $\Xi$ generate different demand functions.

Formally, we say that the observed demand for date 0 consumption and for assets on an open set of incomes and prices identifies beliefs in a set of admissible characteristics $\Xi$ if there are no distinct $\xi_1, \xi_2 \in \Xi, \xi_1 \neq \xi_2$ that, for the cardinal utility $u$, recovered in Proposition 1, generate the same demand function on the observed set of prices and incomes.

Our main results states that $\xi_1$ and $\xi_2$ must generate different demand functions if $(u'(e_s + R_s y))$ are linearly independent for all $s$ for which $(e_s, R_s)$ are distinct.
Recall that functions, \( f_1, \ldots, f_n, f_i : \mathcal{A} \subset \mathbb{R}^m \rightarrow \mathbb{R} \) for all \( i = 1, \ldots, n \), are linearly independent on \( \mathcal{A} \) if there is no \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{S}^{n-1} \) such that \( \sum_{i=1}^{n} \alpha_i f_i(x) = 0 \), for all \( x \in \mathcal{A} \).

If functions \( f_1, \ldots, f_n \) are linearly independent on some set \( \mathcal{A} \), there must exist finitely many points \( x_1, \ldots, x_m \in \mathcal{A} \), such that there is no \( \alpha \in \mathbb{S}^{n-1} \) for which \( \sum_{i=1}^{n} \alpha_i f_i(x_j) = 0 \) for all \( j = 1, \ldots, m \).

We define a differential operator

\[
\Delta_k = \left( \frac{\partial}{\partial x_1} \right)^{j_1} \cdots \left( \frac{\partial}{\partial x_m} \right)^{j_m}, \quad j_1 + \cdots + j_m \leq k,
\]
We say that $f_1, \ldots, f_n$ are *differentiably* linearly independent (on $\mathcal{A}$) if there is some $k \geq n - 1$ and some $\bar{x} \in \mathcal{A}$ such that each $f_i$ is at least $C^k$ at $\bar{x}$ and such that there are differential operators $\Delta_{k_1}, \ldots, \Delta_{k_n}$ with $k_i \leq k$ for all $i = 1, \ldots, n$ such that the matrix

$$
\tilde{W} = \begin{pmatrix}
\Delta_{k_1}(f_1) & \ldots & \Delta_{k_1}(f_i) & \ldots & \Delta_{k_1}(f_n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Delta_{k_j}(f_1) & \ldots & \Delta_{k_j}(f_i) & \ldots & \Delta_{k_j}(f_n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Delta_{k_n}(f_1) & \ldots & \Delta_{k_n}(f_i) & \ldots & \Delta_{k_n}(f_n)
\end{pmatrix}
$$

is non-singular.
If \( f_1, \ldots, f_n \) are differentiably linearly independent on \( \mathcal{A} \), they are linearly independent since differentiable linear independence implies that there cannot be an open neighborhood of \( \bar{x} \) and some \( \alpha \in S^{N-1} \) such that \( \sum_{i=1}^{n} \alpha_i f(x_i) = 0 \) for all \( x \) in the neighborhood.

The converse is generally not true. But, if the functions \( f_1, \ldots, f_n \) are analytic, then, they and linearly independent if and only if they are differentiably linearly independent. In fact in this case one can take \( \Delta_k = \Delta_{i-1} \) for all \( i = 1, \ldots, n \) and obtains the so called Wronskian matrix of \( f_1, \ldots, f_n \). In the analytic case, this matrix is non-singular if and only if the functions are independent.
Given two characteristics

\[ \xi^1 = (S^1, \beta^1, \pi^1, R^1, e^1), \quad \xi^2 = (S^2, \beta^2, \pi^2, R^2, e^2), \]

we define the joint support, \((S, \bar{e}, \bar{R}) = \gamma(\xi^1, \xi^2)\), as \(\bar{e}_s = e^1_s\), \(\bar{R}_s = R^1_s\) for \(s \leq S^1\) and \(\bar{e}_s = e^2_{s-S^1}\), \(\bar{R}_s = R^2_{s-S^1}\) for \(s > S^1\) as well as

\[ S = \{1, \ldots, S^1\} \cup \{ s \in \{S^1 + 1, \ldots, S^1 + S^2\} : (\bar{e}_s, \bar{R}_s) \notin \{(e^1_1, R^1_1), \ldots, (e^1_{S^1}, R^1_{S^1})\} \} . \]
It is now possible to give general sufficient conditions for identification.

**Theorem 1.**
The demand function identifies the unobservable characteristics \( \xi \in \Xi \) if, for any \( \xi^1, \xi^2 \in \Xi \), the joint support \( (S, \bar{e}, \bar{R}) = \gamma(\xi^1, \xi^2) \) is such that the set \( \{ y \in \mathcal{Y} : (\bar{e}_s + y\bar{R}_s) \in \mathcal{X}_0, \text{ for all } s \in S \} \) is non-empty, and the functions \( (u'(\bar{e}_s + \bar{R}_s y))_{s \in S} \) are linearly independent on this set.

Conversely, if there are characteristics \( \xi = (S, \beta, \pi, R, e) \in \Xi \) for which \( (u(e_s + R_s y))_{s=1}^{S} \) are not linearly independent on \( \mathcal{Y} \), then identification is impossible.
Proof.

To prove sufficiency, suppose both characteristics $\xi^1$ and $\xi^2$ rationalize observed asset demand and consider asset demand in a fictitious problem of an investor who faces states $S$ (as defined in the theorem) in the second date.

Define $f_s(y) = u'(\bar{e}_s + \bar{R}_s y)$.

The first order condition with respect to the demand for asset $a = 1$ (that has positive payoffs in all states) can be written as

$$\sum_{s \in S} \beta \pi_s \bar{r}_1s f_s(y) = u'(x_0)q_a$$
We first show that the fact that this first order condition holds on an open neighborhood uniquely identifies $\beta, (\pi_s)_{s \in S}$.

We then argue that this implies that the demand function identifies beliefs.

As pointed out above, if the $f_s$ are linearly independent, we can find $N$ and points $y_1 \ldots y_N \in \mathcal{Y}$ such that the system of equations

$$\sum_{s \in S} \alpha_s \bar{r}_{1s} f_s(y_i) = 0, \quad i = 1, \ldots, N$$

has no solution with $\alpha \neq 0$. 
Since the first order conditions hold on the open set $\mathcal{Y}$ we can find $x_{0i}, q_{ai}, i = 1, \ldots, N$, such that

$$\sum_{s \in S} \beta \pi_s \bar{r}_1 s f_s(y_i) = u'(x_{0i}) q_{ai}, \quad i = 1, \ldots, N$$

This is a linear system in $(\beta \pi_s)_{s \in S}$ and must have a unique solution.

By the construction of the set of distinguishable states $S$, if $\xi^1$ rationalizes the observed demand, this solution must satisfy $\beta \pi_s = 0$ for all $s > S^1$.

But then, if $\xi^2$ also rationalizes the observed demand we must have $S^1 = S^2$ and $(e_s^1, R_s^1) = (e_s^2, R_s^2)$ for all $s = 1, \ldots, S^1$.

Hence, characteristics are uniquely identified.
To prove necessity note that linear dependence of \((u(e_s + R_s y))_{s=1}^S\) implies that there exist \(\alpha_1, \ldots, \alpha_S\) such that
\[
\sum_{s=1}^S \alpha_s r_{as} u'(e_s + R_s y) = 0, \text{ for all } a = 1, \ldots, A.
\]

If asset demand is rationalized for some probabilities \((\pi_1, \ldots, \pi_S) \gg 0\) then, for any \(\epsilon\), the first order conditions can be written
\[
-q_a u'(x_0) + \sum_{s=1}^S (\pi_s + \epsilon \alpha_s) r_{as} u'(e_s + R_s y) = 0 \text{ for all } a = 1, \ldots, A.
\]

For sufficiently small \(\epsilon > 0\) we have that \(\pi_s + \epsilon \alpha_s > 0\) for all \(s\) and we can define alternative probabilities
\[
\tilde{\pi}_s = (\pi_s + \epsilon \alpha_s)/(1 + \epsilon \sum_k \alpha_k) \quad \text{and appropriately adjusted} \quad \tilde{\beta} = \beta (1 + \epsilon \sum_k \alpha_k)
\]
that would rationalize the same demand function. □
Restrictions on fundamentals

The identification theorem obviously raises the question whether there are restrictions on fundamentals, either assumptions on utility or restrictions on \((e_s, R_s)\) that guarantee independence as required.

First, examples show that, even when the support of beliefs, \((S, R, e)\) as well as cardinal utility \(u\) and \(\beta\) are known, identification may not be possible.

Note, that we are concerned with the much more demanding case, where nothing is known about the beliefs of the individual; nevertheless, it shall turn out that understanding these simple examples provides the key to our general identification results.
There is a single risky asset, second date endowments are zero, $e = 0$, and cardinal utility is logarithmic: $u(x) = \ln(x)$ and $\beta = 1$.

A simple computation shows that $q_y = e_0/2$ – the individual invests a fixed fraction of his wealth in the risky asset, and the demand for asset is identical for all $\pi$; beliefs are not identified.
There is a single risk-free asset, there is uncertainty about
second date endowments, \( e \neq 0 \), and utility is CARA: it
exhibits constant absolute risk aversion, and
\( u(x) = -\exp(-x) \) and \( \beta = 1 \).

Direct computation shows that the demand for the risk-free
asset is

\[
y = \frac{1}{1 + q} \left( e_0 - \ln(q) + \ln\left( \sum_{s=1}^{S} \pi_s \exp(-e_s) \right) \right);
\]

beliefs are not identified.

It is useful to note that, with two risky assets, with log-utility,
identification might still be impossible.
There are two obvious ways to guarantee identification

One could make assumptions on utility that rule out these cases; or,

one could assume that there are several assets available for trade.

We shall consider both in detail.
3. There are two risky assets, there are no endowments, $e = 0$, and $u(x) = \ln(x)$ and $\beta = 1$. Recall that $r_{as}$ is the payoff of asset $a$ in state $s$. If, for states $s = 1, 2$,

\[
\frac{r_{11}}{r_{12}} = \frac{r_{21}}{r_{22}},
\]

then $r_{21}/r_{11} = r_{22}/r_{12}$, and the first order conditions that characterize asset demand can be written as

\[
\frac{q_1}{\beta} u'(x_0) = (\pi_1 + \pi_2) \frac{1}{\theta_1 + \theta_2 \frac{r_{21}}{r_{11}}} + \sum_{s=3}^{S} \pi_s r_{1s} u'(x_s),
\]

\[
\frac{q_2}{\beta} u'(x_0) = (\pi_1 + \pi_2) \frac{1}{\theta_1 \frac{r_{11}}{r_{21}} + \theta_2} + \sum_{s=3}^{S} \pi_s r_{2s} u'(x_s);
\]

clearly, $\pi_1$ and $\pi_2$ cannot be identified separately.

Motivated by this example and to simplify the exposition, we will from now on focus on the case where a risk-free asset is available for trade.
The following example shows that identification might still be impossible, even if there is a risky and a risk-less asset.

4. There is a risk-free asset (asset 1) and a risky asset (asset 2). Suppose \( e \neq 0, \ u(x) = -\exp(-x) \) and

\[
    r_{21} = r_{22}, \quad e_1 \neq e_2.
\]

The first order conditions that characterize asset demand can be written as

\[
    q_1 u'(x_0) = \beta \sum_{s=1}^{S} \exp(- (\theta_1 + \theta_2 r_{2s})) \pi_s \exp(-e_s),
\]

\[
    q_2 u'(x_0) = \beta \sum_{s=1}^{S} \exp(- (\theta_1 + \theta_2 r_{2s})) r_{2s} \pi_s \exp(-e_s);
\]

beliefs, \( e_s \) and \( \pi_s \) cannot be identified separately.
In fact, in the example, identification is impossible even if markets are complete.

Existing results on the identification of preferences from demand do not apply when only excess demand is observable, which is the case here: since endowments are unknown, consumption is not observable.