On dynamic pricing*

Ilia Krasikov Rohit Lamba

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Abstract

This paper studies a canonical model of dynamic price discrimination- when firms can endogenously discriminate amongst consumers based on the timing of information arrival and/or the timing of purchase. A seller and buyer trade repeatedly. Buyer's valuation for the trade is private information and it evolves over time according to a renewal Markov process. The seller offers a dynamic pricing contract which options a sequence of forwards. As a first step, we show that this relatively simple dynamic pricing contract achieves the optimum in the two period repeated sales model. We then show that this contract is (a) the optimum when a single object is sold at a fixed time and (b) the optimum under strong monotonicity in the repeated sales model. The gap between the full optimum and our mechanism of simple dynamic pricing instruments is explained through buybacks. Moreover, the general optimal contract is shown to be backloaded and a theoretical bound is provided for the fraction of optimal revenue that can be extracted by the seller from using our mechanism: it achieves more than 70% percent of the total profit uniformly across distributions, and more than 90% for standard ones such as the power distribution. The construction of the mechanism and bounds is then extended to multiple players to study repeated auctions. At every step of the analysis a mapping is established between the pricing model (indirect mechanisms) and the dynamic mechanism design toolkit (direct mechanisms). In this process, novel tools are developed to study dynamic models of mechanism design when global incentive constraints bind.

*Krasikov: Pennsylvania State University, izk113@psu.edu; Lamba: Pennsylvania State University, rlamba@psu.edu. We are deeply indebted to Vijay Krishna for his guidance and encouragement.

1 Introduction

This paper casts the familiar monopolistic screening model from static mechanism design into a repeated allocation setting in the pursuit of a canonical model of dynamic price discrimination. A seller with a linear marginal cost of production wants to repeatedly sell a good or service to a buyer with correlated and sequentially arriving private linear valuations. What is the menu of prices the seller must offer to include and exclude buyer types over time in order to maximize her profit?

Price discrimination refers to the idea of selling two or more different goods at prices that are in different ratios to the marginal cost (Stigler [1987]). In a influential survey on the state of the art on price discrimination, Varian [1989] wrote:

> "In order to lower the price only to the marginal consumer [to increase profit], or more generally to some specific class of consumers, the firm must have a way to sort consumers. The easiest case is where the firm can explicitly sort consumers with respect to some exogenous category such as age. *A more complex analysis is necessary when the firm must price discriminate on the basis of some endogenous category such as time of purchase*. In this case the monopolist faces the problem of structuring his pricing so that consumers 'self-select' into appropriate categories."

Over the years the monopolistic screening model developed as a leading tool in the economists' arsenal for understanding non-linear pricing and price discrimination more generally (See Mussa and Rosen [1978], Börgers [2015] Chapter 2, and Armstrong [2016]). From a modeling perspective, however, the statement italicized above was largely ignored. Booking an airline ticket with the option of cancelation or choosing between monthly and yearly payments for a recurring service or sorting the financial market for traders through a menu of options and forwards are some of potentially many examples where time is an important dimension of screening (a) sequentially arriving private information, and (b) repeated or on-going demand for a good or service.[1]

Courty and Li [2000] delighted economists by providing a first analysis of intertemporal price discrimination, wherein sequential arrival of information is used by the seller to incentivize users to self-select from a menu of partially revealing contracts.[2] This paper builds on the literature that followed after by analyzing a repeated allocation model (as opposed to simply a two-period sequential screening model) where a set of novel and realistic instruments are used to dynamically price discriminate amongst potential buyers. Through an admittedly stylized set up that has some theoretical precedence, we are looking to capture the basic economic forces of intertemporal price discrimination and map them into the much loved Myersonian model of mechanism design.

[1]In the era of big data dynamic considerations are in fact becoming more important. For example, online merchants can now rely on a huge treasure trove of past consumer data which creates enormous opportunities for price discrimination.

[2]To be sure, Baron and Besanko [1984] had elegantly modeled dynamic screening, but its formulation as an endogenous model of price discrimination was discussed extensively in Courty and Li [2000].

We start with the following simple two-period model. A seller can sell two goods to a buyer, one today and another tomorrow. The buyer knows his valuation for the good today and will be informed of the valuation for the second period good tomorrow. The valuation in the first period is drawn form a uniform distribution on the unit interval, and then with probability ρ it stays the same for the second period good and with probability $1-\rho$ it is drawn again uniformly on the unit interval; this is a simple version of the renewal Markov process. The model has two informational channels- the buyer is initially informed of his valuation and the seller is not, and moreover, the buyer will (privately) learn his future valuation later for which he currently posses a signal with 'precision' ρ. What is a reasonable (take it or leave it) contract the seller should offer at the beginning of the first period to maximize the ex ante expected value of her profit?

We build towards our "candidate mechanism" in three steps. First, we document the optimal spot price mechanism wherein the seller simply offers a sequence of prices $\{\alpha_1, \alpha_2\}$ that statically maximize the seller's profit in each period. This is a useful benchmark but one that clearly leaves too much surplus on the table for the buyer since the seller does not exploit the sequential structure of the problem. So, next we allow the seller to charge an upfront payment to give the buyer the right to buy the second period good in the future. The contract space is given by $\{\alpha_1, (\alpha_2, M)\}$, where α_1 is a spot price for the first period good, and (α_2, M) is a classical European call option for the second period good; the latter asks the buyer to pay an option premium of M which grants him the flexibility to buy (or not) the second period good for a strike price α_2.

This second contract invokes a simple version of intertemporal price discrimination. The option premium M can be chosen small enough so that all buyer types in the first period agree to pay up, and then the optimal choice of strike price for the option contract, α_2^o, turns out to be lower than the optimal spot price, α_2^s. Thus, the seller reduces the option strike price below the spot price to include more buyer types' in the second period, but increases her aggregate profit by (a) selling more often, and (b) charging the flat option premium M that is paid by all buyer types.[3]

The seller can of course do even better. The European option increases seller's profit by extracting expected surplus of the buyer that spot pricing leaves on the table. However, the actual screening decision, which buyer types buy the second period good, is still decided upon the realization of v_2; if $v_2 > \alpha_2$ the good is bought. In the third contract, the seller enriches the option contract space to include a forward price: $\{\alpha_1, (M, \alpha_2, y)\}$. The second period good is now sold by the means on an American option on forwards. The buyer can exercise the option in the first period to purchase the second period good at a price y. Alternatively, the buyer can exercise the option in the second period and purchase the second period good for its strike price. The key difference between the two is that the payment y is binding, it is paid no matter the realization of v_2, whereas the buyer can decide upon learning v_2 whether to buy the good at α_2.[4]

[3]In the uniform example with $\rho = 1/2$, we get $\alpha_2^o = 1/3$ and $\alpha_2^s = 1/2$.

[4]Note that in each of these three contracts the optimal first period price remains the same. For the uniform example with $\rho = 1/2$, it is $\alpha_1 = 1/2$.

Well call this third set of dynamic pricing instruments *optioning forwards*. It splits the buyer types into those that would like to commit earlier and those that want to wait to exercise the strike price. Call the optimal strike for the second good in this third contract to be α_2^{fw}. Then, it is shown that all buyer types with with $v_1 > \alpha_2^{fw}$ buy at the optimally chosen forward price y, and types $v_1 < \alpha_2^{fw}$ wait for the second period realization to buy the good if $v_2 > \alpha_2^{fw}$. The seller optimizes through the addition of the forward price y by setting $\alpha_2^{fw} > \alpha_2^o$.[5] The seller trades off serving lesser types through the strike price in period two by making more types in period one commit to buying the good at the forward price.

Perhaps, the most surprising part that adds conceptual heft to the above pricing analysis is that the optimally chosen options with forwards contract is actually the optimal contract. Proving this, or any such results in dynamic mechanism design, has been a blind spot for the literature so far because of its reliance on the so-called first-order approach. Even in the two-period model discussed above global incentive constraints would bind for any interior value of persistence: $0 < \rho < 1$. It is hitherto not clear how to use the standard mechanism design tools to establish the optimum. We do so by proving two critical results: *backloading* and *independent screening*.

Optimal contracts are backloaded, that is, for any sequence of values v_1 and v_2 if trade happens with any positive probability in period 1, then it happens for sure in period 2. This is an intuitive, but fairly strong result. For any fixed value of information rent that the seller has to pay to the agent, she prefers to delay its transfer as much as possible because the shadow price of providing incentives becomes lower as the informational advantage of the buyer decreases.

Backloading allows us to establish that the optimal contract solves the screening problem for the two goods separately. Consider two problems: first where the seller only wants to sell the good in the first period, and second where she wants to sell the good only in the second period; in the latter case the first period value is not payoff relevant but acts as a signal for the second period value. Independent screening demands that solving these two problems separately would yield exactly the same amount of trade as the original problem where both goods are to be allocated across time. Once we have this final result, we can solve for the optimal thresholds of trade which turn out to be α_1^{fw} and α_2^{fw}, the ones calculated for the optimal option contract with forwards. So, the optimal trading rule is to trade in period 1 if and only if $v_1 \geqslant \alpha_1^{fw}$ and trade in period 2 if and only if $\max\{v_1, v_2\} \geqslant \alpha_2^{fw}$. A clear, one-to-one, mapping is established between the optimal set of dynamic pricing instruments and the optimal dynamic contract.

The optimal contract in the two-period model satisfies three key properties: backloading, bang-bang, and independent screening. Of these the optimal contract for more general time horizons and the Markov renewal model satisfies only the first criterion- it is always backloaded. Randomization and linking sales across time improve the seller's profit. However, characterizing the optimal dynamic pattern of randomization and linking of sales of across time is prohibitively complex, and arguably limitedly useful at the margin in increasing the seller's profit. Thus, going forward we take an axiomatic approach to designing a reasonable dynamic mechanism that is simple to describe in that it targets certain simple properties, and still captures

[5] In fact for the uniform example with $\rho = 1/2$, the optimal value is $\alpha_2^{fw} = 7/18$.

a lion share of the optimal value.[6] We take the dynamic pricing route and construct and completely characterize a candidate mechanism that satisfies the aforementioned three properties-backloading, bang-bang and independent screening, and then show that the loss from using this candidate mechanism is actually quite small. Note that independent screening does not imply that the contract is history independent, it just simplifies the exact nature of history dependence in the contract space. For example, the first-order approach, the standard so far in the literature, would also generate a contract that satisfies backloading, bang-bang, and independent screening; however, it would not be incentive compatible in our setting.

Fix a time t and consider the associated sequential screening problem: there is only one allocation that is made in period t, so V_t is the only payoff relevant type and all V_s for $s < t$ act as signals for the payoff relevant type. The types evolve according to a general Markov renewal model. The optimal option contract with forwards is now calculated for this sequential screening problem: $\{\alpha_t, \mathcal{M}_t, (y_t^s)_{s\leq t}\}$ where $\mathcal{M}_t$ is the option premium that would set so as to incentivize all buyer types in the first period to buy in to the option, α_t is the strike price which the buyer would face if he wants to wait till end to make her decision to buy the good, and y_t^s is a set of prices that buyer can commit to paying at the end of period t if he makes the decision to buy the good in period $s \leq t$. Using tools discussed above for the two period, we show that this the optimal contract with forwards is actually the optimal deterministic mechanism for the sequential screening problem. The optimal allocation is given by what we call the *max-threshold mechanism*: the t-th period good is allocated if and only if $\max_{s\leq t} V_s \geq \alpha_t$. Thus, the optimal deterministic mechanism picks the highest realization from the history of types and evaluates it against a threshold determined by distributional parameters.

Our candidate mechanism for the repeated sales problem is then defined by pasting the optimal options with forwards contract for the sequential screening problems: $\left\{(\alpha_t), \mathcal{M}, ((y_t^s)_{s\leq t})_t\right\}$, where the sequence of thresholds (α_t) adopt the same values that they did in their respective sequential screening problems, and $\mathcal{M} = \int_0^\infty e^{-rt}\mathcal{M}_t dt$ refers to the option premium that is paid by all buyer types at the inception of the contract. The buyer can then either wait till time t to buy good t at spot price α_t or make the decision to buy good t at time s for a price y_t^s. In the latter case we have two payoff equivalent choices: the buyer can either pay for the good when he makes a decision to buy it or wait till the good becomes available, linearity assures that adjusted for discounting these two mechanisms generate the same payments, and thus implement the same trading rule. What really matters, in the spirit of Varian's quote, is the timing of the decision of purchase. The set of sequence of valuations is discriminatorily split into bins instrumenting the time s at which the buyer decides to buy good $t \geqslant s$, and the different buyer types are incentivized to select into their respective bins by the appropriately chosen forward prices.

[6] By axiomatic design, we mean that we restrict the class of mechanisms within which we search for the optimum to satisfy certain axioms. This is done typically because the optimum is extremely hard to compute or the instruments required to potentially implement it are impractical. This approach is quite common in combinatorial auction design, see for example Milgrom and Segal [2018], and more recently in dynamic contracts, see Chassang [2013]. We also used this approach in one of our recent papers on dynamic contracts, Krasikov, Lamba, and Schacherer [2018]. A justification of this approach should, of course, include a theoretical and quantitative sense of the distance between the optimum achieved within the restricted class and the actual full optimum.

The comparative statics of the candidate (or max-threshold) mechanism are straightforward. The sequence of thresholds or strike prices α_t are positive and strictly decreasing over time towards to zero. Thus, the contract becomes efficient along any history of realizations in the long run. Moreover, as types become highly persistent, α_t converges to threshold for the spot contract, α^s, and as types converge to being independent across time, α_t converges to zero for all t implying immediate efficiency of the dynamic contract and extraction of the entire surplus by the seller.

Since the candidate mechanism is backloaded, the dynamics play out in the form of a sequence of interdependent optimal stopping problems. At the beginning of time, depending on the realization of the initial valuation V_0, the buyer decides to buy a set of goods from a future date t_0 onwards such that $V_0 \geqslant \alpha_{t_0}$. From then on, goods $[t_0, \infty)$ are considered sold at prices $\{y_t^0\}_{t \geqslant t_0}$, and the stock of goods $S_0 = t_0$ is still available in the market. The second purchase (or decision to trade) happens at the first instance t_1 for which $V_{t_1} > V_0 = \sup_{s<t_1} V_s$. Fixing this value t_1, there exists a number $S_{t_1} \geqslant 0$ such that $V_{t_1} = \alpha_{t_1+S_{t_1}}$, and thus all goods in the interval $[t_1 + S_{t_1}, S_0)$ are sold at prices $\{y_t^{t_1}\}_{t_1+S_{t_1} \leqslant t \leqslant S_0}$. This inductive line of reasoning stops in finite time, say t_τ, leading to sales at times 0, t_1, t_2,...,t_τ at prices $\{y_t^{t_j}\}_{t_j+S_{t_j} \leqslant t \leqslant S_{t_{j-1}}}$ which draws down the total available stock $(S_t)_{t \leqslant t_\tau}$ till we get $\max_{s \leq t_\tau} V_s \geqslant \alpha_{t_\tau}$ culminating in exhaustion of total stock left for sale: $S_{t_\tau} = 0$. Note that the value of stock process S_t at time t and stopping time t_τ are both random variables and we explore the comparative statics of their expected value as a function of persistence of the types process.

As mentioned before, the optimal option with forwards contract, our candidate mechanism, does not achieve the global optimum, that is the optimum in a complete unrestricted class of dynamic mechanisms. The main reasoning for this is that it treats history dependence in a very simple fashion. However, by construction, the max-threshold mechanism for repeated sales is backloaded, deterministic, and it independently screens the sale of each good, and again by construction, it is the optimal contract in the class of mechanisms that satisfy these properties. Moreover, we show that the candidate mechanism is the global optimum in the class of strongly monotone dynamic mechanisms, where allocative distortions are ranked along the history of buyer valuations through the usual partial order on vectors.

As a corollary of these results, we can generate a simple mapping between the world of dynamic pricing and dynamic mechanism design: restricting attention to pricing rules that satisfy the bang-bang property and independent screening is equivalent to searching for the optimum in the class of strongly monotone allocation rules. From a practical perspective, because of its simplicity, strong monotonicity might be a desirable restriction to impose of the class of mechanisms, at least as a benchmark; and on the theoretical side, most of known applications of dynamic mechanism design either satisfy or check for strong monotonicity ex post.

We explore the qualitative nature of the gap between the candidate mechanism and the global optimum through a series of examples. The key lesson here is the optimal contract allows for "buybacks" of goods already sold but not yet consumed to partition the buyer types into even finer bins and thereby increases the seller's profit. Consider the case where in the first period the buyer has exercised the option for the fourth good, but not the third one. To price the third good appropriately, the seller might want to obtain extra information regarding buyer's second period valuation: she wants to lower the threshold for the third good whenever v_2 is sufficiently small, which the max-threshold mechanism does not allow. Our backloading result intuitively means that a purchase of the second good can not provide the seller with such a signal. An improvement here would be to allow the buyer to return the fourth good for some small refund. Only the low buyer types will use this opportunity, therefore marginal cost of this extra information are negligible. On the other hand, the seller will now adjust his selling threshold for the third good selling more often above the marginal type α_3. A formal description of this rather intuitive construction will, however, throw light on the grave complexity involved in completely characterizing the full optimum in general model.

Despite aforementioned gap, the options with forward contract achieves a large fraction of the optimum. As a justification for the use of our specific axiomatic design, we provide a theoretical bound for the seller's profit in our mechanism. The bound is constructed as follows. First we calculate the seller's profit from using our candidate mechanism- say $\mathcal{R}^{fw}$. This forms a lower bound for the optimal profit $\mathcal{R}^*$. Since, the optimal profit is hard to determine in a closed-form, we define a solvable relaxed problem for which it is easy to compute the seller's profit.[7] The optimal profit is then bound from above by this value, say $\overline{\mathcal{R}}$. Our theoretical bound for the fraction of optimal profit that can be expressed as: $1 \geqslant \frac{\mathcal{R}^{fw}}{\mathcal{R}^*} \geqslant \frac{\mathcal{R}^{fw}}{\overline{\mathcal{R}}}$. We go a step further and find a bound $\mathcal{L}$ which is uniform across all distributions for the Markov renewal processes: $\frac{\mathcal{R}^{fw}}{\overline{\mathcal{R}}} \geqslant \mathcal{L}$. The uniform bound is tight in the limit as the types process converge to i.i.d. or perfect persistence, and it never does worse than 70% of the optimum. It is however, not tight in general. We show that for standard class of distributions such as uniform, power and beta, the bound $\frac{\mathcal{R}^{fw}}{\overline{\mathcal{R}}}$ achieves around 90% of the optimum, providing credence to our claim that restricting attention to deterministic and independent screening *does not lead to much loss*, and on the plus *adds considerable simplicity*.

Finally, we extend the single buyer repeated sales problem to the setting with multiple buyers to study repeated auctions.[8] Specifically, we consider the case when each good is sold by the means of a second price auction with a participation fee or required reserve price. Analogous to the single buyer case, we construct a candidate mechanism in three steps. In each pricing rule, the final act of selling each good is a second price auction. The constructive steps deal with how to determine the participation fees so as to dynamically price discriminate the set of potential buyers, how to include or exclude buyers over time so as to maximize expected profit.

First, we derive the optimal spot entry fee. Ignoring dynamics, we document which buyer types to exclude from the auction in order to maximize revenue. The tradeoff here is clear-excluding lower buyer types truncates the distribution of the ones who would participate in the auction which is costly, but it also increases expected payments condition on participation.

[7]This relaxed problem is akin to the approach of solving the seller's problem under orthogonalized type space for the buyer; pioneered by Esö and Szentes [2007]. In our setting it only provides an upper bound which cannot be attained since the contract it produces is not incentive compatible.

[8]We look at the natural dynamic generalization of the classic Myerson optimal auctions [Myerson [1981] to our setting in the sense that buyers' types are assumed to be independent and private (IPV), but each buyer's type is correlated across time.

Second, we consider the setting of a sequential auction where one good is sold at a pre-specified date t, so that the buyers' types before that date act as signals for their valuations. the seller offers a European call option to acquire the right to participate in the auction at date t. The option premium is chosen low enough so as to incentivize each buyer type to take the option contract. At date t, the pre-determined strike price selects the set of buyers that want to participate in the auction. the key improvement here is that the strike price in the second pricing rule will be lower than the first one so as to encourage greater participation, but profit is increased through the upfront extracting of surplus.

In third, truly dynamic, pricing rule, we introduce forward contracts to determine the participation fee. More buyer's types are excluded as the number of buyers increases. Finally, we construct a theoretical upper bound on seller's profit, and argue that even the spot auction can approximate it for large auctions. This should with no surprise- it is well known that in the static setting a second price auction achieves full surplus extraction for a large number of buyers.

To summarize, the rest of the paper is organized as follows. In Section 2, we present a relatively simple two period problem, and build a dynamic pricing mechanism which options forwards. Then, using techniques from dynamic mechanism design, we show that these pricing instruments are sufficient in that they achieve the optimum. In Section 3, we present the general model in continuous time. Next, in Section 4 we establish that a combination of option contract and forward pricing achieves the deterministic optimum for the sequential screening problem, where one object is allocated at a fixed time, but information arrives gradually. In Section 5, we use the optimal contract for the sequential screening model to construct a candidate mechanism for the repeated sales problem , and provide comparative statics. This last contract is not the optimum in general, but is simple and achieves a large fraction of the total revenue for the seller. The optimum is quite complicated and arguably very hard to compute. In Section 6 we explain the gap between the candidate mechanism and the full optimum through two examples, and then in Section 7 we provide a theoretical bound for the loss from using our candidate mechanism and then envelope it into a uniform bound for all possible distributions. In Section 8, we generalize our candidate mechanism to multiple players, and explore two intuitive dynamic auction mechanisms. Again, we theoretically bound the loss from using them vis-a-vis the optimum. Finally, we conclude with a discussion of the literature and questions for future work in Section 9.

2 Fixing ideas: a simple two-period example

Suppose a seller wants to sell one unit of (non-durable) good at dates $k = 1, 2$. The buyer observes his valuations $\{v_k\}$ privately and sequentially. In the first period, it is drawn from a uniform distribution on $[0, 1]$. In the second period, the value stays the same as before with probability $1/2$ and it is independently sampled from a uniform distribution on $[0, 1]$ with probability $1/2$.[9] The seller offers a menu of prices (or allocation and payment rule in case of a direct

[9] For this example we assume that neither party discounts future payoffs.

mechanism) with the objective of maximizing her ex ante profit. In what follows, we first consider a simple set of sequential pricing instruments. Then, using a dynamic mechanism design approach we show that a combination of them achieves the optimum.

2.1 Dynamic pricing

As described in the introduction, there are multiple selling procedures which the seller can use. We represent four candidates in Figure 1; each graph partitions the two dimensional space of values (v_1, v_2) into four possible regions: sell nothing, sell only in period 1, sell only in period 2, and sell in both periods. Perhaps the simplest mechanism is to offer each good for a spot price, α_k. Given this offer, the buyer buys a good at date k if and only if $v_k \geq \alpha_k$. This yields an expected profit of $\alpha_1(1-\alpha_1)$ for period 1. Since v_2 is uniformly distributed from an ex ante perspective, the expected payoff in period 2 is also given by

$$R^s(\alpha_2) = \alpha_2(1-\alpha_2)$$

where R^s stands for the optimal profit from spot pricing. Clearly, $\alpha_1 = \alpha_2 = 1/2$ is the optimal solution, and the seller gets a profit of $1/4$ from selling each good. This statically optimal spot price mechanism is depicted in Figure 1a.

In what follows, we build improvements on the spot contract, eventually describing the optimum. In the sense of classical price discrimination, at each step we will make a trade off between what price to charge and how many (or what measure of) buyers to serve. Moreover, dynamics offer the ability to decide when to exclude or include a future type of the buyer, culminating in an added intertemporal instrument of price discrimination.

Going back to the spot pricing strategy, it is easy to see that each buyer type buyer receives a positive expected surplus from the second good:

$$\mathbb{E}\{(v_2 - 1/2)^+ | v_1\} = 1/2 \times (v_1 - 1/2)^+ + 1/2 \times \underbrace{\mathbb{E}\{(v_2 - 1/2)^+\}}_{=1/8} > 0 \quad \forall v_1$$

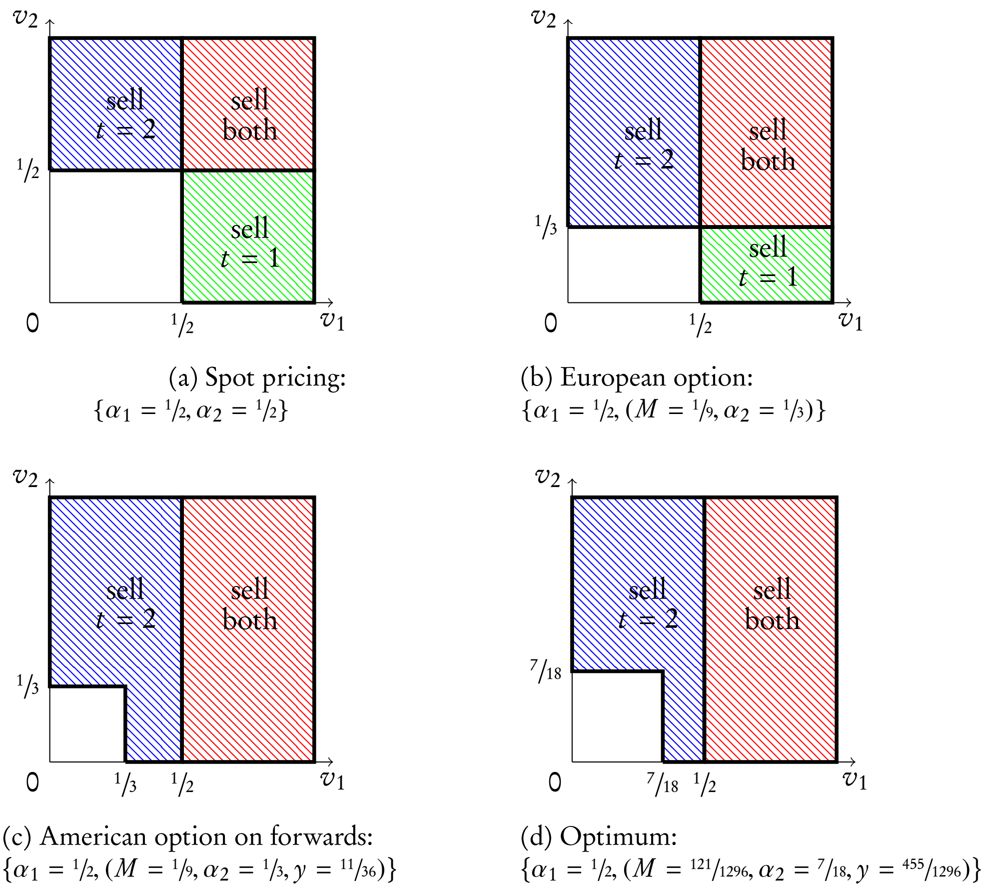
Therefore, the seller can increase her revenue from the second good by extracting some surplus upfront. One way is to charge the ex ante expected surplus of the lowest type, viz. $M(\alpha_2) = \mathbb{E}\{(v_2 - \alpha_2)^+ | v_1 = 0\} = \frac{(1-\alpha_2)^2}{4}$, as a flat upfront fee for the European option to buy the second good later for the spot price, α_2. We will refer to such payment as the option premium.

Buyer's expected payoff from taking this offer is

$$\mathbb{E}\{(v_2 - \alpha_2)^+ | v_1\} - M(\alpha_2) = 1/2 \times (v_1 - \alpha_2)^+ \geqslant 0$$

So, each type of the buyer is willing to accept the option, and pay the premium of $M(\alpha_2)$. Seller's expected profit for the second good is

$$R^o(\alpha_2) = R^s(\alpha_2) + M(\alpha_2) = 1/2 \times \alpha_2(1-\alpha_2) + 1/2 \times \frac{1-\alpha_2^2}{2}$$



(a) Spot pricing:
$\{\alpha_1 = 1/2, \alpha_2 = 1/2\}$

(b) European option:
$\{\alpha_1 = 1/2, (M = 1/9, \alpha_2 = 1/3)\}$

(c) American option on forwards:
$\{\alpha_1 = 1/2, (M = 1/9, \alpha_2 = 1/3, y = 11/36)\}$

(d) Optimum:
$\{\alpha_1 = 1/2, (M = 121/1296, \alpha_2 = 7/18, y = 455/1296)\}$

Figure 1: Selling mechanisms for the two period model, where α_k is the spot price for period k, M is the option premium, and y is the forward price at $k = 1$.

where R^o stands for the seller's profit when the option premium is added to the contract space in addition to the spot price. Now, $R^o(\alpha_2)$ is maximized at $\alpha_2 = 1/3$ with $M(\alpha_2) = 1/9$, resulting in a profit of $1/3 > 1/4$.

In the second period, the buyer is identified by two dimensions: (v_1, v_2), though only the second dimension is ex post payoff relevant. In the option contract described above, the seller chooses to include more buyer types along the second dimension (from $v_2 \geqslant 1/2$ to $v_2 \geqslant 1/3$) by reducing price α_2, but simultaneously increasing her profit by extracting surplus upfront. Since all types agree to pay the option premium, screening of whom to serve is relegated to the second period in the form of spot payment $\alpha_2 = 1/3$. An added dimension of discrimination would be to screen the buyer in the first period. This would increase the participation of the buyer along the v_1-dimension. More specifically, can the seller make it attractive to buy the second period good in the first period itself, while ensuring an increase in her profit?

We allow the seller to offer the second good at a forward price, say y, as part of an American option. If the option is exercised at $k = 1$, then the buyer will have to purchase the good for y at $k = 2$. Notice that the expected value of the forward purchase for the first period buyer type v_1 is given by

$$\mathbb{E}\{v_2|v_1\} - y - m = 1/2 \times v_1 + 1/2 \times 1/2 - y - m$$

What forward price should the seller choose? We start by picking a forward price that makes it optimal for all buyer types $v_1 \geqslant 1/3$ to buy at $k = 1$, culminating in $y = 11/36$:

$$y = \mathbb{E}\{v_2|v_1 = 1/3\} - \mathbb{E}\{(v_2 - 1/3)^+|v_1 = 1/3\} = 1/2 \times 1/3 + 1/2 \times [1/2 - 2/9] = 11/36$$

It is easy to see that when the forward price $y = 11/36$ is added to the option contract ($M = 1/9, \alpha_2 = 1/3$), all buyer types $v_1 \geqslant 1/3$ will be indifferent between choosing the forward contract and waiting for the strike price- both generate the same expected payoff. However, the seller strictly prefers the buyer to take the forward price, because her profit conditional on $v_1 \geq 1/3$ is higher:

$$y = 11/36 > \alpha_2 \mathbb{P}\{v_2 \geqslant \alpha_2 | v_1 \geqslant 1/3\} = 1/3 \times [1/2 + 1/2 \times 2/3] = 5/18$$

So, at the initial date, the buyer can choose either to commit to buy the second period good for $11/36$ or get the spot price $1/3$ in the second period. We shall assume that the buyer will choose the seller optimal option whenever he is indifferent, that is he will choose the forward price.[10]

At $k = 1$, as part of the contract, the first period good is traded if $v_1 \geqslant 1/2$. In addition, the buyer pays the option premium, $M = 1/9$: all types $v_1 \geqslant 1/3$ buy for the forward price; all types $v_1 < 1/3$ exercise the option at $k = 2$ if the second period valuation turns out to be $v_2 \geqslant 1/3$. The outcome of this contract is graphically represented in Figure 1c. Since the seller earns more profit from the types $v_1 \geqslant 1/3$, this third contract generates a strictly higher profit than the second one.

[10]Our criterion of resolving ties is the "right" one in the following sense. Since the seller makes a strictly higher profit from the buyer accepting the forward price, she can transfer a very small amount of money to break buyer's indifference in favor of taking the forward price. This keeps the seller's profit approximately the same and makes all buyer types $v_1 \geqslant 1/3$ strictly prefer the forward price.

Now, the third contract showed that adding forward pricing to the option strictly improves seller's profit. But what is the *optimal* combination?

It is not a coincidence that we choose a menu where the strike price in the second period, $\alpha_2 = 1/3$, is exactly equal to the forward price threshold of the first period types: $v_1 \geqslant 1/3$. In fact, it can be shown that when the option contract and forward price are simultaneously offered, it is optimal to set $y(\alpha_2) = \mathbb{E}\{v_2|v_1 = \alpha_2\} - \mathbb{E}\{(v_2 - \alpha_2)^+|v_1 = \alpha_2\}$. Thus, the choice of α_2 implicitly defines both the option premium and forward price. Therefore, for (M, y) such that $M = M(\alpha_2)$ and $y = y(\alpha_2)$, the profit of the seller from the second good is given by:

$$\begin{aligned} R^{fw}(\alpha_2) &= M(\alpha_2) + 1/2 \times \alpha_2 R^s(\alpha_2) + (1 - \alpha_2)y(\alpha_2) \\ &= 1/2 \times \alpha_2(1 - \alpha_2) + 1/2 \times \frac{1 - \alpha_2^2}{2} + 1/2 \times (1 - \alpha_2)\frac{\alpha_2^2}{2} \end{aligned}$$

where R^{fw} refers to the seller's profit under the combination of option contract and forward pricing. Thus, the first-order condition for α_2 is as it follows: $2\alpha_2 - 1 + \frac{3\alpha_2}{2} = 0$.

Only one of the roots lies in the interval $[0, 1]$, which is given by $\alpha_2 = 7/18$. It is easy to see that the second order condition is satisfied. Plugging in α_2 into the expressions for M and y, we get the optimal combination of option premium and forward price: $M = 121/1296$ and $y = 455/1296$.

The allocation rule implemented by the optimal dynamic pricing mechanism is depicted in Figure 1d. It provides the seller with an expected profit of 0.25 from the selling the first good and 0.387 from selling the second good. Comparing the optimal values of R^s and R^{fw}, it can be concluded that dynamic pricing instruments increase the seller's profit by about about 55 %. In the next section we show that this final contract is in fact optimal.

We would like to emphasize two features of the allocation rule achieved by the optimal dynamic pricing mechanism. First is that the allocation rules can be summarized as follows: trade in first period iff $v_1 \geqslant 1/2$, and trade in the second period iff $\max\{v_1, v_2\} \geqslant 7/18$. The thresholds for trade are decreasing over time ($1/2 > 7/18$) and the threshold for trade in the second period seeks the maximum value between the first two periods. Thus, the allocation rule is backloaded; that is, trade in the first period implies trade for sure in the second period, and moreover history dependence enters the contract is a fairly simple way.

Second, the measure of types for no trade shrinks for the contracts depicted in Figures 1a, 1b and 1d, thus each added pricing instrument is finding the optimal way to incentivize more buyer types to buy the second period good, while finding a way to extract the surplus generated from added sales. However, thresholds for trade in period 2, viz. α_2, given by $1/2 > 7/18 > 1/3$, are non-monotonic. By adding forward pricing, the seller finds it optimal to serve lesser types through the strike price in period two and trade it off by making more types in period one commit to buying the good at the forward price.

2.2 Optimality: a mechanism design approach

Here we show that the allocation rule depicted in Figure 1d is in fact the optimum. To that end, we tread the dynamic mechanism design path, and bring to its shore the novelty of handling global incentive constraints. The description of the model here is brief, and formal gaps are filled in Section 3 when we discuss the general model.

For $k = 1, 2$ the price of transaction and probability of trade are denoted respectively by $p_k \in \mathbb{R}$ and $q_k \in [0, 1]$. The flow utility of the seller is given by p_k and that of the buyer is given by $v_k q_k - p_k$.[11] The ex ante aggregate payoffs are simply the sum of the flow utility over the two periods. We also assume that the seller can commit to a dynamic contract, while the buyer needs to be provided with a minimal expected utility each period, which is normalized to zero.

For the direct mechanism, given by $\{\mathbf{q}, \mathbf{p}\} = \{q_1(v_1), q_2(v_1, v_2), p_1(v_1), p_2(v_1, v_2)\}$, the flow utility of the buyer from truthtelling is given by

$$u_1(v_1) = v_1 q_1(v_1) - p_1(v_1) \quad \text{and} \quad u_2(v_1, v_2) = v_2 q_2(v_1, v_2) - p_2(v_1, v_2)$$

and expected utility in the first period is given by

$$U_1(v_1) = u_1(v_1) + \mathbb{E}\{u_2(v_1, v_2) \mid v_1\}$$

The seller's objective to offer a contract that maximizes her ex ante expected profit subject to individual rationality and incentive compatibility for the buyer. Individual rationality simply demands that $U_1(v_1) \geq 0$ and $u_2(v_1, v_2) \geq 0$ for all v_1, v_2. In what follows, we describe the restrictions imposed by incentive compatibility on the set of attainable allocation rules. For brevity, we shall write $v^2 = (v_1, v_2)$.

For any v_1, incentive compatibility in the second period requires that for all v_2 and v:[12]

$$u_2(v^2) = v_2 q_2(v^2) - p_2(v^2) \geq v_2 q_2(v_1, v) - p_2(v_1, v)$$

By standard arguments in (static) mechanism design, this is equivalent to:

(i) envelope condition: $u_2(v_1, .)$ is a.e. differentiable with $\frac{\partial}{\partial v_2} u_2(v^2) = q_2(v^2)$, and

(ii) monotonicity: $q_2(v^2)$ is non-decreasing in v_2.

Analogously, in the first period, incentive compatibility requires that for all v_1 and v:

$$U_1(v_1) = v_1 q_1(v_1) - p_1(v_1) + \mathbb{E}\{u(v^2)|v_1\} \geq v_1 q_1(v) - p_1(v) + \mathbb{E}\{u_2(v, v_2)|v_1\}$$

which can re-written as

$$U_1(v_1) - U_1(v) \geq \underbrace{(v_1 - v) q_1(v)}_{\text{static information rent}} + \underbrace{{}^1\!/\!_2 \times [u_2(v, v_1) - u_2(v, v)]}_{\text{dynamic information rent}} \tag{1}$$

Note that here we have invoked the one-shot deviation principle by looking at a specific deviation where the buyer reverts back to truthtelling in the second period. Again, by standard arguments in (dynamic) mechanism design this is equivalent to:

[11] For a marginal cost of production c, the flow utility of the seller is given by $p_k - c q_k$. We put $c = 0$ here for simplicity, the extension to $c > 0$ is straightforward.

[12] This condition characterizes incentive compatibility off-path as well, because buyer's payoffs are time-separable.

(iii) dynamic envelope condition: U_1 is a.e. differentiable with $\frac{\partial}{\partial v_1}U_1(v_1) = q_1(v_1) + {}^1\!/\!_2 \times q_1(v_1, v_1) = Z_1(v_1)$ (say), and

(iv) integral monotonicity: $\int_{v_1}^{v} [q_1(v_1) + {}^1\!/\!_2 \times q_2(v_1, x)]dx \leq \int_{v_1}^{v} Z_1(u)du$

The integral monotonicity constraint has been derived as it follows. Starting from Equation (1), use the envelope formula to substitute for $u_2(v, v_1) - u_2(v, v)$, and the dynamic envelope formula to substitute for $U_1(v_1) - U_1(v)$, where all the substitutions are carried out in the integral form of the envelope formulas. Simple re-arranging of terms then gives us the integral monotonicity constraint. In Figure 2 we graphically exposit the integral monotonicity condition: the "average allocation" along the diagonal is greater than the "average allocation" along the vertical line.[13] An equivalence result follows.

Lemma 1. *A mechanism* $\{\mathbf{q}, \mathbf{p}\}$ *is incentive compatible if and only if (i)-(iv) above hold.*

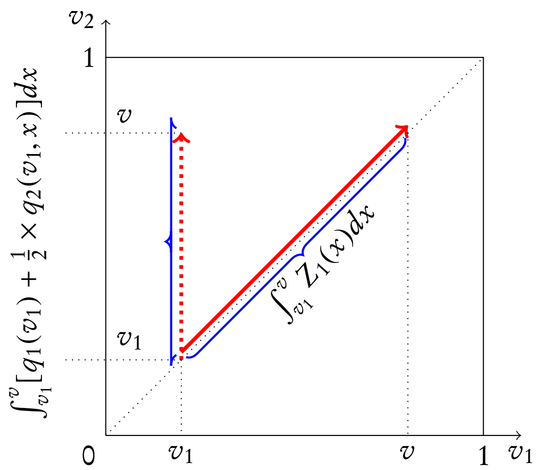


Figure 2: Integral monotonicity for the two period model

Now, using the dynamic envelope condition, seller's expected profit can be expressed only in terms of allocations:

$$
\begin{aligned}
\mathbb{E}\{p_1(v_1) + p_2(v^2)\} &= \mathbb{E}\{v_1 q_1(v_1) + v_2 q(v^2) - U(v_1)\} \\
&= \int_0^1 \left[(2v-1)q_1(v) + {}^1\!/\!_2 \times (2v-1)q_2(v,v) + {}^1\!/\!_2 \times v\mathbb{E}\{q_2(v_1, v)\}\right] dv \\
&= \int_0^1 \left[(2v-1)Z_1(v) + {}^1\!/\!_2 \times v\mathbb{E}\{q_2(v_1, v)\}\right] dx
\end{aligned}
$$

The term $v - \frac{1-F(v)}{f(v)} = 2v - 1$ is the (regular) virtual value in the first period; in the second period the virtual value continues to be same with probability $^1\!/\!_2$, and it is simply the efficient value v with probability $^1\!/\!_2$. The virtual value measures the information rent versus efficiency trade-off:

[13] Integral monotonicity can be seen as the dynamic version of the cyclical monotonicity constraint in multidimensional screening (see Rochet [1987]). Pavan, Segal, and Toikka [2014] show that the dynamic envelope formula and the integral monotonicity condition are necessary and sufficient for incentive compatibility in Markovian models of dynamic mechanism design.

allocations are made on the basis of efficient value minus a sufficient statistic of information rent. Note that if the buyer's type changes in the second period, then the buyer has no more information than the seller about its realization at the time of signing the contract.

Inspired from Myerson [1981], we prefixed the virtual values with *regular* to signify the standard approach wherein information rents are pinned down solely by the local incentive constraints summarized by the envelope condition, and the above expression is maximized pointwise to derive: $q_1(v_1) = \mathbb{1}\{v_1 \geq {}^1\!/\!_2\}$ and $q_2(v^2) = \mathbb{1}\{v_1 = v_2 \geq {}^1\!/\!_2 \wedge v_1 \neq v_2\}$, where $\mathbb{1}$ is the indicator function. Therefore, in the second period, distortions persist only if the first period type is less than ${}^1\!/\!_2$ and types remain constant; dynamic price discrimination is executed only along the lower "diagonal" types. Unfortunately, this contract does *not* satisfy the dropped constraints- it violates both monotonicity constraint in period two and the integral monotonicity constraint in period one. So far the dynamic mechanism design literature does not give us any guidance on how to tackle such problems.

To move beyond the local approach, as a first step, observe that $Z_1(\cdot)$ must be monotonic: for $v_2 > v_1$

$$(v_2 - v_1)Z_1(v_1) \leqslant \int_{v_1}^{v_2} [q_1(v_1) + {}^1\!/\!_2 \times q_2(v_1, x)]dx \leqslant \int_{v_1}^{v_2} Z_1(x)dx \leqslant (v_2 - v_1)Z_1(v_2).$$

where the first and third inequalities follow from monotonicity of $q(v_1, .)$ and the second inequality is simply the integral monotonicity constraint. Thus, we have the following simple result.

Lemma 2. *For* $\{\mathbf{q}\}$ *satisfying (ii) and (iv),* $Z_1(v_1) = q(v_1) + {}^1\!/\!_2 \times q(v_1, v_1)$ *is a non-decreasing function.*

Next, we prove that the seller finds it optimal to backload allocations as much as possible. We say that an allocation is backloaded if $q_1(v_1) > 0$ implies $q_2(v_1, v_2) = 1$ for all v_2. In fact we prove a stronger result here: for any incentive compatible allocation $\{\mathbf{q}\}$, there exists another incentive compatible allocation $\{\hat{\mathbf{q}}\}$ that weakly increases the seller's profit and strictly so if $\{\mathbf{q}\}$ is not backloaded. This construction ensures that keeping the aggregate allocation along the diagonal constant, we can push the weight maximally towards the second period in an incentive compatible way that improves the seller's profit. Backloading s formalized in the next lemma, the proof of which exploits the monotonicity of Z_1.

Lemma 3. *The optimal allocation is backloaded.*

Backloading permits us to split the problem of repeated sales into two independent screening problems. Suppose that the seller solved two distinct problems: (a) the static screening problem of selling one good in the first period, and (b) the sequential screening problem where types are realized in both periods, but an allocation is only made in the second period. In the second problem, the type in period 1 simply acts as a signal for the payoff relevant type in the second period. By independent screening, we mean that the optimal allocations in the two distinct problems: say $q_1(v_1)$ in the first problem, and $q_2(v^2)$ in the second problem, are exactly the same for the joint problem of determining the optimal mechanism repeated sales.

Lemma 4. *A backloaded allocation* $\{\mathbf{q}\}$ *satisfies (ii) and (iv) defined above if only if* q_1, $q_2(v_1, .)$ *are non-decreasing and*

$$\int_{v_1}^{v} q_2(v_1, x)dx \leqslant \int_{v_1}^{v} q_2(x, x)dx$$

Lemma 4 establishes that a backloaded allocation rule is incentive compatible for the original problem if the allocation rule for the first period is incentive compatible in the static model and if the allocation in the second period is incentive compatible in the corresponding sequential screening model.

Finally, suppose that the seller can only use deterministic contracts that is $q_1(v_1)$ and $q_2(v^2) \in \{0, 1\}$. From Lemmata 2 and 4, we get that both $q_1(v_1)$ and $q_2(v_1, v_1)$ are non-decreasing functions. Since the optimum is backloaded (see Lemma 3), we must have two thresholds $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$ such that $q_1(v_1) = \mathbb{1}\{v_1 \geqslant \alpha_1\}$ and $q_2(v_1, v_1) = \mathbb{1}\{v_1 \geqslant \alpha_2\}$. Clearly, $q_2(v^2) = 0$ for $\max\{v_1, v_2\} < \alpha_2$ by (iv). Since the virtual value in the second period is positive for $v_2 \neq v_1$, the optimal allocation satisfies $q(v^2) = \mathbb{1}(\max\{v_1, v_2\} \geqslant \alpha_2)$.

It is easy to show that the optimal first period threshold is α_1 is $1/2$. To find α_2, we compute seller's profit from the second period allocation as a function of α_2:

$$R^{fw}(\alpha_2) = 1/2 \times \int_{\alpha_2}^{1} \left[(2x - 1) + \frac{3x^2}{2}\right] dx$$

The optimal threshold is $\alpha_2 = 7/18$.

We can summarize the main findings of this section in the following result.

Proposition 1. *Consider the two period repeated sales problem where buyer's private valuation is distributed uniformly in the first period, assumes the same value in the second period with probability* $1/2$ *and is drawn again from a uniform distribution with probability* $1/2$*. The optimal allocation is given by:*

$$q_1(v_1) = \mathbb{1}\{v_1 \geqslant 1/2\} \quad \textit{and} \quad q_2(v^2) = \mathbb{1}\{\max\{v_1, v_2\} \geqslant 7/18\}$$

We provide the missing formal details in proving the result in the appendix. In particular, we show that the seller can not gain by using stochastic allocations. Moreover, we state the model and results for a general renewal types process with persistence parameter ρ as opposed to the special case $\rho = 1/2$ assumed here.

3 The general model and direct mechanism

3.1 Primitives

A seller wants to repeatedly sell a (non-durable) good or service to a buyer. Time is continuous and infinite, $t \in \mathbb{R}_+$ and both agents discount future at the same rate, r. The buyer's valuations for the good, $\{V_t\}$, is private and follow a stationary pseudo-renewal process; that is, $V_t = X_{N_t}$ where

- $\{X_n\}$ is a sequence of i.i.d. draws from F,

- $\{N_t\}$ is a Poisson process with intensity λ.

We shall assume that F is continuous, supported on $[0,1]$ and admits a density f.

The buyer gets to observe a current state of his valuation process at discrete points, $\{\Delta k : k \in \mathbb{N}\}$. The process observed by the buyer is discrete, and it is defined as $v_k = V_{\Delta(k-1)}$. This process is Markov with the following law of motion:

$$\mathbb{P}\{v_{k+1} \le v' | v_k = v\} = \rho \mathbb{1}\{v' \geqslant v\} + (1-\rho)F(v')$$

where $\rho = e^{-\lambda\Delta}$ is a probability that there was no arrival between two consecutive discrete time points. Note that in discrete time this stochastic process is a direct generalization of of the one we saw in Section 2- first period type is drawn from a prior F and then every period the type is the same with probability ρ or is drawn again from F with probability $1-\rho$. Since ρ assumes values between 0 and 1 (as λ goes from ∞ to 0), it captures the level of persistence in the types' process. Moreover, in continuous time, as $\Delta \to 0$, $\{v_{\lfloor t/\Delta \rfloor + 1}\}$ converges almost surely to $\{V_t\}$.

The seller can supply one unit of the good at each instance of time at no cost. The buyer's flow payoff is given by $V_t Q_t dt - dP_t$ and the seller's is given by dP_t where $Q_t \in [0,1]$. The seller wants to design a contract that maximizes her expected profit. The direct mechanism and the associated design problem are described next.

3.2 The Myersonian optimization problem

A contract specifies a history-dependent allocation and a payment for each instance of time, $\{Q_t, dP_t\}$. Since buyer's information is arriving discretely, there is no loss of generality in restricting attention to allocations which are constant on intervals $[\Delta(k-1), \Delta k)$ and payments which are made at discrete time points. Invoking the revelation principle (Myerson [1986], and see also Sugaya and Wolitzky [2018]), we focus on direct mechanisms of the form $\{\mathbf{q}, \mathbf{p}\} = \left\{q_k(v^k), p_k(v^k)\right\}_{k=1}^{\infty}$, where $q_k(v^k) = Q_{\Delta(k-1)}\left(V^{\Delta(k-1)}\right)$ and $p_k(v^k) = dP_s\left(V^{\Delta(k-1)}\right)$. For a later use, define discounting between two consecutive time points $\delta = e^{-r\Delta}$ and $\hat{\Delta} = \frac{1-\delta}{r}$.

Buyer's reporting strategy prescribes a history-dependent report for each discrete time point, $\{\sigma_k\}$ with $\sigma_k(v^k) \in [0,1]$. A contract is incentive compatible if the buyer can not gain by misreporting his information; that is, for any reporting strategy σ_k,

$$\mathbb{E}\left\{\sum_{k=1}^{\infty} \delta^{k-1}\left[v_k q_k(v^k)\hat{\Delta} - p_k(v^k)\right]\right\} \geqslant \mathbb{E}\left\{\sum_{k=1}^{\infty} \delta^{k-1}\left[v_k q_k[\sigma_k(v^k)]\hat{\Delta} - p_k[\sigma_k(\tilde{v}^k]\right]\right\}$$

Define buyer's expected payoffs from truth-telling by

$$U_k(v^k) = v_k q_k(v^k)\hat{\Delta} - p_k(v^k) + \delta\mathbb{E}\left\{U_{k+1}(v^{k+1})|v^k\right\} \tag{2}$$

Using the one-shot deviation principle, incentive compatibility can be expressed as it follows:

$$U_k(v^k) \geqslant U_k(v^{k-1}, v) + \underbrace{(v_k - v)q_k(v^{k-1}, v)\hat{\Delta}}_{\text{static information rent}} + \underbrace{\delta\rho\left\{U_k(v^{k-1}, v, v_k) - U_k(v^{k-1}, v, v)\right\}}_{\text{dynamic information rent}} \tag{3}$$

If $\delta = 0$ the equation above is equivalent to the incentive compatibility familiar from static mechanism design. For the renewal Markov process, the dynamic information rent has two components, one along which the true type remains constant (with probability ρ) and another where the type changes (with probability $1-\rho$). Since a change in type leads to a history independent draw of the new type, that term cancels out from the expression above. Thus, if we have $\rho = 0$, the types' process would be i.i.d. and information rent would then be purely static. Also, a contract is said to be individually rational if

$$U_k(v^k) \geq 0 \tag{4}$$

Seller's optimization problem ($\star$) can then be written as

$$(\star) \qquad \max_{\{\mathbf{q},\mathbf{p}\}} \quad \mathbb{E}\left\{\sum_{k=1}^{\infty} \delta^{k-1} p_k(v^k)\right\} \quad \text{subject to (2), (3), (4)}$$

In what follows, we pursue a Myersonian approach towards characterizing the optimal contract. As in the two period model, it is useful to define the average allocation along the "diagonal", that is along the persistent histories, as $Z_k(v^k) = q_k(v^k)\hat{\Delta} + \delta\rho Z_{k+1}(v^k, v_k)$. The following lemma characterizes buyer's incentives.

Lemma 5. *A mechanism $\{\mathbf{q}, \mathbf{p}\}$ is incentive compatible if and only if it satisfies the following set of conditions:*

$$U_k(v^{k-1}, .) \quad \textit{is differentiable a.e. with} \quad \frac{\partial}{\partial v_k} U_k(v^k) = Z_k(v^k) \tag{IC-FOC}$$

$$\int_{v_k}^{v} Z_k(v^{k-1}, x)dx \geqslant \int_{v_k}^{v} \left[q_k(v^k)\hat{\Delta} + \delta\rho Z_{k+1}(v^k, x)\right] dx \tag{IM}$$

Here (IC-FOC) is the first-order condition implied by incentive compatibility, referred commonly as the dynamic envelope formula, since it is derived from the envelope theorem applied to such dynamic models of mechanism design. (IM) is the integral monotonicity constraint which is derived by substituting the formula for expected utility from (IC-FOC) to the inequality (3). It is the dynamic counterpart of the cyclical monotonicity condition for multidimensional screening in Rochet [1987].

Next, we state the appropriate generalization of Lemma 2 that will be invoked repeatedly in proofs that concern the direct mechanism.

Lemma 6. *Take* $\mathbf{q}$ *satisfying (IM), then $Z_k(v^{k-1}, .)$ is non-decreasing.*

Unlike the static screening problem and akin to the multidimensional screening problem, types in dynamic screening cannot be ordered. Thus, there is no complete order that can characterize a notion of monotonicity for the allocation rules. Lemma 6 simplifies that problem to some extent by allowing us to rank "average" allocations along the "diagonal" for any history of types.

The next Lemma reduces problem ($\star$) to one that allocates the good to the buyer according to his (local) virtual utility subject to potentially binding global incentive constraints. We say that an allocation rule $\mathbf{q}$ solves problem ($\star$) if there exists $\mathbf{p}$ such that $\{\mathbf{q}, \mathbf{p}\}$ is a solution to ($\star$).

Lemma 7. *An allocation rule* $\mathbf{q}^*$ *solves* ($\star$) *if and only if* $\{\mathbf{q}^*\}$ *is a solution to the following problem* (*):

$$(*) \qquad \max_{\{\mathbf{q}\}} \quad \mathbb{E}\left\{\sum_{k=1}^{\infty} \delta^{k-1} w_k(v^k) q_k(v^k)\right\}$$

where

$$w_k(v^k) = \begin{cases} v_1 - \frac{1-F(v_1)}{f(v_1)} & \text{if} \quad v_1 = \ldots = v_k \\ v_k & \text{otherwise} \end{cases}$$

subject to (IM).

4 Sequential screening: selling one object at a specified time

In this section, we consider a special model where seller makes a sale of one good only at time t. Only buyer's type at time t is payoff relevant and all previous types are relevant to the extent that they informative about the payoff relevant type. The seller designs a contract where payments can be charged at any given point in time, but allocation decisions are made at t. This model is a direct generalization (along the time dimension) of the two-period sequential screening model studied by Courty and Li [2000].

We first describe the optimum under a restricted set of pricing instruments, and then establish that a simple combination of these instruments achieves the deterministic optimum.

4.1 Static spot pricing

Perhaps, the simplest selling strategy is to offer a fixed spot price. Suppose that the buyer can purchase the t-th good for a price αdt; he will buy the good if and only if $V_t \geq \alpha$ (resolving ties in favor of trade). Since V_t is distributed according to F, the probability of making a sale is $1 - F(\alpha)$. It follows that seller's profit is $e^{-rt} R^s(\alpha) dt$, where

$$R^s(\alpha) = \alpha[1 - F(\alpha)]$$

4.2 European option

As we showed in the two period model, the seller can increase her profit by requiring the buyer to pay an upfront fee at the initial date. Note that type V_0 buyer's expected payoff from the fixed price in Section 4.1 given by is $e^{-rt}\mathbb{E}\{V_t - \alpha)^+ | V_0\} dt$,

$$\mathbb{E}\{V_t - \alpha)^+ | V_0\} = e^{-\lambda t}(V_0 - \alpha)^+ + (1 - e^{-\lambda t})\mathbb{E}\{(V_t - \alpha)^+\}$$

The first term corresponds to the event of no arrival of new information, and the second refers to its complement. Conditional of wanting all buyer types' to partake in the contract, the seller can charge up to $e^{-rt}\mathcal{M}_t(\alpha)dt$, where

$$\mathcal{M}_t(\alpha) = (1 - e^{-\lambda t})\mathbb{E}\{(V_t - \alpha)^+\} \tag{5}$$

Seller's revenue is given by $e^{-rt}\mathcal{R}_t^o(\alpha)dt$,

$$\mathcal{R}_t^o(\alpha) = \mathcal{R}^s(\alpha) + \mathcal{M}_t(\alpha) = e^{-\lambda t}\alpha[1 - F(\alpha)] + (1 - e^{-\lambda t})\int_\alpha^1 v dF(v) \tag{6}$$

Clearly, $\mathcal{R}_t^o(\alpha) \geqslant \mathcal{R}^s(\alpha)$ for all α, and optimizing Equation (6) over α gives us the optimal value say, α_t^o.

4.3 Optioning forwards

Now, we show that the seller can do even better by selling the t-th good at earlier dates, in addition to charging the upfront fee introduced in Section 4.2. In particular, suppose that as a function of strike price α, the buyer can commit at time $s \leqslant t$ to purchase the good for a price $y_t^s(\alpha)dt$, where y_t^s is defined as

$$y_t^s(\alpha) = e^{-\lambda(t-s)}\alpha + (1 - e^{-\lambda(t-s)}) \left[\mathbb{E}\{V_t\} - \mathbb{E}\{(V_t - \alpha)^+\}\right] \tag{7}$$

This is to be interpreted as an American option on forwards $\{y_t^s\}_{s \leq t}$ with premium $\mathcal{M}_t$.

Once the premium $\mathcal{M}_t$ paid, the buyer is facing a stopping a problem with a deadline t and gain process $e^{-rt}[\mathbb{E}\{V_t|V_s\} - y_t^s]ds$. We make an assumption here that when buyer is indifferent between waiting and buying today, he will buy today. This selection resolves ties in favor of the seller optimal outcome.[14] A characterization of the smallest optimal stopping rule $\tau_t = \tau_t(\alpha)$ is as it follows.

Lemma 8. *Consider the problem of selling a single good at date t. The pricing mechanism constituting of an option contract with premium $\mathcal{M}_t$ and a schedule of forwards $\{y_t^s\}_{s \leq t}$, defined respectively in equations (5) and (7), induces each buyer type to pay the upfront fee $e^{-rt}\mathcal{M}_t dt$ and implements the following optimal stopping rule, τ_t:*

$$\textit{buy the } t\textit{-th good at time } s \quad \textit{iff} \quad V_s \geq \alpha_t$$

Proof. Suppose that the upfront fee was paid, then buyer's problem is to choose the stopping rule $\tau \leq t$.

Fix $s \leq t$, and define Snell's envelope

$$W_t^s = ess \sup_{\tau \in [s,t]} \mathbb{E}\left\{\mathbb{E}\{V_t|V_\tau\} - y_t^\tau | V^s\right\}$$

Clearly, $W_t^t = (V_t - \alpha)^+$. Since waiting until date t is available,

$$W_t^s \geqslant \mathbb{E}\{W_t^t|V^s\} = e^{-\lambda(t-s)}(V_s - \alpha)^+ + (1 - e^{-\lambda(t-s)})\mathbb{E}\{(V_t - \alpha)^+\}$$

[14]We explained in Footnote 10 the sense in which this is the "right" way of resolving ties.

Notice that $\mathbb{E}\{W_t^t|V^s\}$ is a martingale, and it dominates $\mathbb{E}\{V_t|V_s\} - y_t^s$ as

$$\mathbb{E}\{V_t|V_\tau\} - y_t^\tau = e^{-\lambda(t-s)}(V_s - \alpha) + (1 - e^{-\lambda(t-s)})\mathbb{E}\{(V_t - \alpha)^+\}$$

Since W_t^s is the smallest super-martingale dominating $\mathbb{E}\{V_t|V_\tau\} - y_t^\tau$, we must have $W_t^s = \mathbb{E}\{W_t^t|V^s\}$. It follows that the smallest optimal stopping time is $\tau_t = \inf\{s \in [0,t] : \mathbb{E}\{V_t|V_s\} - y_t^s = \mathbb{E}\{W_t^t|V^s\}\}$ where

$$\mathbb{E}\{V_t|V_s\} - y_t^s = \mathbb{E}\{W_t^t|V^s\} \quad \text{iff} \quad V_s \geqslant \alpha$$

Finally, observe that buyer's expected payoff net the option premium is always positive:

$$W_t^0 - \mathcal{M}_t = e^{-\lambda t}(V_0 - \alpha)^+ \geqslant 0$$

In other words, all types of the buyer will take the option. □

Next, we define seller's revenue from offering an option contract with forward prices.

Lemma 9. *The seller's expected revenue from offering the contract* $\left\{\alpha, \mathcal{M}_t(\alpha), \{y_t^s(\alpha)\}_{s\leq t}\right\}$ *is given by* $e^{-rt}\mathcal{R}_t^{fw}(\alpha)dt$ *where*

$$\mathcal{R}_t^{fw}(\alpha) = e^{-\lambda t}\alpha[1 - F(\alpha)] + (1 - e^{-\lambda t})\int_\alpha^1 vdF(v) + (1 - e^{-\lambda[1-F(\alpha)]t})\int_0^\alpha vdF(v) \quad (8)$$

Proof. See Appendix. □

It easy to that $\mathcal{R}_t^{fw}(\alpha) \geqslant \mathcal{R}_t^o(\alpha)$ for all α, in fact:

$$\mathcal{R}_t^{fw}(\alpha) = \mathcal{R}_t^o(\alpha_t) + (1 - e^{-\lambda[1-F(\alpha)]t})\int_0^\alpha xf(x)dx$$

Finally, denote by α_t^{fw}, the threshold that maximizes Equation (8) when the date of sale of the single object is t.

4.4 Characterizing dynamic pricing

In this section, we discuss the thresholds for sale generated by the dynamic pricing instruments introduced above. We also take a closer look at dynamics of forward prices and distribution of sales time.

It is important to note that time t and rate of transition λ enter symmetrically in all the expressions; thus what really matters for R_t^o and R_t^{fw} is the normalized time λt. A characterization of these thresholds is as it follows.

Proposition 2. *The optimal thresholds satisfy the following properties:*

(a) $\alpha_0^o = \alpha_0^{fw} = \alpha^s$ *and* $\lim_{\lambda\to 0}\alpha_t^o = \lim_{\lambda\to 0}\alpha_t^{fw} = \alpha^s$*;*

(b) *α_t^o and α_t^{fw} are strictly positive;*

(c) *α_t^o and α_t^{fw} are strictly decreasing in λ and t;*

(d) *$\lim_{t\to\infty} \alpha_t^o = \lim_{t\to\infty} \alpha_t^{fw} = 0$ and $\lim_{\lambda\to\infty} \alpha_t^o = \lim_{\lambda\to\infty} \alpha_t^{fw} = 0$;*

(e) *suppose that $v \mapsto \frac{1-F(v)}{f(v)}$ is non-decreasing, then $\alpha^s > \alpha_t^{fw} > \alpha_t^o$ for all $t > 0$.*

Proof. See Appendix. □

Part (a) states that as $t \to 0$, so that model becomes static, or as $\lambda \to 0$, that is as types become perfectly persistent, the optimal strike price from the option contract converges to the optimal (static) spot price. Further, these thresholds are positive and strictly decreasing in normalized time. Thus, as the date of sale of the object increases in the sequential screening problem, the (ex ante) probability of trade goes up. Part (d) shows that in the limit the good is always sold. When $t \to \infty$ the initial information advantage of the agent go to zero, and when $\lambda \to \infty$ the stochastic process becomes i.i.d; in both cases the efficient contract becomes optimal and the seller can extract all the expected surplus as profit. Note that in each of these cases the limit of time or transition probability is taken while keeping the other fixed at some positive finite value.

Finally, Part (e) makes a subtle point about the increase in strike price or threshold of sale from the simple option contract to the option with forwards: $\alpha_t^{fw} > \alpha_t^o$. In particular, it implies that the European option contract is more expensive than the American option on forwards; that is, $\mathcal{M}_t(\alpha_t^o) \geqslant \mathcal{M}_t(\alpha_t^{fw})$.

As for dynamics of forward prices, observe that $\mathbb{E}\{(V_t - \alpha)^+\} = \int_\alpha^1 [1 - F(v)]dv$ for any α. Therefore, we can rewrite forward prices of the optimal American option on forwards as

$$y_t^s(\alpha_t^{fw}) = \alpha_t^{fw} - \underbrace{(1 - e^{-\lambda(t-s)}) \int_0^{\alpha_t^{fw}} F(v)dv}_{\text{forward discount}} \leqslant \alpha_t^{fw}$$

We term the difference between the spot price and forward price at date s as forward discount. It immediately follows that the forward discount progressively decreases over time.

Proposition 3. *$\alpha_t^{fw} - y_t^s(\alpha_t^{fw})$ is strictly decreasing in s.*

For the option with forward prices, as described in Lemma 8, the first time the buyer has a draw of value greater than α_t^{fw} he buys the good. Observe that $\max_{s\leq t} V_s$ is distributed according to $\mathcal{G}_t$ defined by

$$\mathcal{G}_t(v) = F(v)e^{-\lambda[1-F(v)]t}$$

It immediately follows that the probability that the t-th good is sold by time s is $1 - \mathcal{G}_s(\alpha_t^{fw})$,

$$1 - \mathcal{G}_s(\alpha_t^{fw}) = \underbrace{1 - F(\alpha_t^{fw})}_{\text{prob. of sale at } t=0} + F(\alpha_t^{fw}) \underbrace{\left[1 - e^{-\lambda[1-F(\alpha_t^{fw})]s}\right]}_{\text{prob. of at least one arrival} \geqslant \alpha_t^{fw} \text{ given } V_0 \leqslant \alpha_t^{fw}}$$

Notice that the good can be unsold which happens with the probability $G_t(\alpha_t^{fw})$. The expected time of sale (given it happened) can be computed as

$$\mathbb{E}\{\tau_t|\tau_t \neq t\} = \int_0^t s d\left[\frac{1-\mathcal{G}_s(\alpha_t^{fw})}{1-\mathcal{G}_t(\alpha_t^{fw})}\right] = t - \int_0^t \frac{1-\mathcal{G}_s(\alpha_t^{fw})}{1-\mathcal{G}_t(\alpha_t^{fw})} ds$$

A simple set of results, complimentary to Proposition 2 follows.

Proposition 4.

(a) $\mathcal{G}_t(\alpha_t^{fw})$ *is strictly decreasing in* λ *with* $\lim_{\lambda\to 0}\mathcal{G}_t(\alpha_t^{fw}) = F(\alpha^s)$ *and* $\lim_{\lambda\to\infty}\mathcal{G}_t(\alpha_t^{fw}) = 0$;

(b) $\mathcal{G}_t(\alpha_t^{fw})$ *is strictly decreasing in* t *with* $\lim_{t\to 0}G_t(\alpha_t^{fw}) = F(\alpha^s)$ *and* $\lim_{t\to\infty}G_t(\alpha_t^{fw}) = 0$;

(c) $\lim_{\lambda\to 0}\mathbb{E}\{\tau_t|\tau_t \neq t\} = \lim_{\lambda\to\infty}\mathbb{E}\{\tau_t|\tau_t \neq t\} = 0$;

(d) $\lim_{t\to 0}\mathbb{E}\{\tau_t|\tau_t \neq t\} = \lim_{t\to\infty}\mathbb{E}\{\tau_t|\tau_t \neq t\} = 0$.

Proof. See Appendix. □

We plot these statistics as functions of t for the uniform distribution and $\lambda = 1$.

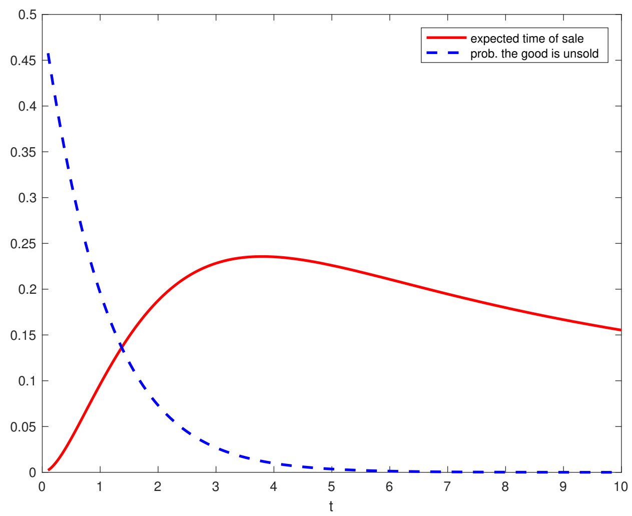


Figure 3: red- expected time of sale, blue- probability that the good is unsold; $F(x) = x$ and $\lambda = 1$.

4.5 Optimality of forward pricing.

In this section, we show that optioning forwards implements the optimal deterministic allocation for the sequential screening problem. Specifically, Proposition 5 below establishes that the dynamic pricing contract derived in Section 4.3 achieves the optimal deterministic allocation rule. The intuition for the result is similar to selling of the second good in the two-period problem.

Proposition 5. *Suppose that the seller can sell only the k-th good, there exists a_k^{fw} such that the optimal deterministic allocation is*

$$q_k(v^k) = \begin{cases} 1 & \textit{whenever} \quad \max_{n \le k} v_n \geqslant a_k^{fw} \\ 0 & \textit{otherwise} \end{cases}$$

Moreover, $a_{\lfloor t/\Delta \rfloor + 1}^{fw} \to \alpha_t^{fw}$ as $\Delta \to 0$.

Proof. We split the proof into two parts. First, we establish that the optimal determenistic allocation is to allocate the good if and only if $\max_{n \leqslant k} v_n \geq a$ for some threshold a. Second, we show how to choose the value of a optimally.

Step 1. Take any allocation $q_k \in \{0, 1\}$ satisfying (IM). By assumption, the allocation can happen only at k, therefore $Z_1(v_1)$ could take only two value: 0 and $(\delta\rho)^{k-1}\hat{\Delta}$.

Lemma 6 implies that there exists a threshold a such that $Z_1(v_1) = (\delta\rho)^{k-1}\hat{\Delta}$ whenever $v_1 > a$ and $Z_1(v_1) = 0$ whenever $v_1 < a$. By induction, suppose that for $n < k$, $Z_n(v^n) = 0$ for $\max_{m \leqslant n} v_m < a$. For $v_n < v \leqslant a$, (IM) requires

$$\int_{v_n}^{v} Z_n(v^{n-1}, x)dx = 0 \geq (\delta\rho) \int_{v_n}^{v} Z_{n+1}(v^n, x)dx$$
$$Z_{n+1}(v^{n-1}, v, v) = 0 \geq Z_{n+1}(v^{n-1}, v, v_n)du$$

which implies that $Z_{n+1}(v^{n+1}) = 0$ for $\max_{m \leqslant n+1} v_m < a$.

Consider $\hat{q}_k(v^k) = \mathbb{1}\{\max_{n \leqslant k} v_n \geqslant a\}$. It is easy to see that $\hat{q}_k$ satisfies (IM) as $\hat{Z}_n(v^{n-1}, \max\{v_n, v\}) = \delta\rho\hat{Z}_{n+1}(v^n, v, v)$ for $n < k$. We claim that seller's profit is higher. On the one hand, buyer's rents are the same in both cases, because $Z_1(v_1) = \hat{Z}_1(v_1)$. On the other hand, the surplus is higher, because $\hat{q}_k(v^k) \geqslant q_k(v^k)$.

Step 2. We have established that the optimal deterministic allocation must take the form: $q_k(v^k) = \mathbb{1}\{\max_{n \leqslant k} v_n \geq a\}$ for some a. It remains to choose the threshold optimally. Write seller's profit from k-th good as a function of a:

$$\mathbb{E}\{v_k \mathbb{1}\{\max_{n \leqslant k} v_n \geq a\}\} - \rho^{k-1} \int_a^1 [1 - F(v)]dv$$

Denote by G_k the distribution of $\max_{n \le k} v_n$; that is, $G_k(v) = F(v)[\xi(v)]^{k-1}$ for $\xi(v) =$

$(1-\rho)F(v)+\rho$. Rewrite the first term of this expression as it follows:

$$\begin{aligned}
\mathbb{E}\{v_k \mathbb{1}\{\max_{n\leqslant k} v_n \geq a\}\} &= \rho^{k-1}\int_a^1 vf(v)dv + (1-\rho)\sum_{n=2}^{k}\rho^{k-n}\int_0^1 vf(v)\mathbb{P}\{\max\{\max_{m\leq n-1} v_m, v\} \geqslant a\}dv \\
&= \rho^{k-1}\int_a^1 vf(v)dv + (1-\rho)\sum_{n=2}^{k}\rho^{k-n}\int_a^1 \left[\int_0^v xf(x)dxG_{n-1}(v)\right]' dv \\
&= \rho^{k-1}\int_a^1 \phi_k(v)dv
\end{aligned}$$

where ϕ_k is defined recursively by $\phi_1(v) = vf(v)$ and

$$\begin{aligned}
\phi_{k+1}(v) &= \phi_k(v) + \frac{1-\rho}{\rho}\left(\int_0^v xf(x)dxF(v)\left[\frac{\xi(v)}{\rho}\right]^{k-1}\right)' \\
&= \phi_1(v) + \frac{1-\rho}{\rho}\left(\int_0^v xf(x)dx\left(1-\left[\frac{\xi(v)}{\rho}\right]^{k}\right)\right)'
\end{aligned}$$

Clearly, seller's profit is continuous function of a, therefore the maximizer a_k^{fw} is well defined. Moreover, observe that $\phi_{\lfloor t/\Delta\rfloor+1}$ uniformly converges to $\left(e^{F(v)\lambda t}\int_0^v xf(x)dx\right)'$. It follows that seller's profit converges uniformly as well:

$$\rho^{\lfloor t/\Delta\rfloor}\int_a^1 [\phi_{\lfloor t/\Delta\rfloor+1}(v) + F(v) - 1]dv \to \mathcal{R}_t^{fw}(a)$$

As a result, $a_{\lfloor t/\Delta\rfloor+1}^{fw} \to \alpha_t^{fw}$ as $\Delta \to 0$. □

5 Dynamic pricing for repeated sales

Now, we look at dynamics of sales for the repeated sales model when the seller is optioning forwards. Forward pricing mechanism has the same allocation as in the sequential screening setting, namely $Q_t(V^t) = 1\{\max_{s\leq t} V_s \geq \alpha_t^{fw}\}$. Moreover, a sale of each good is still characterized by the stopping time, τ_t. The novelty is that these stopping times are correlated, because buyer's purchases depend on the same sample path of valuations.

We describe timing of sales using the concept of stock, S_t; that is, the set of remaining goods available at time t. Since forward sales are backloaded, we identify the set of remaining goods with the latest available good. Formally, $S_t = s$ means that each good in the interval $[t, t+s]$ is still available. And, $S_t = 0$ means that no goods are available anymore.

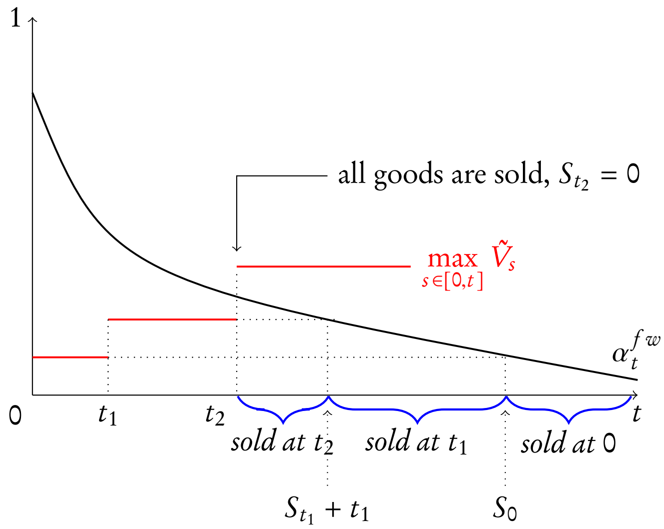
The process of stock is Markov and can be described as it follows: $S_t = 0$ if and only if $\max_{t'\leq t} V_{t'} \geq \alpha_t$ and $S_t = s > 0$ if and only if $\max_{t'\leq t} V_{t'} = \alpha_{t+s}^{fw}$. Therefore, conditional on $S_{t-} = 0$,

$S_t = 0$ with probability one. Conditional on $S_{t-} = s > 0$,

$$\begin{cases} S_t = s - dt & \text{with probability} \quad 1 - \lambda[1 - F(\alpha^{fw}_{t+s})]dt + o(dt) \\ S_t \in [0, s], \quad \mathbb{P}(S_t \le s') = \frac{1-F(\alpha^{fw}_{t+s'})}{1-F(\alpha^{fw}_{t+s})} & \text{with probability} \quad \lambda[1 - F(\alpha^{fw}_{t+s})]dt + o(dt) \end{cases}$$

The former corresponds to the case of no sale, when the stock depreciates at rate -1. In the latter case, there is a sale and the stock decreases at random.

The following figure visualizes the process of stock. At the initial date, it is value drawn to be S_0 and all goods with $t \geqslant S_0$ are sold. Then, the stock is changing depreciates detemistically until date t_1 reaching $S_{t_1-} = -t_1 + S_0$. At t_1, the stock drops to S_{t_1}; that is, all goods with $t \in [t_1 + S_{t_1}, S_0)$ are sold. Finally, it depreciates determenistically until date t_2 where all availble goods are sold.



We look at unconditional properties of the stock process. It is easy to see that the expected stock at t is given by

$$\mathbb{E}\{S_t\} = \int_0^\infty s d\left[1 - \mathcal{G}_t(\alpha^{fw}_{t+s})\right]$$

Let τ^S be the stopping time when the stock drops to zero; that is, $\tau^S = \inf\{t : S_t = 0\}$. Clearly, τ^S is distributed according to $1 - \mathcal{G}_t(\alpha^{fw}_t)$, thus

$$E\{\tau^S\} = \int_0^\infty t d\left[1 - \mathcal{G}_t(\alpha^{fw}_t)\right]$$

The following proposition summarizes behavior of the stock process.

Proposition 6.

(a) S_t *converges a.s. to* 0;

(b) $\mathbb{E}\{S_t\}$ *is strictly decreasing in* λ *with* $\lim_{\lambda\to 0} \mathbb{E}\{S_t\} = \infty$ *and* $\lim_{\lambda\to\infty} \mathbb{E}\{S_t\} = 0$;

(c) $\mathbb{E}\{S_t\}$ *is strictly decreasing in* t *with* $\lim_{t\to 0}\mathbb{E}\{S_t\} = \int_0^\infty s d\left[1 - F(\alpha_s^{fw})\right] > 0$ *and* $\lim_{t\to\infty}\mathbb{E}\{S_t\} = 0$;

(d) $\mathbb{E}\{\tau^S\}$ *is strictly decreasing in* λ *with* $\lim_{\lambda\to 0}\mathbb{E}\{\tau^S\} = \infty$ *and* $\lim_{\lambda\to\infty}\mathbb{E}\{\tau^S\} = 0$;

Proof. See Appendix. □

6 Max-threshold versus optimum: understanding the gap

So far we have solved for the optimal (deterministic) mechanism for the sequential screening problem and then used it to construct a candidate dynamic pricing mechanism for the repeated sales problem. We will refer to this candidate mechanism as option contract with forwards in the pricing world and the max-threshold mechanism in the (direct) mechanism design world. The former name takes after the pricing instruments used in our construction, and the latter is based on the fact that the corresponding direct mechanism implements the allocation rule: $Q_t(V^t) = 1\{\max_{s\le t} V_s \ge \alpha_t^{fw}\}$; thus, trade happens if the maximum of the sequence of values is above a strictly decreasing sequence of thresholds.

It can be shown that the max-threshold contract is in fact not optimal, beyond the two-period model. In this section we explore the gap between our candidate mechanism and the optimum. First we show that the optimal contract is backloaded, and argue that it is a property that any candidate mechanism should satisfy in this setting. Second, we establish that the max-threshold mechanism is in fact optimal under a restricted class of mechanisms that demand a strong form of monotonicity of allocation rule, a condition typically invoked in the literature to ex post check the validity of the first-order approach. And, third, we describe two key properties of the optimal mechanism that our candidate mechanism does not satisfy, and why not requiring them is a reasonable form of simplicity to seek in a candidate mechanism.

6.1 Optimal contract is backloaded

We say an allocation $\{\mathbf{q}\}$ is backloaded if:

$$q_k(v^k) > 0 \Rightarrow q_{k+1}(v^k, v) = 1\ \forall\ v \tag{B}$$

This is a fairly strong backloading requirement. It demands that for any sequence of realizations of the buyer's types, the first instance of positive allocation will automatically lead to trade for sure in all future periods. One can easily envision a more permissive bacloading criterion, but that will not be required since the optimal allocation actually satisfies the aforesaid notion of backloading.

Lemma 10. *The optimal contract satisfies backloading.*

Lemma 10 implies that for any sequence of value realizations, the seller will find it optimal to randomize at most at one point, all allocations before and after this instant are bang-bang, rasing the plausible hypothesis that gains from randomization are probably not very large. Moreover, in the technique pursed to prove this result, a stronger version is established: starting from any incentive compatible allocation that is not backloaded, we can construct another incentive compatible and backloaded allocation that keeps the expected utility of the buyer the same but strictly increases the profit of the seller.

6.2 Max-threshold is optimal under strong monotonicity

Most of the literature (theoretical and applied) that studies dynamic screening problems with persistent private information relies on the first-order approach which then checks its validity ex post by verifying a set of sufficient conditions on allocation rules that are simpler than the (IM).[15] Pavan et al. [2014] (Corollary 1 therein) enlist a set of such sufficient conditions: the stringiest amongst them being strong monotonicity and the most permissive being what they term single-crossing. We define them next in the context of our model.

An allocation rule $\{\mathbf{q}\}$ is said to satisfy strong monotonicity if

$$v^k \geqslant \hat{v}^k \Rightarrow q_k(v^k) \geqslant q_k(\hat{v}^k) \tag{SM}$$

And, an allocation rule $\{\mathbf{q}\}$ is said to satisfy single-crossing if

$$\left[v' - v\right] \left[Z_k(v^{k-1}, v') - q_k(v^{k-1}, v)\hat{\Delta} - \delta \rho Z_{k+1}(v^{k-1}, v, v')\right] \geqslant 0 \tag{SC}$$

A clear hierarchy in the criterion for incentive compatibility can be easily checked:

$$\text{(SM)} \Rightarrow \text{(SC)} \Rightarrow \text{(IM)}$$

In the next proposition, we show that the max-threshold mechanism is actually the optimal one under (SC). Moreover, there is no gap between (SM) and (SC), since the max-threshold mechanism trivially satisfies (SM).

Proposition 7. *Suppose that the seller sells at $k = 1, 2, ..., \infty$. The optimal allocation that satisfies either (SC) is as the max-threshold mechanism, defined as:*

$$q_k(v^k) = \begin{cases} 1 & \textit{whenever} \quad \max_{n \leq k} v_n \geqslant a_k^{fw} \\ 0 & \textit{otherwise} \end{cases}$$

We do not think there is an immediately deep economic reason to demand that the optimal allocation satisfy (SM), though it is indeed a simple heuristic in a very complicated design problem. However, as the next section will illuminate, (SC) actually generates an economically justifiable and logically intuitive set of restrictions on the optimal mechanism. In particular, the exact gap between the optimal mechanism and the optimal mechanism under (SC) is precisely that the latter kills randomization and more importantly simplifies the exact nature of history dependence in the set of feasible allocation rules, both of which are desirable properties

[15] In a recent survey, Bergemann and Välimäki [2018] write: "The method of analysis for the dynamic contracting problem above relies heavily on the payoff equivalence theorem, also known as the first-order approach.

in the search for simpler dynamic mechanisms that can still manage to achieve a lion share of the optimum.

6.3 The gap: stochastic and linked sales

The option contract with forwards implements the max-threshold mechanism, which is optimal under (SC). Is the gap between (SC) and (IM) binding? In this section, we establish that it indeed is and characterize through examples, the qualitative nature of the gap. Recollect a feasible contract is one which satisfies incentive compatibility and individual rationality.

Proposition 8.

(a) Suppose that the seller sells only at $k = 3$, and $v \mapsto \frac{1-F(v)}{f(v)}$ is non-decreasing. There exists a feasible stochastic contract which yields a higher profit than the max-threshold mechanism.

(b) Suppose that the seller sells only at $k = 3, 4$ and f. There exists a feasible deterministic contract which yields a higher profit than the max-threshold mechanism.

In the proof of this result, presented in the appendix, we start with the max-threshold mechanism and construct small (local) improvements that are incentive compatible and ..

For intuition consider the following:

Now, consider the perturbed allocation $\{\tilde{q}_3, \tilde{q}_4\}$. Modify this implementation as it follows.

1. at $k = 1$, the buyer has to pay the option premium $\delta^2[M_3(\alpha_3^{fw}) + \delta M_4(\alpha_4^{fw})]\hat{\Delta}$ to be eligible for future purchases;
2. at $k = 1$, the buyer can purchase the fourth good for $\delta^3 y_4^1(\alpha_4^{fw})\hat{\Delta}$ or both goods for $\delta^2[y_3^1(\alpha_3^{fw}) + \delta y_4^1(\alpha_4^{fw})]\hat{\Delta}$;
3. at $k = 2$, the buyer can return the fourth good, provided that he bought only it, and receive v back,

$$v = \delta^2\rho^2\varepsilon_1 + \delta^2(1-\rho^2)\left[\mathbb{E}\{v_4\} - \mathbb{E}\{(v_4-\varepsilon_1-\varepsilon_2)^+\}\right] + \delta(1-\rho)\left[\mathbb{E}\{v_3\} - \mathbb{E}\{(v_3+\lambda\varepsilon_2-\alpha_3^{fw})^+\}\right]$$

4. alternatively, at $k = 2$, the buyer can purchase the third good for $\delta y_3^2(\alpha_4^{fw})\hat{\Delta}$;
5. at $k = 3$, given the return, the bundle is priced at $[y_3^3(\alpha_3^{fw} - \lambda\varepsilon 2) + \delta y_4^3(\varepsilon_1 + \varepsilon_2)]\hat{\Delta}$;
6. at $k = 3, 4$, given the return, the forth good is priced at $\delta^{4-k} y_4^k(\varepsilon_1 + \varepsilon_2)\hat{\Delta}$
7. at $k = 3$, given no return, the third is priced at $y_3^3(\alpha_3^{fw})\hat{\Delta}$;

Starting from the options with forwards contract that impelements the max-trhehold mechanism the above steps build an improvement that strictly increases the seller's profit.

7 A theoretical bound

We observed in the previous section that calculating the optimal contract and hence the exact theoretical value of optimal profit for the seller can be a prohibitively complex exercise. Instead the approach we have taken in this paper is to explore a class of reasonable dynamic pricing instruments which are optimal under a restricted set of mechanisms. As described in the introduction, this can be can be referred to as an axiomatic design approach, where the key properties we desired from our mechanism were- backloading, bang-bang, and independent screening, of which only backloading is satisfied by the optimal contract. A natural question to ask then is: What is the (approximate) loss in using our mechanism and thereby giving up on randomization and linked sales? Providing theoretical bounds for the loss is the subject matter of this section.

Define $\mathcal{R}^*$ to be the profit of the seller in the repeated sales problem in continuous time, and recall that $\mathcal{R}^s$, $\mathcal{R}^o$ and $\mathcal{R}^{fw}$ refer to the seller's optimal aggregate profits under static spot prices, option contracts, and options with forwards. By construction:

$$\mathcal{R}^* \geqslant \mathcal{R}^{fw} \geqslant \mathcal{R}^o \geqslant \mathcal{R}^s$$

Moreover, in order to evaluate the performance of the dynamic pricing instruments we bound the optimal profit from above through a calculable expression. A simple way to do that is to solve problem (*) described in Lemma 7 without the (IM): essentially the value of the optimal profit derived under the first-order approach provides an upper bound to $\mathcal{R}^*$, since the objective of optimization problems is exactly the same but the first-order approach considers a subset of constraints.

Our approach in Lemma 11 is slightly more general. We calculate the optimal profit under the assumption that the seller can actually observe if the buyer's type has changed or not, so the seller observes the arrival of the Poisson shock, however, she does not observe what the buyer's type has changed to. This optimization problem has strictly less constraints than the original problem but more than the first-order approach- in fact our new problem is equivalent to the first-order approach under monotone hazard rate.

Lemma 11.

$$\mathcal{R}^* \leqslant \int_0^\infty e^{-rt}\Big[e^{-rt}R^s(\alpha^s) + (1-e^{-rt})\int_0^1 vdF(v)\Big]dt = \overline{\mathcal{R}}$$

where $\alpha^s = arg\max_{\alpha} R^s(\alpha)$ *for* $R^s(\alpha) = \alpha[1-F(\alpha)]$.

Proof. We prove the claim for fixed $\Delta > 0$, then the limiting result follows immediately taking $\limsup_{\Delta\to 0}$ on both sides of the inequality.

By Lemma 6, $Z_k(v^{k-1}, .)$ must be non-decreasing for the allocation satisfying (IM). Consider seller's problem without any constraints, but Z_1 being non-decreasing; that is,

$$\max_{\{q\}} \quad \mathbb{E}\left\{\sum_{k=1}^{\infty}\delta^{k-1}w_k(v^k)q_k(v^k)\right\}$$

subject to Z_1- non-decreasing.

In the proof of Proposition 7, we showed that seller's profit can be written as

$$\int_0^1 \left[\Big(vf(v) + F(v) - 1\Big) Z_1(v) + \delta(1-\rho) vf(v) \sum_{k=1}^{\infty} \delta^{k-1} \mathbb{E}\left\{ Z_{k+1}(v^k, v) \right\} \right] dv$$

Integrate by parts the first term of seller's profit,

$$\int_0^1 \Big(vf(v) + F(v) - 1\Big) Z_1(v) dv = - \int_0^1 Z_1(u) d\left(\int_u^1 \Big(vf(v) - [1 - F(v)]\Big) dv \right) =$$

$$= \int_0^1 u[1 - F(u)] dZ_1(u) \leqslant \frac{R^s(\alpha^s)}{1-\delta\rho} \hat{\Delta}$$

The last inequality holds, because $Z_1(\cdot)$ is non-decreasing. The second term is clearly bounded from above by $\frac{1}{1-\delta} \frac{\delta(1-\rho)}{1-\delta\rho} \int_0^1 v dF(v) \hat{\Delta}$. It follows that seller's profit is at most

$$\frac{1}{1-\delta\rho} \left(R^s(\alpha^s) + \frac{\delta(1-\rho)}{1-\delta} \int_0^1 v dF(v) \right) \hat{\Delta} = \sum_{k=1}^{\infty} \delta^{k-1} \left[\rho^{k-1} R^s(a^s) + (1 - \rho^{k-1}) \int_0^1 v dF(v) \right] \hat{\Delta}$$

Take $\Delta \to 0$ to obtain $\overline{\mathcal{R}}$. □

Finally, we can provide a theoretical bound for the seller's profit which is uniform across all distributions F.

Proposition 9. *Consider the repeated sales problem in continuous time. Let $\mathcal{R}^*$ be the optimal profit of the seller, and $\mathcal{R}^s$, $\mathcal{R}^o$ and $\mathcal{R}^{fw}$ respectively be the profits from spot pricing, European option and the American option one forwards. Then we have the following set of bounds for the optimal profit:*

$$1 \geqslant \frac{\mathcal{R}^{fw}}{\mathcal{R}^*} \geqslant \frac{\mathcal{R}^{fw}}{\overline{\mathcal{R}}} \geqslant \frac{\mathcal{R}^o}{\overline{\mathcal{R}}} \geqslant \mathcal{L}$$

where $\mathcal{L}$ is a function of $\frac{r}{\lambda}$ and provides a uniform lower bound for $\frac{\mathcal{R}^{op}}{\overline{\mathcal{R}}}$, across all distributions F. Moreover, in the limit for $\frac{r}{\lambda}$ the bound is tight:

$$\lim_{\frac{r}{\lambda} \to 0} \mathcal{L} = \lim_{\frac{r}{\lambda} \to \infty} \mathcal{L} = 1$$

The exact mathematical expression for $\mathcal{L}$ is provided in the appendix. In Figure 4, $\mathcal{L}$ is plotted as a function of $\frac{r}{\lambda}$; it can be seen that its value is never below 70%.

Now, $\mathcal{L}$ is a very demanding bound in that it holds for all possible distributions F. One could ask a simpler, but less general questions: how the simple option contract and options with forwards contract perform for specific distributions? We can easily answer this question since $\frac{\mathcal{R}^{fw}}{\overline{\mathcal{R}}}$ and $\frac{\mathcal{R}^{op}}{\overline{\mathcal{R}}}$ are closed form expressions that provides lower bounds for $\frac{\mathcal{R}^{fw}}{\mathcal{R}^*}$, viz. the fraction of optimal profit attained by our dynamic pricing instruments. We present the two theoretical lower bounds, $\frac{\mathcal{R}^{fw}}{\overline{\mathcal{R}}}$ and $\frac{\mathcal{R}^{op}}{\overline{\mathcal{R}}}$, as functions of $\frac{r}{\lambda}$ for the power distribution in Figure 5.

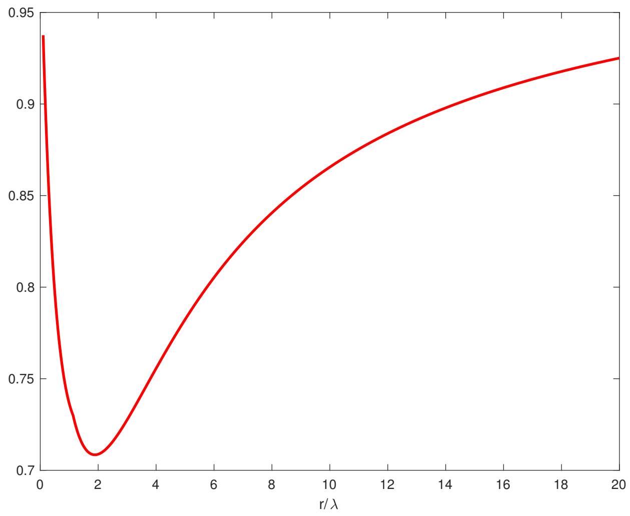
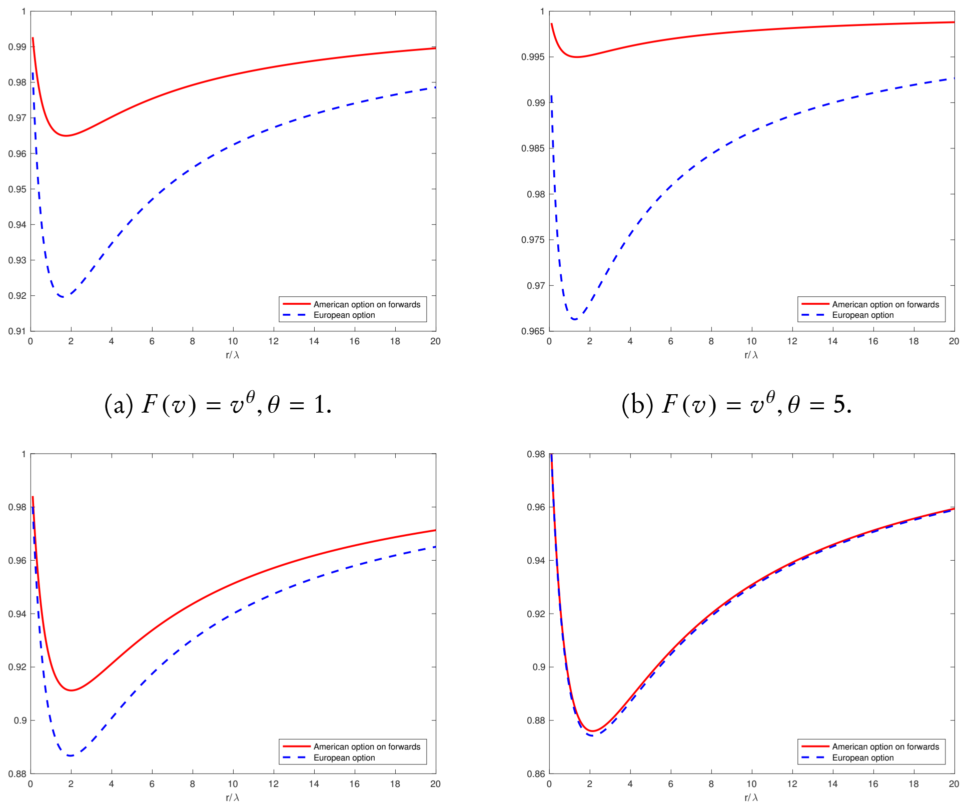


Figure 4: $\mathcal{L}$ as a function of r/λ.



(a) $F(v) = v^{\theta}, \theta = 1$.

(b) $F(v) = v^{\theta}, \theta = 5$.

(c) $F(v) = v^{\theta}, \theta = 0.2$.

(d) $F(v) = v^{\theta}, \theta = 0.001$.

Figure 5: red- ratio of profit from the American option on forwards $\mathcal{R}^*$, blue- ratio of profit from the European to $\mathcal{R}^*$ as functions of r/λ.

8 Extension to multiple players: repeated auctions

In this section, we extend the pricing strategies introduced above to the setting of multiple buyers. As before, a seller wants to repeatedly sell a non-durable good. Time is continuous and infinite. Now, there are $i = 1, \ldots, N$ ex ante identical buyers who share the same discount rate as the seller, r. Buyers' valuations are independent across players, but correlated across time. Specifically, we assume that buyer i's valuations $\{V_{it}\}$ follows a stationary pseudo-renewal process as defined in Section 3.

In what follows, we first consider extensions of three pricing strategy to sell a single good at a fixed date t. Then, we look at performance of these pricing strategies as compared to a theoretical upper bound on seller's revenue.

8.1 Spot auction.

Suppose that the buyer can purchase the good at a second price auction at date t. As an instrument of exclusion, the seller uses an entry fee, say $\alpha F^{N-1}(\alpha)dt$. We look at a symmetric equilibrium where every bidder bids one's valuation and participates if and only if $V_{it} \geqslant \alpha$.

Clearly, if bidder i participates, then he wins the auction whenever $V_{it} \geqslant \max_{j \neq i} \{V_{jt} \mathbb{1}\{V_{jt} \geqslant \alpha\}\}$ and pays $\max_{j \neq i} \{V_{jt} \mathbb{1}\{V_{jt} \geqslant \alpha\}\}$. It follows that bidder i's payoff from participation is given by $e^{-rt}\pi(V_{it})dt$,

$$\pi(v) = \begin{cases} vF^{N-1}(\alpha) & \text{for} \quad v < \alpha \\ vF^{N-1}(V_{it}) - F^{N-1}(\alpha) - \displaystyle\int_{\alpha}^{v} s dF^{N-1}(s) & \text{for} \quad v \geqslant \alpha \end{cases}$$

It is easy to see that bidder i's optimal strategy is to participate if and only if $V_{it} \geq \alpha$.

Then, seller's revenue as a function of α can be computed as $e^{-rt}R^s(\alpha)dt$ where

$$\begin{aligned} R^s(\alpha) &= N \int_{\alpha}^{1} \underbrace{\left[\alpha F^{N-1}(\alpha) + \int_{\alpha}^{v} x dF^{N-1}(x) \right]}_{\text{bidder } i\text{'s expected payment: } V_{it} = v \geqslant \alpha} f(v)dv \\ &= \int_{\alpha}^{1} \left[v - \frac{1 - F(v)}{f(v)} \right] dF^N(v) \end{aligned}$$

Observe that the optimal threshold, α^s, is independent of a number of buyers.

8.2 European option

Suppose, as before, that the buyer can purchase the good at a second price auction at date t. At $t = 0$, the seller sells an option which permits the buyer to decide whether or not to pay $\alpha F^{N-1}(\alpha)dt$ at t to enter the auction. We consider a symmetric equilibrium every buyer bids one's valuation, enters the auction if $V_{it} \geqslant \alpha$ and always pays the option premium, say $e^{-rt}\mathcal{M}_t(\alpha)dt$.

By the previous argument, entering for $V_{it} \geqslant \alpha$ is optimal. Bidder i's expected payoff at $t = 0$ is $e^{-rt}\mathbb{E}\{[\pi(V_{it}) - \pi(\alpha)]^{+}|V_{i0}\}dt$ where

$$\mathbb{E}\{[\pi(V_{it}) - \pi(\alpha)]^{+}|V_{i0}\} = e^{-\lambda t}[\pi(V_{i0}) - \pi(\alpha)]^{+} + (1 - e^{-\lambda t})\mathbb{E}\{[\pi(V_{it}) - \pi(\alpha)]^{+}\}$$

The seller can charge at most $e^{-rt}\mathbb{E}\{[\pi(V_{it}) - \pi(\alpha)]^{+}\}dt$ to induce the buyer to always participate; that is,

$$\mathcal{M}_t(\alpha) = \mathbb{E}\{[\pi(V_{it}) - \pi(\alpha)]^{+}\} = \int_{\alpha}^{1}\int_{\alpha}^{v} F^{N-1}(x)dxdv = \int_{\alpha}^{1}[1 - F(v)]F^{N-1}(v)dv$$

Conclude that seller's revenue as a function of α can be computed as $e^{-rt}R^{o}(\alpha)dt$ where

$$\begin{aligned}\mathcal{R}_t^{o}(\alpha) &= \mathcal{M}_t(\alpha) + R^{s}(\alpha)\\ &= e^{-\lambda t}\int_{\alpha}^{1}\Big[v - \frac{1 - F(v)}{f(v)}\Big]dF^{N}(v) + (1 - e^{-\lambda t})\int_{\alpha}^{1} vdF(v)(v)\end{aligned}$$

The optimal threshold, α^{o}, is independent of a number of buyers.

8.3 Optioning forwards

The seller sells the good at t by the means of second price auction. To participate the buyer needs to pay an entry fee. In particular, the buyer can choose to commit at date $s \leq t$ to pay the entry fee $y_t^{s}(\alpha)dt$ defined by

$$y_t^{s}(\alpha) = e^{-\lambda(t-s)}\pi(\alpha) + (1 - e^{-\lambda(t-s)})\Big(\mathbb{E}\{\pi(V_{it})\} - \mathbb{E}\{[\pi(V_{it}) - \pi(\alpha)]^{+}\}\Big)$$

Moreover, the to be eligible the buyer needs to pay the fee $\mathcal{M}_t(\alpha)$ at $t = 0$ at $t = 0$.

The same argument as in the proofs of Lemma 8 can be used to establish that there exists a symmetric equilibrium where each buyer participates and pays the entry fee at date s if and only if $V_{is} \geq y_t^{s}(\alpha)$. Similarly to Lemma 9, seller's profit can be written as $e^{-rt}R_t^{fw}(\alpha)dt$,

$$\begin{aligned}R_t^{fw}(\alpha) = {}& e^{-\lambda t}\int_{\alpha}^{1}\Big[v - \frac{1 - F(v)}{f(v)}\Big]dF^{N}(v) + (1 - e^{-\lambda t})\int_{\alpha}^{1} vdF^{N}(v)\\ &+ (1 - e^{-\lambda[1-F(\alpha)]t})\int_{0}^{\alpha} vdF^{N}(v)\end{aligned}$$

As before, let α_t^{fw} be the optimal threshold.

Next, we show that the optimal threshold is increasing in N when the hazard rate is monotone. The first-order condition for α_t^{fw} can be written as

$$\left[\frac{1 - F(\alpha)}{f(\alpha)} - e^{\lambda F(\alpha)t}\alpha\right] = \lambda t f(\alpha)e^{\lambda F(\alpha)t}\int_{0}^{\alpha}\frac{v[F^{N}(v)]'}{[F^{N}(\alpha)]'}dv$$

Since the hazard rate is monotone, the left-hand side is clearly a decreasing function. On the other hand, the right hand is an increasing function. Observe that the ratio $\frac{[F^N(v)]'}{[F^N(\alpha)]'} = \frac{f(v)}{f(\alpha)}\left[\frac{F(v)}{F(\alpha)}\right]^{N-1}$ is decreasing in N, therefore α_t^{fw} is increasing in N. In particular, as $N \to 0$, α_t^{fw} converges to the root of

$$\frac{1-F(\alpha)}{f(\alpha)} - e^{\lambda F(\alpha)t}\alpha = 0$$

For $t \neq 0$, $\lim_{N\to\infty} \alpha_t^{fw} < \alpha^s$, thus $\alpha_t^o < \alpha_t^{fw} < \alpha^s$ for all N.

8.4 A theoretical bound

We extend the theoretical bound of Section 7 to the setting with multiple players. As before, we consider the discrete setting where each buyer gets to observe his valuations at discrete time points $\{(k-1)\Delta : k = 1, 2, ..., \infty\}$.

Denote by $q_{ik}^e(v_i^k)$ expected probability that bidder i gets the k-good when the history is v_i^k. As before, define the average allocations along "persistent" path $Z_{ik}(v_i^k)$.

Seller's profit can be written as

$$\max_{\{q\}} \quad \sum_{i=1}^{N} \mathbb{E}\left\{\sum_{k=1}^{\infty} \delta^{k-1} w_k(v_i^k) q_{ik}^e(v_i^k)\right\} \hat{\Delta}$$

where

$$w_k(v_i^k) = \begin{cases} v_{i1} - \frac{1-F(v_{i1})}{f(v_{i1})} & \text{if} \quad v_{i1} = ... = v_{ik} \\ v_{ik} & \text{otherwise} \end{cases}$$

Let $\psi(v)$ be the ironed version of $v - \frac{1-F(v)}{f(v)}$; that is, $\psi(v) = -\frac{1}{f(v)}\frac{\partial}{\partial v}J(v)$ where $J(v)$ is the lowest concave majorant of $v[1-F(v)]$.

We drop all incentive constraints, but the requirement that Z_{i1}. Using ideas from Myerson [1986], the optimal solution to the relaxation can be shown to allocate based on ironed virtual valuations:

$$\hat{w}_k(v_i^k) = \begin{cases} \psi(v_{i1}) & \text{if} \quad v_{i1} = ... = v_{ik} \\ v_{ik} & \text{otherwise} \end{cases}$$

Consider a distribution of $\hat{w}_k(v_i^k)$; that is, $H_k(w) = \rho^{k-1}F[\psi^{-1}(v)] + (1-\rho^{k-1})F(w)$. Then, seller's profit can be written succinctly as

$$\sum_{k=1}^{\infty} \delta^{k-1} \int_0^{\infty} w d\left[\rho^{k-1}F[\psi^{-1}(v)] + (1-\rho^{k-1})F(w)\right]^N \hat{\Delta}$$

As $\Delta \to 0$, seller's profit clearly converges to $\overline{\mathcal{R}}$

$$\overline{\mathcal{R}} = \int_0^{\infty} e^{-rt} \int_0^{\infty} w d\left[e^{-\lambda t}F[\psi^{-1}(v)] + (1-e^{-\lambda t})F(w)\right]^N dt$$

We illustrate performance of our selling strategies with respect to $\overline{\mathcal{R}}$ as a function of N.

The figure shows that both strategies achieve a progressively higher fraction of revenue as N

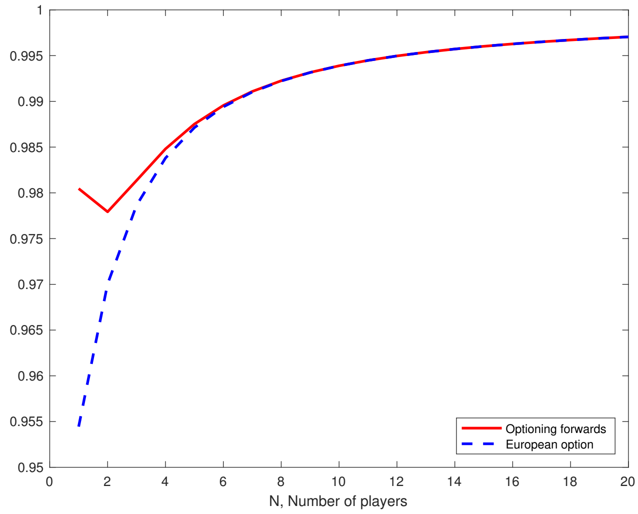


Figure 6: red- ratio of profit from the American option on forwards to $\overline{\mathcal{R}}$, blue- ration of profit from the European option to $\overline{\mathcal{R}}$; $F(x) = x$ and $r = \lambda = 1$.

increases. This is intuitive, because with a large number of buyers, even a spot second price auction guarantees to the seller a profit arbitrarily close to the highest possible valuation which is 1.

9 Final remarks

At the core this paper is about a dynamic model of pricing where the seller can endogenously discriminate between different buyer types by instrumenting on the timing of purchase. Price discrimination has a rich history in economics, starting at least from Pigou [1920].[16] Its invocation as an instrument of accessing consumers within and across various markets in now almost axiomatic. On the wide prevalence of price discrimination, Varian [1980] famously wrote: âĂIJEconomists have belatedly come to recognize that the 'law of one price' is no law at all. Most retail markets are instead characterized by a rather large degree of price dispersion."

While the "cross-section" of price discrimination gained much prominence through the study of bundling (see for example Adams and Yellen [1976] and McAfee et al. [1989]), the "time series" of it came to be studied later with a rise in interest in dynamic contracts and dynamic mechanism design. Our paper is a contribution to this rapidly growing body of work, reviewed masterfully in recent surveys by Krähmer and Strausz [2015a] and Bergemann and

[16]Interestingly, the classic political economy treatise from approximately the 2nd century BC called Arthshahtra (literally meaning the *study of money*), typically attributed to Kautilya, talks in some detail about offering a menu of loans varied by interest rates and maturity structure so as to discriminate potential consumers on the basis of their demand, risk, and capacity to repay (see Kautilya [1992]). In a forthcoming book, Rajan [2019] discusses in-depth the prohibition of the debt contract in the West in medieval times and its rise post Renaissance, providing at the time the intellectual and moral underpinnings of the use of market instruments, price discrimination as we know it today being one its modern renditions.

Välimäki [2018]. We discuss a small subset of papers here which speak directly to our results.

Courty and Li [2000] studied the sequential screening model of price discrimination which is akin to the two period model presented in Section 2 with the allocation in the first period restricted to be zero. They explain the use of evolving information as an instrument of price discrimination and discuss its applicability to airline ticket booking. Towards this end, they use the classical mechanism design approach. Pavan, Segal, and Toikka [2014] unified this approach by providing a necessary conditions for incentive compatibility in the form of a general dynamic envelope theorem, and showed that the sufficient part can be written as the integral monotonicity condition, both of which we employ here in the paper. We build on this line of work in many ways.

First, in the two period model presented in Section 2 we explicitly solve for the optimal dynamic pricing mechanism, which is different from the implementation used in Courty and Li [2000], and then show that this indirect mechanism is actually optimal. This involves two separate non-trivial optimization problems that are then shown to be equivalent in allocations and payoffs. The techniques used here also very different because the first-order approach does not work here, and we directly prove novel results for the two period model- backloading and independent screening.

Second, our model tackles both sequential screening and repeated sales for general time horizons. While repeated sales have been modelled elegantly in Battaglini [2005], Boleslavsky and Said [2013], and Bergemann and Strack [2015]; the analysis has mostly been restricted to either two types or AR(1) process where evolving information can be reconducted to an orthogonalized information structure. In such a scenario Esö and Szentes [2017] argue for the irrelevance of dynamics, but this irrelevance of dynamics is not extendable to models where the optimal contract under the orthogonalized information structure is not incentive compatible, which is the case in our setting. Dynamics are alive and kicking in our setting, which is explained simply by the fact that in the option with forwards contract, each type decides to buy into the option contract and is then discriminated endogenously over time as the values evolve. We cannot decide what the optimal distortions would be on observing the first period type- we view this as an important missing feature of modeling dynamic price discrimination so far.

Third, the results here, to the best of our knowledge, are the first of its kind to throw light on the general structure of dynamic mechanisms when global incentive contracts bind- they involve backloading, randomization and specific geometry of linking across time, of which we relax the last two criterions, to nonetheless produce a globally incentive compatible contract that is approximately optimal.[17] As mentioned in the introduction, this has been a blind spot in the literature on dynamic pricing and dynamic mechanism design more generally. Moreover, the extension to repeated auctions provides some promise of tackling economically relevant multi-

[17] Battaglini and Lamba [2018] show that in a general Markovian dynamic mechanism design model global incentive constrains bind generically for frequent interactions and/or high types' persistence. Ours is a more specialized set up in that we assume types evolve according to a renewal Markov model and therefore global incentive constraints always bind for any interior value of the persistence parameter ρ. They also solve a three types-two period model as a first analysis of the optimal dynamic screening contract when global incentive constraints bind. On that front the analysis here in much more general with economically meaningful mechanisms that either achieve or approximate the optimum.

agent mechanism design problems with evolving information beyond the efficient allocation benchmarks pursued so far.[18]

Fourth, so far the economic implications of imposing stronger incentive compatibility requirements on the set of allocation rules, such as strong monotonicity or single-crossing have not been explored. These conditions can be imposed ex ante on the set of mechanisms to pursue simplicity, or they have typically been invoked to check incentive compatibility (often numerically) ex post. We show that imposing either of those conditions amounts to (a) no randomization at the optimum, and (b) simplification of history dependence in the form of independent screening of each sale.

Fifth, our use of options as an instrument of implementing dynamic screening mechanisms builds on Esö and Szentes [2007] and Boleslavsky and Said [2013].[19] They use a different version of the simple option contract or the European call option described in this paper. We give option pricing a greater role by adding forwards so that each good can be sold at an earlier date in a very general manner. To be precise, Esö and Szentes [2007] study a two period problem and use a menu of European call options to implement the optimal allocation. In the first period, buyer's types are simply screened by their choice of option from the menu. It turns out in our setting a simpler menu is sufficient- the American option on forwards which screen buyers' types only using timing of purchases. Importantly, the implementation of Esö and Szentes [2007] can not be easily ported to a model with more than two periods, whereas our dynamic pricing strategy has a clear generalization to any time horizon. Boleslavsky and Said [2013] study a model where the buyer's types are i.i.d conditional on the first period information, akin to the orthogonalized information structure. Thus, the first period and current type are sufficient statistics for allocative distortions and can be implemented by a menu of European call options. Our pricing strategy of using forwards is shown to be useful even when this sufficient statistic is not available. In addition to these improvements, option on forwards also throws light on the fact that going all the way to the global optimum will involve enrichment of the contract space to include buybacks of the goods previously sold as a forward.

Sixth, we can divide the economic content of our analysis into short-term and long-term predictions. For the latter, Battaglini [2005] showed that in the two-types infinite horizon model efficiency is achieved along every history of types' realizations. There is a folk sense in which this result is robust.[20] The thresholds for exclusion in our candidate mechanism converge to zero along every history (for any interior level of discounting and persistence), and we conjecture that the same is true for the optimal mechanism, pointing towards a confirmation of the folk result. But, we view the short-run properties as being more salient in our understanding of dynamic price discrimination, since in the short-run the assumption of commitment seems

[18]An exception to this is the recent paper Mirrokni et al. [2018], which looks at revenue maximizing dynamic mechanisms with multiple agents where the seller is only allowed to use allocation and pricing rules at each period which do not depend on the type distributions in the future periods.

[19]Another elegant model of intertemporal price discrimination is presented in Deb [2014]. It studies the sequential screening version of our model where there is only one Poisson shock which changes the buyer's type. A partial characterization of the optimum is provided with an intuitive implementation of introductory pricing.

[20]In fact Garrett, Pavan, and Toikka [2018] argue the same by showing that on "average" optimal distortions in a monopolistic screening model with Markov types converge to zero in the long-run.

more plausible. In that realm, our candidate mechanism posits a precise pricing and allocation rule even when the first-order approach cannot be applied. It is important to note that the option with forwards contract will implement an incentive compatible allocation rule for any Markov evolution of types- the renewal model allows us to precisely understand its mapping into the world of unrestricted class of all direct mechanisms.

Going forward it would worthwhile to explore the qualitative properties of our candidate mechanism, or a close cousin, for other (or more general) Markov evolution of types. While seeking full optimality is a welcome ambitious goal, we think emphasizing an axiomatic approach to design and searching for incentive compatible mechanisms that satisfy certain desirable properties and ensure a minimal loss is equally attractive. Moreover, a large part of the price discrimination exercise here is driven by a relatively permissible notion of feasibility in the form of a forward looking individual rationality constraint. It allows for the agent to have the cash to pay for an option premium at the begging. Krähmer and Strausz [2015b] point out that the two period sequential screening problem essentially reduces to a static one if ex post individual rationality is imposed. Krasikov and Lamba [2018] and Grillo and Ortner [2018] have explored different versions of the repeated sales problem with the ex post individual rationality (or limited liability) constraint. Understanding further the implications of such financial constraints on price discrimination is also an interesting question for future work.

10 Appendix

The appendix is divided into three sections. We start with a generalized version of the two-period example covered in Section xx, which includes all the missing steps and proofs from main text. Second we provide proofs for the results that were stated in the main text. And third, we state and prove supplementary results that were not formally stated in the main text.

Generalized two period example

Consider the setting with two discrete periods $k = 1, 2$. Suppose that buyer's valuations satisfy $v_1 \sim F$, and

$$\mathbb{P}(v_2 \leq v|v_1) = \rho \mathbb{1}\{v \geqslant v_1\} + (1-\rho)F(v_2)$$

where F is continuous distribution supported on $[0, 1]$ with density f.

Proof of Lemmata 1 and 2. For $k = 2$, incentive compatibility requires

$$u_2(v^2) = \max_{v} \quad v_2 q_2(v_1, v) - p_2(v_1, v)$$

It follows that $u_2(v_1, .)$ is the maximum of linear functions, therefore it is convex and a.e. differentiable with $\frac{\partial}{\partial v_2} u_2(v^2) = q_2(v^2)$.

Using the envelope condition, rewrite incentive compatibility as

$$u_2(v^2) - u_2(v_1, v) = \int_{v}^{v_2} q_2(v_1, x)dx \geqslant (v_2 - v)q_2(v_1, v)$$

Clearly, (ii) holds; that is, monotonicity of $q_2(v_1, .)$ is sufficient for incentive compatibility.

Now, consider $k = 1$. Incentive compatibility demands

$$\begin{aligned} U_1(v_1) &= \max_v \quad v_1 q_1(v) - p_1(v) + \mathbb{E}\{u_2(v, v_2)|v_1\} \\ &= \max_v \quad v_1 q_1(v) - p_1(v) + \left[\rho u_2(v, v_1) + (1-\rho)\mathbb{E}\{u_2(v, v_2)\}\right] \end{aligned}$$

We showed that $u_2(v, \cdot)$ is convex, therefore U_1 is the maximum of convex functions. It follows that U_1 is convex, a.e. differentiable with $\frac{\partial}{\partial v_1} U_1(v_1) = q_1(v_1) + \rho \frac{\partial}{\partial v_2} u_2(v_1, v_1) = q_1(v_1) + \rho q_2(v_1, v_1) = Z_1(v_1)$.

Using the envelope condition, rewrite incentive compatibility as

$$U_1(v_1) - U_1(v) = \int_v^{v_1} Z_1(x) dx \geqslant \int_v^{v_1} [q_1(v) + \rho q_2(v, x)] dx$$

Unfortunately, monotonicity of Z_1 alone is no longer sufficient for incentive compatibility. Condition (iv) though is clearly sufficient. □

Proof of Lemmata 3 and 4. Take any allocation $\{\mathbf{q}\}$ satisfying (ii) and (iv). First, we solve for seller's profit using (i) and (iii):

$$\mathbb{E}\{v_1 q_1(v_1) + \rho v_2 q_2(v^2) - U_1(v_1)\} = \int_0^1 \Big([v f(v) + F(v) - 1] Z_1(v) + (1-\rho) v f(v) \mathbb{E}\{q_2(v_1, v)\}\Big) dv$$

Next, we construct a new allocation which yields a higher profit. Define $\{\hat{\mathbf{q}}\}$ by $\hat{q}_1(v_1) = [Z_1(v_1) - \rho)]^+$ and

$$\hat{q}_2(v^2) = \begin{cases} 1 & \text{whenever} \quad Z_1(v_1) \geqslant \rho \\ \min\{1, q_2(v^2) + q_1(v_1)/\rho\} & \text{otherwise} \end{cases}$$

Observe that $\hat{q}_1(v_1) + \rho \hat{q}_2(v_1, v_1) = Z_1(v_1)$, but $\hat{q}_2(v_2) \geqslant q_2(v_2)$. Clearly, $\{\hat{\mathbf{q}}\}$ gives a higher profit than $\{\mathbf{q}\}$.

By construction, $\{\hat{\mathbf{q}}\}$ is backloaded. Now, we show Lemma 4 which will imply that $\{\hat{\mathbf{q}}\}$ is incentive compatible.

Take any backloaded allocation $\{\tilde{\mathbf{q}}\}$ satisfying (ii) and (iv). Since Z_1 is non-decreasing, there exists a such that $Z_1(v_1) \geqslant \rho$ if and only if $v_1 \geqslant a$. By backloading, $\tilde{q}_1(v_1) = [Z_1(v_1) - \rho]^+$, which is non-decreasing.

For $v_1 \geqslant a$,

$$\begin{aligned} \int_{v_1}^{v} \tilde{q}_2(v_1, x) dx = (v - v_1) = \int_{v_1}^{v} \tilde{q}_2(x, x) dx \quad \forall v > v_1 \\ \int_{v_1}^{v} \tilde{q}_2(v_1, x) dx = (v - v_1) \leqslant \int_{v_1}^{v} \tilde{q}_2(x, x) dx \quad \forall v < v_1 \end{aligned}$$

For $v_1 < a$, $\tilde{q}_1(v_1) = 0$, thus

$$\rho \int_{v_1}^{v} \tilde{q}_2(v_1, x)dx \leqslant \int_{v_1}^{v} \min\{\tilde{Z}_1(x), \rho\}dx = \rho \int_{v_1}^{v} \tilde{q}_2(x, x)dx \quad \forall v > v_1$$
$$\rho \int_{v_1}^{v} \tilde{q}_2(v_1, x)dx \leqslant \int_{v_1}^{v} \tilde{Z}_1(x)dx = \rho \int_{v_1}^{v} \tilde{q}_2(x, x)dx \quad \forall v < v_1$$

□

Proof of Proposition 1. By Lemma 4, we can look at two independent screening problems.

For $k = 1$, the argument is standard. Seller's revenue is given by

$$\int_0^1 [vf(v) + F(v) - 1]q_1(v)dv$$

The objective is linear, therefore it is maximized for $q_1(v_1) = \mathbb{1}\{v_1 \geqslant a\}$ for some a. Write seller's revenue as a function of a; that is, $a[1 - F(a)]$. For the uniform prior, the revenue is maximized at $a_1^{fw} = 1/2$.

For $k = 2$, the argument is a little bit more involved. First, we consider the deterministic optimum; that is, q_2 is restricted to take values in $\{0, 1\}$. By Lemmata 2 and 4, $q_2(v_1, v_1)$ must be a non-decreasing function. Therefore, there exists a such that $q_2(v_1, v_1) = 1\{v_1 \geqslant a\}$. Notice that $v_1, v < a$, $\int_{v_1}^{v} q_2(x, x)dx = 0$, therefore $q_2(v^2) = 0$ for all v^2 such that $\max\{v_1, v_2\} < a$. In the proof of Lemmata 3 and 4, we showed that seller's profit can be written as

$$\int_0^1 \Big(\rho[vf(v) + F(v) - 1]q_2(v, v) + (1 - \rho)vf(v)\mathbb{E}\{q_2(v_1, v)\}\Big)dv$$

Define $\hat{q}_2(v^2) = \mathbb{1}\{\max\{v_1, v_2\} \geqslant a\}$. Clearly, $\hat{q}_2$ yields a higher profit as $\hat{q}_2(v_1, v_1) = q_2(v_1, v_1)$ and $\hat{q}_2(v^2) \geq q_2(v^2)$. Next, we write seller's profit as a function of a,

$$\rho \int_a^1 \left[f(v) + \frac{1-\rho}{\rho}\left(\int_0^v xf(x)dxF(v)\right)' + F(v) - 1\right]dv$$

For the uniform distribution, the expression for seller's profit is $\rho \displaystyle\int_a^1 \left[(2v - 1) + \frac{1-\rho}{\rho}\frac{3v^2}{2}\right]dv$. Clearly, the integrand is increasing, thus the first-order condition is sufficient. It is easily checked that there exists unique a_2^{fw} such that

$$(2a_2^{fw} - 1) + \frac{1-\rho}{\rho}\frac{3[a_2^{fw}]^2}{2} = 0$$

Now, we verify that the optimum is deterministic by constructing appropriate dual variables. As standard, we suppose that $v \mapsto \frac{1-F(v)}{f(v)}$ is non-decreasing. Then, $v \mapsto vf(v)$ is non-decreasing on $[0, \alpha^s]$. To see it formally, take $0 < \beta < \alpha \leqslant \alpha^s$, then $\alpha[1 - F(\alpha)] > \beta[1 - F(\beta)]$, because $v[1 - F(v)]$ is strictly increasing on $[0, \alpha^s)$. Since the inverse hazard rate is non-decreasing,

$$\frac{\alpha}{\beta} > \frac{1-F(\beta)}{1-F(\alpha)} \geq \frac{f(\beta)}{f(\alpha)}$$

which implies that $\alpha f(\alpha) > \beta f(\beta)$.

Consider the relaxation where we require only two sets of constraints: $q_2(v^2) \leqslant q_2(v_1, v_1)$ for $v_2 \leqslant v_1$, and

$$\int_{v_1}^{1} q_2(v_1, x)dx \leqslant \int_{v_1}^{1} q_2(x, x)dx$$

Clearly, the former constraint shall bind, so we can solve for $q_2(v^2)$ with $v_2 < v_1$ as $q_2(v^2) = q_2(v_1, v_1)$. Let $\eta(v_1) = (1-\rho)f(v_1)a_2^{fw} f(a_2^{fw})$ be the dual variable attached to the remaining constraint. The Lagrangian is as it follows:

- coefficient for $q_2(v^2)$ with $v_2 > v_1$ is

$$(1-\rho)f(v_1)[v_2 f(v_2) - a_2^{fw} f(a_2^{fw})]$$

- coefficient for $q_2(v_1, v_1)$ is

$$\rho[vf(v) + F(v) - 1] + (1-\rho)\int_0^v xf(x)dxf(v) + (1-\rho)a_2^{fw} f(a_2^{fw})F(v)$$

We claim that the first coefficient is positive if and only of $v_2 \geqslant a_2^{fw}$, because $a_2^{fw} < a^s$ and $v \mapsto vf(v)$ is non-decreasing on $[0, a^s]$. By the same argument, the second coefficient is positive if and only of $v_2 \geqslant a_2^{fw}$. □

Omitted proofs

Proof of Lemma 6. Consider some v^k an take $v > v_k$. Use the identity $q_k(v^k)\hat{\Delta} = Z_k(v^k) - \delta\rho Z_{k+1}(v^k, v_k)$ to rewrite (IM) as

$$\int_{v_k}^{v} \left[Z_k(v^{k-1}, x) - Z_k(v^k)\right] dx \geqslant \delta\rho \int_{v_k}^{v} \left[Z_{k+1}(v^k, x) - Z_{k+1}(v^k, v_k)\right] dx$$

Iterate forward to obtain that

$$\int_{v_k}^{v} \left[Z_k(v^{k-1}, x) - Z_k(v^k)\right] du \geqslant (\delta\rho)^m \int_{v_k}^{v} \left[Z_{k+m}(v^{k-1}, \underbrace{v_k, ..., v_k}_{m \text{ times}}, x) - Z_{k+m}(v^{k-1}, \underbrace{v_k, ..., v_k}_{m+1 \text{ times}})\right] dx$$

Since allocations are in $[0, 1]$, $Z_{k+m}(v^{k+m}) - Z_{k+m}(\tilde{v}^{k+m})$ is uniformly bounded from above by $\frac{\hat{\Delta}}{1-\delta\rho}$. Take $m \to \infty$ and conclude that

$$\int_{v_k}^{v} Z_k(v^{k-1}, x)dx \geqslant [v - v_k]Z_k(v^k)$$

By the similar reasoning,

$$\int_{v}^{v_k} Z_k(v^{k-1}, x)dx \geqslant [v - v']Z_k(v^{k-1}, v)$$

Add up these two inequalities to get $[v - v_k][Z_k(v^k) - Z_k(v^{k-1}, v)] \leq 0$; that is, $Z_k(v^{k-1}, v) \geqslant Z_k(v^k)$. □

Proof of Lemma 9. Let $\mathcal{G}_t$ be a distribution of $\max\limits_{s \leqslant t} V_s$; that is,

$$\mathcal{G}_t(x) = F(x)e^{-\lambda[1-F(x)]t}$$

The good is sold by $\tau \neq 0$ if and only if $\max\limits_{s \leqslant \tau} V_\tau \geqslant \alpha$ which happens with probability $1 - \mathcal{G}_\tau(\alpha)$. It follows that seller's profit, $\mathcal{R}_t^{fw}(\alpha)$, can be written as

$$\mathcal{R}_k^{fw}(\alpha) = \mathcal{M}_t(\alpha) + \int_0^t y_t^s(\alpha)d[1 - \mathcal{G}_s(\alpha)]$$

According to Equation 5,

$$\mathcal{M}_t(\alpha) = (1 - e^{-\lambda t})\mathbb{E}\{(V_t - \alpha)^+\} = (1 - e^{-\lambda t}) \left[\int_\alpha^1 vf(v)dv - \alpha[1 - F(\alpha)]\right]$$

According to Equation 7,

$$\begin{aligned} y_t^s(\alpha) &= e^{-\lambda(t-s)}\alpha + (1 - e^{-\lambda(t-s)})\left[\mathbb{E}\{V_t\} - \mathbb{E}\{(V_t - \alpha)^+\}\right] \\ &= e^{-\lambda(t-s)}\alpha + (1 - e^{-\lambda(t-s)})\left[\alpha[1 - F(\alpha)] + \int_0^\alpha vf(v)dv\right] \end{aligned}$$

Integrate $e^{-\lambda s}$ with respect to $1 - \mathcal{G}_s(\alpha)$ to obtain

$$\int_0^t e^{\lambda s}d[1 - \mathcal{G}_s(\alpha)] = 1 - F(\alpha) + [1 - F(\alpha]\int_0^t de^{\lambda F(\alpha)s} = [1 - F(\alpha)]e^{\lambda F(\alpha)t}$$

Therefore,

$$\int_0^t (1 - e^{-\lambda(t-s)})d[1 - \mathcal{G}_s(\alpha)] = G_t(\alpha) - e^{-\lambda t}\int_0^t e^{\lambda s}d[1 - \mathcal{G}_s(\alpha)] = 1 - e^{-\lambda[1-F(\alpha)]t}$$

Taking all things together,

$$\mathcal{R}_t^{fw}(\alpha) = e^{-\lambda t}\alpha[1 - F(\alpha)] + (1 - e^{-\lambda t})\int_\alpha^1 vdF(v) + (1 - e^{-\lambda[1-F(\alpha)]t})\int_0^\alpha vdF(v)$$

□

Proof of Proposition 2.

Part (a). For $\lambda t = 0$, $\mathcal{R}_t^o(\alpha) = \mathcal{R}_t^{fw}(\alpha) = \mathcal{R}^s(\alpha)$ for all α.

Part (b). Clearly, α_t^o and α_t^{fw} shall satisfy the following first-order conditions:

$$\alpha_t^o: \quad \frac{1-F(\alpha)}{f(\alpha)} - e^{\lambda t}\alpha = \begin{cases} \geqslant 0 & \text{if} \quad \alpha = 1 \\ = 0 & \text{if} \quad \alpha \in (0,1) \\ \leqslant 0 & \text{if} \quad \alpha = 0 \end{cases}$$

$$\alpha_t^{fw}: \quad \frac{1-F(\alpha)}{f(\alpha)} - e^{F(\alpha)\lambda t}\left[\alpha + \lambda t \int_0^\alpha v f(v) dv\right] = \begin{cases} \geqslant 0 & \text{if} \quad \alpha = 1 \\ = 0 & \text{if} \quad \alpha \in (0,1) \\ \leqslant 0 & \text{if} \quad \alpha = 0 \end{cases}$$

In both cases, the first term at $\alpha = 0$ is $1/f(0) > 0$, whereas the second term at $\alpha = 0$ is 0. Therefore, these thresholds are strictly positive.

Part (c). First, we consider the European option. Observe that $\mathcal{R}_t^o$ can be rewritten as

$$\mathcal{R}_t^o(\alpha) = e^{-\lambda t} \int_\alpha^1 [e^{\lambda t} v f(v) + F(v) - 1] dv$$

Suppose there exist $t > t'$ such that $\alpha_t^o \geqslant \alpha_{t'}^o$. By definition,

$$\mathcal{R}_t^o(\alpha_t^o) - \mathcal{R}_t^o(\alpha_{t'}^o) \geqslant 0 \geqslant \mathcal{R}_{t'}^o(\alpha_t^o) - \mathcal{R}_{t'}^o(\alpha_{t'}^o)$$

Therefore,

$$e^{\lambda t} \int_{\alpha_t^o}^{\alpha_{t'}^o} v f(v) dv \geqslant e^{\lambda t'} \int_{\alpha_t^o}^{\alpha_{t'}^o} v f(v) dv$$

which is a contradiction.

Next, we consider the option with forward pricing. Observe that $\mathcal{R}_t^{fw}$ can be rewritten as

$$\mathcal{R}_t^{fw}(\alpha) = e^{-\lambda t} \int_\alpha^1 \left[e^{F(x)\lambda t}\left(x + \lambda t \int_0^x v f(v) dv\right) f(x) + F(x) - 1\right] dx$$

Suppose there exist $t > t'$ such that $\alpha_t^{fw} \geq \alpha_{t'}^{fw}$. By the similar reasoning,

$$\int_{\alpha_t^{fw}}^{\alpha_{t'}^{fw}} e^{F(x)\lambda t}\left(x + \lambda t \int_0^x v f(v) dv\right) f(x) dx \geq \int_{\alpha_t^{fw}}^{\alpha_{t'}^{fw}} e^{F(x)\lambda t'}\left(x + \lambda t' \int_0^x v f(v) dv\right) f(x) dx$$

which is a contradiction.

Part (d). The thresholds depend only on the normalized time $\tau = \lambda t$. In both cases, for any $\alpha \neq 0$, there exists τ such that the for all $\tau > \tau'$, the first order conditions holds (see Part (b)) as the "$\leqslant$" inequality. It follows that the optimal thresholds shall converge to zero as $\tau \to \infty$.

Part (e). Consider the first-order conditions presented in Part (b). Clearly, these conditions must hold as equalities, because, in both cases, the first term is 0 at $\alpha = 1$, whereas the second term is strictly positive at $\alpha = 1$. The first order condition for the α^s is

$$\frac{1 - F(\alpha)}{f(\alpha)} - \alpha = 0$$

Observe that $\alpha < e^{\lambda t}\alpha$ and $\alpha < e^{F(\alpha)\lambda t}\left[\alpha + \lambda t \int_0^\alpha vf(v)dv\right]$ for all $\alpha \in (0,1)$. By assumption, $v \mapsto \frac{1-F(v)}{f(v)}$ is non-decreasing, which implies $\alpha^s > \alpha_t^o$ and $\alpha^s > \alpha_t^{fw}$.

Next, we shall show that $\alpha_t^{fw} > \alpha_t^o$ by establishing

$$1 > e^{-\lambda[1-F(\alpha)]t}\left[1 + \lambda t \int_0^\alpha \frac{vf(v)dv}{\alpha}\right]$$

for all $\alpha \in (0, \alpha^s]$.

Take $0 < \beta < \alpha \leq \alpha^s$, then $\alpha[1 - F(\alpha)] > \beta[1 - F(\beta)]$, because $v[1 - F(v)]$ is strictly increasing on $[0, \alpha^s)$. Since the inverse hazard rate is non-decreasing,

$$\frac{\alpha}{\beta} > \frac{1 - F(\beta)}{1 - F(\alpha)} \geq \frac{f(\beta)}{f(\alpha)}$$

which implies that $\alpha f(\alpha) > \beta f(\beta)$. In other words, $xf(x)$ is strictly increasing on $[0, \alpha^s)$. It follows that on $[0, \alpha^s)$,

$$e^{-\lambda[1-F(\alpha)]t}\left[1 + \lambda t \int_0^\alpha \frac{vf(v)dv}{\alpha}\right] < e^{-\lambda[1-F(\alpha)]t}\Big(1 + \lambda[1 - F(\alpha)]t\Big)$$

Observe that the right-hand side is strictly-increasing in α, and it is at most 1 for $\alpha = 1$. □

Proof of Proposition 4.

Part (a). By proposition 2, α_t^{fw} is strictly decreasing in λ. Since $\mathcal{G}_t(v)$ is strictly decreasing in λ for all v, $\mathcal{G}_t(\alpha_t)$ is strictly decreasing in λ as well.

For $\lambda \to 0$, we have $\alpha_t^{fw} \to \alpha^s$ and $\mathcal{G}_t(v) \to F(v)$ for all v, therefore $\mathcal{G}_t(\alpha_t) \to F(\alpha^s)$.

For $\lambda \to \infty$, we have $\alpha_t^{fw} \to 0$ and $\mathcal{G}_t$ converges to the degenerate distribution with mass point at $v = 1$, therefore $\mathcal{G}_t(\alpha_t^{fw}) \to 0$.

Part (b). The argument is the same as in Part (a), because both α_t^{fw} and $\mathcal{G}_t$ depend only on normalized time λt.

Parts (c). By the reasoning of Part (a), $\frac{1-\mathcal{G}_s(\alpha_t^{fw})}{1-\mathcal{G}_t(\alpha_t^{fw})}$ converges to 1 as λ goes to 0. Therefore, $\mathbb{E}\{\tau_t | \tau_t \neq t\} \to 0$.

Part (d). By the reasoning of Part (b), $\frac{1-\mathcal{G}_s(\alpha_t^{fw})}{1-\mathcal{G}_t(\alpha_t^{fw})}$ converges to 1 as t goes to 0. Therefore, $\mathbb{E}\{\tau_t | \tau_t \neq t\} \to 0$. □

Proof of Proposition 6.

Part (a). S_t is non-decreasing, therefore it has to converge. Since only 0 is absorbing, S_t can not converge to any positive number.

Part (b). By proposition 2, α_{t+s}^{fw} is strictly decreasing in λ for all s. Observe that $1-\mathcal{G}_s(v)$ is strictly increasing in λ for all v. It follows that the distribution $1-\mathcal{G}_s(\alpha_{t+s}^{fw})$ is strictly decreasing in λ in the sense of first-order stochastic dominance. Thus, $1-\mathcal{G}_s(\alpha_{t+s}^{fw})$ is strictly decreasing in λ in the sense of second-order stochastic dominance which implies that $\mathbb{E}\{S_t\}$ is strictly decreasing in λ.

For $\lambda \to 0$, we have $\alpha_{t+s}^{fw} \to \alpha^s$ and $\mathcal{G}_t(v) \to F(v)$ for all v, therefore $\mathcal{G}_t(\alpha_{t+s})$ converges to the degenerate distribution with mass points at $s = 0$ and $s = \infty$. It follows that $\mathbb{E}\{S_t\}$ diverges to ∞.

For $\lambda \to \infty$, we have $\alpha_{t+s}^{fw} \to 0$ and $\mathcal{G}_t$ converges to the degenerate distribution with mass point at $v = 1$, therefore $\mathbb{E}\{S_t\} \to 0$.

Part (c). The argument is the same as in Part (b), because both α_{t+s}^{fw} and $\mathcal{G}_t$ depend only on normalized time λt.

For $t \to 0$, we have $\alpha_{t+s}^{fw} \to \alpha_s^{fw}$ and $\mathcal{G}_t(v) \to F(v)$ for all v, therefore $\mathcal{G}_t(\alpha_{t+s})$ converges to $F(\alpha_s^{fw})$. It follows that $\lim\limits_{t\to\infty} \mathbb{E}\{S_t\} = \int_0^\infty s d[1-F(\alpha_s)]$.

For $t \to \infty$, we have $\alpha_{t+s}^{fw} \to 0$ and $\mathcal{G}_t$ converges to the degenerate distribution with mass point at $v = 1$, therefore $\mathbb{E}\{S_t\} \to 0$.

Part (d). The argument is analogous to Part (b). □

Proof of Lemma 10. Take $\{\mathbf{q}\}$ satisfying (IM). Define new allocations $\{\hat{\mathbf{q}}\}$ by setting $\hat{Z}_1(v_1) = Z_1(v_1)$, and recursively:

$$\hat{Z}_{k+1}(v^{k+1}) = \frac{\hat{\Delta}}{1-\delta\rho} \quad \text{if} \quad \hat{Z}_k(v^k)/\hat{\Delta} > \frac{\delta\rho}{1-\delta\rho}$$
$$\hat{Z}_{k+1}(v^{k+1}) = \min\left\{Z_{k+1}(v^{k+1}) + \sum_{m=1}^{k} \frac{q_m(v^m)\hat{\Delta}}{(\delta\rho)^{k-m+1}}, \frac{\hat{\Delta}}{1-\delta\rho}\right\} \quad \text{otherwise}$$

Moreover, let $\hat{q}_k(v^k) = \hat{Z}_k(v^k)/\hat{\Delta} - \delta\rho\hat{Z}_{k+1}(v^k, v_k)/\hat{\Delta}$.

Clearly, $\hat{Z}_k(v^k) \geqslant Z_k(v^k)$. We next show that $\hat{q}_k(v^k) \in [0,1]$ and that these allocations satisfy both desired properties.

Case 1. Consider v^k such that $\hat{Z}_k(v^{k-1})/\hat{\Delta} > \frac{\delta\rho}{1-\delta\rho}$. By construction, $\hat{Z}_k(v^k)/\hat{\Delta} \leqslant \frac{1}{1-\delta\rho}$, thus $\hat{q}_k(v^k) = \hat{Z}_k(v^k)/\hat{\Delta} - \frac{\delta\rho}{1-\delta\rho} \in (0,1]$. It follows that $\hat{Z}_{k+1}(v^k, v)/\hat{\Delta} = \hat{Z}_{k+2}(v^k, v, v)/\hat{\Delta} = \frac{1}{1-\delta\rho}$ for all v implying that $\hat{q}_{k+1}(v^k, v) = \hat{Z}_{k+1}(v^k, v)/\hat{\Delta} - \delta\rho\hat{Z}_{k+2}(v^k, v_k, v)/\hat{\Delta} = 1$ for all v. By construction $\hat{Z}_k(v^{k-1}, .)$ is non-decreasing, thus (IM) is satisfied as for all v,

$$\int_{v_k}^{v} \left[\hat{q}_k(v^k)\hat{\Delta} + \delta\rho\hat{Z}_{k+1}(v^k, x)\right]dx = \int_{v_k}^{v} \hat{Z}_k(v^k)dx \leqslant \int_{v_k}^{v} \hat{Z}_k(v^{k-1}, x)dx$$

Case 2. Consider v^k such that $\hat{Z}_k(v^k)/\hat{\Delta} \leq \frac{\delta\rho}{1-\delta\rho}$; that is, either $k = 1$ and $\hat{Z}_1(v_1) = Z_1(v_1)$ or $k \neq 1$ and $\hat{Z}_k(v^k) = Z_k(v^k) + \sum_{m=1}^{k-1} \frac{q_m(v^m)\hat{\Delta}}{(\delta\rho)^{k-m}}$. In either case, $\hat{Z}_{k+1}(v^k, v_k) = Z_{k+1}(v^k, v_k) +$

$\sum_{m=1}^{k} \frac{q_m(v^m)\hat{\Delta}}{(\delta\rho)^{k-m+1}}$ which implies

$$\hat{q}_k(v^k) = Z_k(v^k)/\hat{\Delta} - [q_k(v^k) + \delta\rho Z_{k+1}(v^k, v_k)/\hat{\Delta}] = 0$$

Finally, we verify (IM).

Take $v < v_k$. By monotonicity of $Z_{k+1}(v^k, .)$, $\hat{Z}_{k+1}(v^k, x) = Z_{k+1}(v^k, x) + \sum_{m=1}^{k} \frac{q_m(v^m)\hat{\Delta}}{(\delta\rho)^{k-m+1}}$ for all $x \in [v, v_k]$, and

$$\int_{v_k}^{v} \left[\hat{q}_k(v^k)\hat{\Delta} + \delta\rho\hat{Z}_{k+1}(v^k, x) - \hat{Z}_k(v^{k-1}, x)\right]dx =$$
$$= \int_{v_k}^{v} \left[q_k(v^k)\hat{\Delta} + \delta\rho Z_{k+1}(v^k, x) - Z_k(v^{k-1}, x)\right]dx \leqslant 0$$

Take $v_k > v$. Clearly, it suffices to consider only v such that $\hat{Z}_k(v^{k-1}, v)/\hat{\Delta} < \frac{1}{1-\delta\rho}$. By construction, $\hat{Z}_{k+1}(v^k, x) = Z_{k+1}(v^k, x) + \sum_{m=1}^{k} \frac{q_m(v^m)\hat{\Delta}}{(\delta\rho)^{k-m+1}}$ for all $x \in [v_k, v]$, and

$$\int_{v_k}^{v} \left[\hat{q}_k(v^k)\hat{\Delta} + \delta\rho\hat{Z}_{k+1}(v^k, x) - \hat{Z}_k(v^{k-1}, x)\right]dx \leq$$
$$\leqslant \int_{v_k}^{v} \left[q_k(v^k)\hat{\Delta} + \delta\rho Z_{k+1}(v^k, x) - Z_k(v^{k-1}, x)\right]dx \leqslant 0$$

□

Proof of Proposition 7. The proof has three steps. First, we show that the optimum must satisfy a particular property. Then, we use this property to turn the problem into a single-dimensional one. Finally, we characterize the optimum.

Step 1. Take $\{\mathbf{q}\}$ satisfying (SC) Define $\{\hat{\mathbf{q}}\}$ by

$$\hat{Z}_k(v^k) = \min\left\{\frac{Z_1\left(\max_{n\leq k} v_n\right)}{(\delta\rho)^{k-1}}, \frac{\hat{\Delta}}{1-\delta\rho}\right\}$$

$$\hat{q}(v^k) = \max\left\{0, \min\left\{1, \frac{Z_1\left(\max_{n\leq k} v_n\right)/\hat{\Delta}}{(\delta\rho)^{k-1}} - \frac{\delta\rho}{1-\delta\rho}\right\}\right\}$$

Clearly, $\hat{Z}_k(v^k) = \hat{q}_k(v^k)\hat{\Delta} + \delta\rho\hat{Z}_{k+1}(v^k, v_k)$ and $\hat{q}_k(v^k) \in [0, 1]$. We claim that $\{\hat{\mathbf{q}}\}$ satisfies (SC) and increases seller's profit.

To verify (SC), we will repeatedly use the fact that $\hat{Z}_k(v^{k-1}, \cdot)$ is non-decreasing which follows from Lemma 6. There are two cases two consider:

1. Take v^k such that $\hat{Z}_k(v^k) = \frac{\hat{\Delta}}{1-\delta\rho}$. Then, $\hat{Z}_{k+1}(v^k, v) = \frac{\hat{\Delta}}{1-\delta\rho}$ and $\hat{q}_k(v^k) + \delta\rho\hat{Z}_{k+1}(v^k, v) =$

$\hat{Z}_k(v^k)$ for all v. Since Z_1 is non-decreasing,

$$\left[v - v_k\right]\left[\hat{Z}_k(v^{k-1}, v) - \hat{Z}_k(v^k)\right] \geqslant 0$$

2. Take v^k such that $\hat{Z}_k(v^k) \neq \frac{\hat{\Delta}}{1-\delta\rho}$; that is, $\hat{q}_k(v^k) = 0$ and $\hat{Z}_k(v^{k-1}, v) = \frac{Z_1\left(\max_{n\leq k} v_n\right)/\hat{\Delta}}{(\delta\rho)^{k-1}}$. Then, $\hat{q}_k(v^k)+\delta\rho\hat{Z}_{k+1}(v^k, v) = \hat{Z}_k(v^{k-1}, \max\{v_k, v\})$ for all v. Since Z_1 is non-decreasing,

$$\left[v_k - v\right]\left[\hat{Z}_k(v^{k-1}, v) - \hat{Z}_k(v^k)\right] \geqslant 0$$

Next, we establish that $\{\hat{\mathbf{q}}\}$ increases seller's profit. Rewrite seller's profit as it follows. Let k^* be a discrete time point of the last arrival, and v^{k^*-1} be the history preceding it, then

$$\mathbb{E}\left\{\sum_{k^*=1}^{\infty}\delta^{k^*-1}\mathbb{E}\left\{\sum_{k=1}^{\infty}\delta^{k-1}w_{k^*-1+k}(v^{k^*-1+k})q_{k-1+k}(v^{k^*-1+k})\hat{\Delta}\middle|v^{k^*-1}\right\}\right\} =$$
$$= \int_0^1\left[\left(vf(v) - [1-F(v)]\right)Z_1(v) + \delta(1-\rho)vf(v)\sum_{k=1}^{\infty}\delta^{k-1}\mathbb{E}\left\{Z_{k+1}(v^k, v)\right\}\right]dv$$

Observe that $\hat{Z}_1 = Z_1$. Fix some v^k. Suppose it has been established that $\hat{Z}_k(v^{k-1}, .) \geq Z_k(v^{k-1}, .)$. Notice that the new contract satisfies $\hat{Z}_{k+1}(v^k, v) = \min\left\{\frac{\hat{Z}_k(v^{k-1}, \max\{v_k, v\})}{\delta\rho}, \frac{\hat{\Delta}}{1-\delta\rho}\right\}$. Since $\{\mathbf{q}\}$ satisfies (SC),

$$\hat{Z}_{k+1}(v^k, v) \geq \min\left\{\frac{Z_k(v^{k-1}, \max\{v_k, v\})}{\delta\rho}, \frac{\hat{\Delta}}{1-\delta\rho}\right\} \geq Z_{k+1}(v^k, v)$$

Step 2. We show how seller's problem can be reduced to a single-dimensional one. Recall that G_k is a distribution of $\max_{n\leq k} v_n$ given by $G_k(v) = F(v)[\xi(v)]^{k-1}$ for $\xi(v) = (1-\rho)F(v)+\rho$. Then, $\{\hat{\mathbf{q}}\}$ satisfies

$$\int_0^1 xf(x)\mathbb{E}\{\hat{Z}_{k+1}(v^k, x)\}dx = \int_0^1\int_0^1 uf(u)\min\left\{\frac{\hat{Z}_1(\max\{v, u\})}{(\delta\rho)^k}, \frac{\hat{\Delta}}{1-\delta\rho}\right\}dG_k(v)du =$$
$$= \int_0^1\left[\int_0^u vf(v)dvG_k(u)\right]'\min\left\{\frac{\hat{Z}_1(u)}{(\delta\rho)^k}, \frac{\hat{\Delta}}{1-\delta\rho}\right\}du$$

Express $\hat{Z}_1(\cdot)$ as an average of allocations along the persistent path:

$$Z_1(u) = \sum_{k=1}^{\infty}(\delta\rho)^{k-1}\hat{q}_k(\underbrace{u, ..., u}_{k \text{ times}})\hat{\Delta}$$

By construction, $\hat{q}_{k+1}(\underbrace{u, ..., u}_{k+1 \text{ times}}) = 1$ whenever $\hat{q}_k(\underbrace{u, ..., u}_{k \text{ times}}) \neq 0$. Using this insight, seller's profit can be rewritten as

$$\sum_{k=1}^{\infty}(\delta\rho)^{k-1}\int_0^1 [\phi_k(v)+F(v)-1]\hat{q}_k(\underbrace{u,...,u}_{k\text{ times}})\hat{\Delta}dv$$

where $\phi_1(v)=vf(v)$ and $\phi_{k+1}(v)$ is defined as in the proof of Proposition 5,

$$\begin{aligned}\phi_{k+1}(v) &= \phi_k(v)+\frac{1-\rho}{\rho}\left(\int_0^v xf(x)dxF(v)\left[\frac{\xi(v)}{\rho}\right]^{k-1}\right)' \\ &= \phi_1(v)+\frac{1-\rho}{\rho}\left(\int_0^v xf(x)dx\left(1-\left[\frac{\xi(v)}{\rho}\right]^{k}\right)\right)'\end{aligned}$$

Recall that $\hat{Z}_1(\cdot)$ must be non-decreasing, therefore each $v\mapsto \hat{q}_k(\underbrace{v,...,v}_{k\text{ times}})$ is non-decreasing as well. It follows that seller's problem reduces to a choice of a sequence of thresholds $\{\alpha_k\}$:

$$\max_{\{a_k\}}\quad \sum_{k=1}^{\infty}(\delta\rho)^{k-1}\int_{a_k}^1[\phi_k(v)+F(v)-1)]\hat{\Delta}dv \quad \text{s.t. } \{a_k\} \quad \text{is non-increasing}$$

By the same argument as in the proof of Proposition 2, monotonicity constraint does not bind. The optimal sequence of threshold is the same as in the sequential screening model discussed in Proposition 5. □

Proof of Proposition 8. First we prove Part (a). We start with the deterministic optimum; that is, $q_3(v^3)=\mathbb{1}\{\max\{v_1,v_2,v_3\}\geqslant a_3^{fw}\}$. We construct an improvement using two consecutive perturbations.

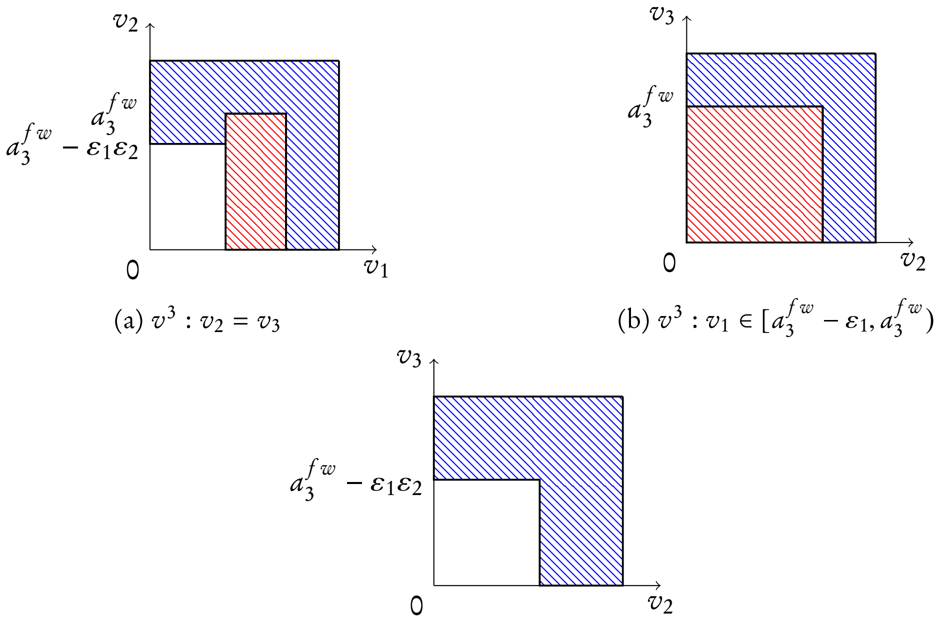
Fix small $\varepsilon_1,\varepsilon_2>0$. Consider the new allocation $\tilde{q}_3$ which differs from the original one only when $v_1<a_3^{fw}$:

- For $v_1<a_3^{fw}-\varepsilon_1$, $\tilde{q}_3(v^3)=\mathbb{1}\{\max\{v_2,v_3\}\geqslant a_3^{fw}-\varepsilon_1\varepsilon_2\}$, whereas $q_3(v^3)=\mathbb{1}\{\max\{v_2,v_3\}\geqslant a_3^{fw}\}$;
- For $v_1\in[a_3^{fw}-\varepsilon_1,a_3^{fw})$, $\tilde{q}_3(v^3)=\varepsilon_2+(1-\varepsilon_2)\mathbb{1}\{\max\{v_2,v_3\}\geqslant a_3^{fw}\}$, whereas $q_3(v^3)=\mathbb{1}\{\max\{v_2,v_3\}\geqslant a_3^{fw}\}$.

The following figure illustrates the new allocation.

It is easy to see that the new allocation satisfies (IM). Recall that $\phi_1(v)=vf(v)$, $\phi_2(u)=\phi_1(v)+\frac{1-\rho}{\rho}\left(F(v)\int_0^v xf(x)dx\right)'$ and

$$\phi_3(v)=\phi_2(v)+\frac{1-\rho}{\rho}\left(\left(1-\left[1+\frac{1-\rho}{\rho}F(v)\right]^2\right)\int_0^v xf(x)dx\right)'$$



(a) $v^3 : v_2 = v_3$

(b) $v^3 : v_1 \in [a_3^{fw} - \varepsilon_1, a_3^{fw})$

(c) v^3: $v_1 < a_3^{fw} - \varepsilon_1$

Figure 7: blue- $\tilde{q}_3(v^3) = 1$, red- $\tilde{q}_3(v^3) = \varepsilon_2$ and white- $\tilde{q}_3(v^3) = 0$

The net change of seller's profit is given by $\delta^2 \tilde{D}(\varepsilon_1, \varepsilon_2)\hat{\Delta}$

$$\tilde{D}(\varepsilon_1, \varepsilon_2) = \rho^2 \int_{a_3^{fw}}^{a_3} [\phi_3(v) + F(v) - 1] dv \varepsilon_2 - \rho(1-\rho)F(a_3^{fw} - \varepsilon_1) \int_{a_3^{fw} - \varepsilon_1}^{a_3^{fw}} \phi_2(v) dv \varepsilon_2 +$$
$$+ \rho(1-\rho)F(a_3^{fw} - \varepsilon_1) \int_{a_3^{fw} - \varepsilon_1\varepsilon_2}^{a_3^{fw}} \phi_2(v) dv$$

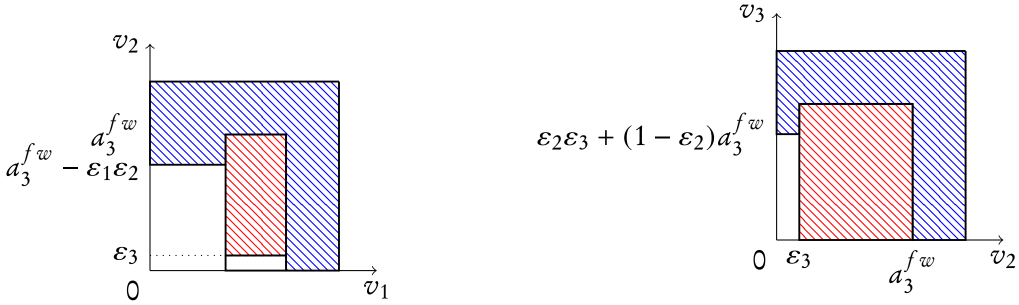
Observe that $\tilde{D}$ is twice differentiable in the neighborhood of $(0,0)$, and $\tilde{D}(0,0) = \frac{\partial}{\partial \varepsilon_1}\tilde{D}(0,0) = \frac{\partial}{\partial \varepsilon_2}\tilde{D}(0,0) = \frac{\partial^2}{\partial(\varepsilon_1)^2}\tilde{D}(0,0) = \frac{\partial^2}{\partial(\varepsilon_2)^2}\tilde{D}(0,0) = 0$. Moreover, by the definition of a_3^{fw},

$$\frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\tilde{D}(0,0) = \rho^2[\phi_3(a_3^{fw}) + F(a_3^{fw}) - 1] = 0$$

The allocation $\tilde{q}_3$ yields the same revenue (up to the second order) as the deterministic optimum for small $\varepsilon_1, \varepsilon_2 > 0$.

Next, we perturb $\tilde{q}_3$. Fix small $\varepsilon_3 > 0$. Consider the new allocation $\hat{q}_3$ which differs from the original one only when $v_1 \in [a_3^{fw} - \varepsilon_1, a_3^{fw})$ and $v_2 < \varepsilon_3$: $\hat{q}_3(v^3) = \mathbb{1}\{v_3 \geqslant \varepsilon_2\varepsilon_3 + (1-\varepsilon_2)a_3^{fw}\}$, whereas $\tilde{q}_3(v^3) = \varepsilon_2 + (1-\varepsilon_2)\mathbb{1}\{v_3 \geqslant a_3^{fw}\}$. The following figure illustrates the allocation perturbed twice.

It is easy to see that the new allocation satisfies (IM). Then, the net change of seller's profit



(a) $v^3 : v_2 = v_3$ (b) $v^3 : v_1 \in [a_3^{fw} - \varepsilon_1, a_3^{fw})$

Figure 8: blue- $\hat{q}_3(v^3) = 1$, red- $\hat{q}_3(v^3) = \varepsilon_2$ and white- $\hat{q}_3(v^3) = 0$

(as compared to $\tilde{q}_3$) is given by $\delta^2(1-\rho)[F(a_3^{fw}) - F(a_3^{fw} - \varepsilon_1)]\hat{D}(\varepsilon_2, \varepsilon_3)\hat{\Delta}$ where

$$\hat{D}(\varepsilon_2, \varepsilon_3) = (1-\rho)F(\epsilon_3)\left[\int_{\varepsilon_2\varepsilon_3+(1-\varepsilon_2)a_3^{fw}}^{1} \phi_1(v)dv - \int_0^{a_3^{fw}} \phi_1(v)dv\varepsilon_2\right] - \rho\int_0^{\varepsilon_3} \phi_1(v)dv\varepsilon_2$$

Again, $\hat{D}$ is twice differentiable in the neighborhood of $(0,0)$, and $\hat{D}(0,0) = \frac{\partial}{\partial\varepsilon_2}\hat{D}(0,0) = \frac{\partial}{\partial\varepsilon_3}\hat{D}(0,0) = \frac{\partial^2}{\partial(\varepsilon_2)^2}\hat{D}(0,0) = \frac{\partial^2}{\partial(\varepsilon_3)^2}\hat{D}(0,0) = 0$,

$$\frac{\partial^2}{\partial\varepsilon_2\varepsilon_3}\hat{D}(0,0) = (1-\rho)f(0)a_3^{fw}\int_0^{a_3^{fw}} [a_3^{fw}f(a_3^{fw}) - vf(v)]dv > 0$$

The last inequality is implied by $v \mapsto vf(v)$ by being strictly increasing on $[0, a^{fp}]$, see Part (e) of Proposition 2.

Now, we prove part (b) of Proposition 8. We start with the allocation implemented by forward pricing; that is, $q_3(v^3) = \mathbb{1}\{\max\{v_1, v_2, v_3\} \geqslant a_3^{fw}\}$ and $q_4(v^4) = \mathbb{1}\{\max\{v_1, v_2, v_3\} \geqslant a_3^{fw}\}$.

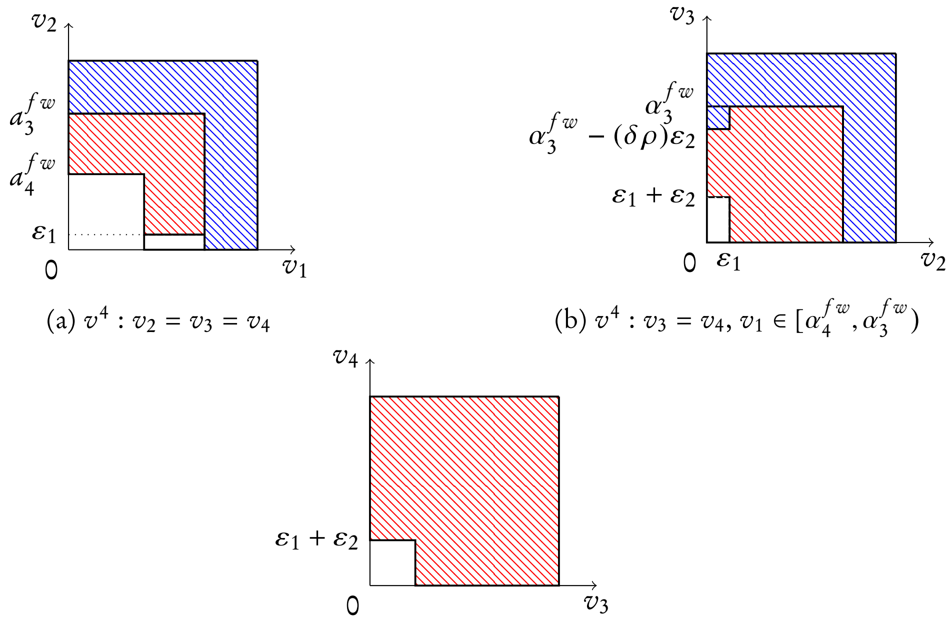
Fix small $\varepsilon_1, \varepsilon_2 > 0$. Consider the new allocation $\{\tilde{q}_3, \tilde{q}_4\}$ which differs from the original one only for $v_1 \in [a_4^{fw}, a_3^{fw})$ and $v_2 \in [0, \varepsilon_1)$:

- $\tilde{q}_3(v^3) = 1$ if and only if $v_3 \geqslant a_3^{fw} - (\delta\rho)\varepsilon_2$, whereas $q_3(v^3) = 1$ if and only if $v_3 \geqslant a_3^{fw}$;
- $\tilde{q}_4(v^4) = 1$ if and only if $\max\{v_3, v_4\} \geqslant \varepsilon_1 + \varepsilon_2$, whereas $q_4(v^4) = 1$ for all v_3, v_4.

The following figure visualizes the new allocation.

It is easy to see that the new allocation satisfies (IM). Recall that $\phi_1(v) = vf(v)$ and $\phi_2(v) = \phi_1(v) + \dfrac{1-\rho}{\rho}\left(F(v)\displaystyle\int_0^v xf(x)dx\right)'$. Then, the net change of seller's profit can be succinctly written as $[\delta(1-\rho)]^2[F(a_3^{fw}) - F(a_4^{fw})]\tilde{D}(\varepsilon_1, \varepsilon_2)\hat{\Delta}$ where

$$D(\varepsilon_1, \varepsilon_2) = F(\varepsilon_1)\left[\int_{a_3^{fw}-(\delta\rho)\varepsilon_2}^{a_3^{fw}} \phi_1(v)dv - (\delta\rho)\int_0^{\varepsilon_1+\varepsilon_2} (\phi_1(v) + \phi_2(v))dv\right]$$



(a) $v^4 : v_2 = v_3 = v_4$

(b) $v^4 : v_3 = v_4,\ v_1 \in [\alpha_4^{fw}, \alpha_3^{fw})$

(c) v^4: $v_1 \in [a_4^{fw}, a_3^{fw})$, $v_2 \in [0, \varepsilon_1)$

Figure 9: blue- $\tilde{q}_3(v^3) = \tilde{q}_4(v^4) = 1$, red- $\tilde{q}_3(v^3) = 0$, $\tilde{q}_4(v^4) = 1$ and white- $\tilde{q}_3(v^3) = \tilde{q}_4(v^4) = 0$

Observe that $\tilde{D}$ is twice differentiable in the neighborhood of $(0,0)$, and $\tilde{D}(0,0) = \frac{\partial}{\partial\varepsilon_1}\tilde{}(0,0) = \frac{\partial}{\partial\varepsilon_2}\tilde{D}(0,0) = \frac{\partial^2}{\partial(\varepsilon_1)^2}\tilde{D}(0,0) = \frac{\partial^2}{\partial(\varepsilon_2)^2}\tilde{D}(0,0) = 0$. Moreover,

$$\frac{\partial^2}{\partial\varepsilon_1\varepsilon_2}D(0,0) = f(0)(\delta\rho)\phi_1(a_3^{fw}) > 0$$

□

Proof of Proposition 9. We first define a uniform bound for fixed $\Delta > 0$. As in Section xxx, construct seller's profit $\delta^{k-1}R_k^o(\alpha)\hat{\Delta}$ from the k-th good sold by the means of European option with strike price $\alpha\hat{\Delta}$ and premium $\delta^{k-1}M_k(\alpha)\hat{\Delta}$:

$$M_k^o(\alpha) = (1-\rho^{k-1})\left[\mathbb{E}\{(v_k-\alpha)^+\}\right]$$
$$R_k^o(\alpha) = M_k^o + R^s(\alpha) = \rho^{k-1}R^s(\alpha^s) + (1-\rho^{k-1})\int_0^1 v\,dF(v)$$

Denote the optimal strike price by α_k^o.

Observe that for all k, $\rho^{k-1}R^s(\alpha^s) + (1-\rho^{k-1})\int_0^1 v\,dF(v) \geq R_k^o(a_k^o) \geq R^s(a^s)$. Our goal is to find a lower bound for the following ratio:

$$\frac{R_k^o(a_k^o)}{\rho^{k-1}R^s(\alpha^s) + (1-\rho^{k-1})\int_0^1 v\,dF(v)}$$

We construct two different bounds and take the pointwise maximum of them. First of all, by definition, $R_k^o(a_k^o) \geq R_k^o(0) = (1-\rho^{k-1})\int_0^1 v\,dF(v)$, and $R^\sigma(\alpha^s) \geq (1-\rho^{k-1})\int_0^1 v\,dF(v)$. The equality holds if and only if F is a point mass distribution, therefore for any F,

$$\frac{R^s_k(a^o_k)}{\rho^{k-1}R^{fp}(\alpha^s) + (1-\rho^{k-1})\int_0^1 vdF(v)} \geq 1-\rho^{k-1}$$

The second bound is based on Tamuz [2013] who showed $R^s(\alpha^s) \geq exp\left(\int_0^1 \log vdF(v) - 1\right)$. We replicate the result for completeness. By definition of α^s, for all v,

$$\log R^s(\alpha^s) = \log \alpha^s + \log[1-F(\alpha^s)] \geq \log v + \log[1-F(v)]$$

Take an expectation on both sides to get

$$\log \alpha^s + \log[1-F(\alpha^s)] \geq \int_0^1 \log vdF(v) + \int_0^1 \log[1-F(v)]dF(v) = \int_0^1 \log vdF(v) - e$$

Take an exponent on both sides to obtain the result.

By Jensen's inequality,

$$exp\left(\int_0^1 \log vdF(v)\right) \geq \int_0^1 vdF(v)$$

And, the equality holds if and only if F is a point mass distribution, therefore for any F,

$$\frac{R^o_k(a^{sp}_k)}{\rho^{k-1}R^s(\alpha^s) + (1-\rho^{k-1})\int_0^1 vdF(v)} \geq L_k = \max\left\{\frac{1}{\rho^{k-1} + (1-\rho^{k-1})e}, 1-\rho^{k-1}\right\}$$

Combine Proposition 11 and the previous result to get:

$$\frac{\sum_{k=1}^{\infty}\delta^{k-1}R^o_k(a^a_k)\hat{\Delta}}{\sum_{k=1}^{\infty}\delta^{k-1}\left[\rho^{k-1}R^s(\alpha^s) + (1-\rho^{k-1})\int_0^1 vdF(v)\right]\hat{\Delta}} \geq \frac{A\times R^{fp}(\alpha^{fp}) + B\times\int_0^1 udF(u)}{C\times R^{fp}(\alpha^{fp}) + D\times\int_0^1 udF(u)}$$

where

$$A = \sum_{k=1}^{\infty}\delta^{k-1}\rho^{k-1}L_k\hat{\Delta}, \quad B = \sum_{k=1}^{\infty}\delta^{k-1}(1-\rho^{k-1})L_k\hat{\Delta}, \quad C = \frac{\hat{\Delta}}{1-\delta\rho}, \quad D = \frac{\delta(1-\rho)}{1-\delta}\frac{\hat{\Delta}}{1-\delta\rho}$$

By the argument from Step 1, for any F,

$$\frac{A\times R^{fp}(\alpha^{fp}) + B\times\int_0^1 udF(u)}{C\times R^{fp}(\alpha^{fp}) + D\times\int_0^1 udF(u)} \geq \min\left\{\frac{A+B}{C+D}, \frac{A+Be}{C+De}\right\}$$

Now, we take $\Delta \to 0$. First of all,

$$L_{\lfloor k/\Delta\rfloor - 1} \to \mathcal{L}_t = \max\left\{\frac{1}{e^{-\lambda t} + (1-e^{-\lambda t})e}, 1-e^{-\lambda t}\right\}$$

Then, $C \to \mathcal{C} = \frac{1}{r+\lambda} = \frac{1/\lambda}{r/\lambda+1}$, $D \to \mathcal{D} = \frac{\lambda}{r}\frac{1/\lambda}{r/\lambda+1}$, and

$$A \to \mathcal{A} = \int_0^\infty e^{-rt}e^{-\lambda t}L_t dt = (1/\lambda)\int_0^\infty e^{-r/\lambda t}e^{-t}L_t dt$$
$$B \to \mathcal{B} = \int_0^\infty e^{-rt}(1-e^{-\lambda t})L_t dt = (1/\lambda)\int_0^\infty e^{-r/\lambda t}(1-e^{-t})L_t dt$$

So, $\mathcal{L}$ can be defined as

$$\mathcal{L} = \min\left\{\frac{\mathcal{A}+\mathcal{B}}{\mathcal{C}+\mathcal{D}}, \frac{\mathcal{A}+\mathcal{B}e}{\mathcal{C}+\mathcal{D}e}\right\}$$

Observe that $\mathcal{L}$ depends only on r/λ. □

Supplementary results

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