Risk premia and unemployment fluctuations[*]

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Abstract

We study the role of fluctuations in discount rates for the joint dynamics of expected returns in the stock market and employment dynamics. We construct a non-parametric bound on the predictability and time-variation in conditional volatility of the firm's profit flow that must be met to rationalize the observed business-cycle fluctuations in vacancy-filling rates. A stochastic discount factor consistent with conditional moments of the risk-free rate and expected returns on risky assets only partly alleviates the need for an excessively volatile model of the expected profit flow.

1   Introduction

Stock market valuation and the dynamics of labor market variables are highly volatile relative to conventional measures of macroeconomic risk and strongly correlated with the business cycle. The volatility in the stock market variables in excess of the subsequent movements in dividends is the basis of the Shiller (1981) excess volatility puzzle in the asset pricing literature. On the other hand, Shimer (2005) identified the discrepancy between the volatility in labor market tightness and measures of labor productivity as an excess unemployment volatility puzzle in search and matching models of the labor market.

Campbell and Shiller (1988) used linear approximations of present value budget constraints to argue that variation in price-dividend ratios unexplained by movements in subsequent dividends have to be attributed to fluctuations in discount rates applied to these dividends. However, given the relative smoothness of yields on riskless assets, in particular the real risk-free rate, these fluctuations in discount rates have to come mainly in the form of time-varying compensation for risk, manifested in fluctuations in the conditional volatility of the stochastic discount factor.

The search and matching literature dealt with the unemployment volatility puzzle by introducing mechanisms that increase the volatility and cyclicality of the profit flow earned by the firm from hiring the marginal worker (Hall and Milgrom (2008), Hagedorn and Manovskii (2008), Kehoe et al. (2015), Christiano et al. (2016), and others). At the same time, preferences used in this work typically imply minimal compensations for risk, and hence are not able to match standard asset pricing facts including large risk premia for risky cash flows and fluctuations in the valuation of the stock market.

In this paper, we provide a quantitative connection between valuation of risky financial assets and valuation of profit flows earned by hiring workers in the labor market. The question is to which extent can fluctuations in discount rates, disciplined by financial markets data, serve as an explanation for the observed business cycle fluctuations in labor market variables.

The central issue when building this connection is the distinction between cash flows priced in financial markets and cash flows that are relevant for the firm that decides to hire a worker. Stock market valuation, when normalized by workforce, can be thought of as valuation of profit flow generated by the *average* worker. On the other hand, the hiring decision is based on the profits earned by the firm from hiring the *marginal* worker. While the formed can be constructed using observed data on profits or dividends and employment, the latter is inherently unobservable.

Hence, rather than explicitly modeling the profit flow from the marginal worker, we

characterize a set of restrictions that every such profit flow must satisfy in order to be consistent with optimality of firms' hiring decisions. In particular, we construct a non-parametric bound on two moments of firm's profit dynamics.

One of the moments is the volatility of the conditional expectation of firm's profits from the marginal worker. The standard approach in the search and matching literature dictates to make this volatility high, so that business-cycle fluctuations in expected profits are able to explain changes in incentives of firms to hire workers. The other moment is the average conditional volatility of the profit flow. Business cycle fluctuations in the conditional volatility of innovations to profits that are correlated with innovations to the stochastic discount factor lead to fluctuations in risk premia associated with the firm's profit flow and again generate procyclical incentives to hire workers.

The bound that we construct reveals that a successful model of the labor market dynamics that relies on the optimality condition for hiring must satisfy this condition through a combination of the above two channels. Introducing a properly modeled stochastic discount factor alleviates the burden imposed by the optimality condition on generating an excessively volatile expected profit flow, and allows to shift part of the weight on the risk compensation channel.

We construct alternative parametric and non-parametric models of the stochastic discount factor to study the magnitude of the tradeoff between fluctuations in expected profit flow and fluctuations in its conditional volatility. The main conclusion is that a successful stochastic discount factor must have a large and strongly countercyclical conditional volatility in order for risk premia to play a quantitatively meaningful role in explaining labor market fluctuations. We explain that these labor market restrictions imposed on plausible stochastic discount factors are in fact stronger than in pricing of financial assets, due to short duration of profit flows associated with hired workers.

The bound that we derive depends on the model of the stochastic discount factor and observed variation in labor market variables, in particular vacancy-filling rates and separation rates, but not on the profit flow from the marginal worker itself. This allows us to construct the bound and then investigate whether alternative empirically observable proxies for this profit flow are consistent with the bound. We construct measures of aggregate average profit per worker and conclude that they do not meet the bound.

We further study to which extent our analysis could be affected by the presence of financial constraints faced by firms that desire to hire workers. Our analysis assumes that the same stochastic discount factor that prices assets in financial markets is also used by firms to value cash flows from hired workers. If firms are facing financial constraints and investors in financial markets are not, then the optimal hiring decision of the firms will be

distorted by shadow prices assigned to these constraints. We form portfolios of firms sorted on measures and financial constraints and indeed observe that cash flows of financially less constrained firms are closer to meeting the bound.

Finally, we investigate whether the bound provides meaningful information in the sense that profit flow processes that are consistent with it also do well in generating relevant unemployment fluctuations. To do so, we construct counterfactual profit flow processes that lie exactly on the bound and embed them in a search and matching framework with a calibrated matching function. Optimal hiring decisions implied by the dynamics of the profit flow process then imply a counterfactual time series for the unemployment rate that we can compare with historical data. We show that this counterfactual time series fits very well the U.S. data since 1950. The bound is therefore not only necessary but also sufficiently tight.

This paper contributes to the growing body of work that incorporates insights from asset pricing literature to study labor market dynamics. Petrosky-Nadeau et al. (2015), Favilukis and Lin (2016), Kuehn et al. (2014) Favilukis et al. (2015), Belo et al. (2014), Donangelo et al. (2016), or Kilic and Wachter (2015) are recent examples. Perhaps the closest paper to ours is Hall (2017), who explicitly highlights the economic forces that fluctuations in discount rates play in the search and matching framework, and provides insights into the interaction between the stochastic discount factor and profit flow. However, his model of preferences implies that most of the time-variation in discount rates is manifested as time-variation in the mean, as opposed to the dispersion, of the stochastic discount factor, and hence implies quantitatively implausible movements in the risk-free rate.

The paper is organized as follows. We start in Section 2 with a back-of-the-envelope calculation that highlights the quantitative challenge that the risk-compensation channel faces as an explanation of labor market fluctuations. In Section 3 we introduce the theoretical framework, and in Section 4 we derive and interpret the bound that we use to analyze restrictions on profit flow processes. Section 5 introduces alternative models of stochastic discount factors, and Sections 6 and 7 describe the data and discuss empirical results. In Section 8 we study implied labor market dynamics. Section 9 concludes.

2   A back-of-the-envelope calculation

Before delving into the construction of our bound, we want to motivate the quantitative challenge studied in this paper by a simple approximate calculation. This includes time-variation in risk premia to a log-linearization result outlined in Shimer (2005).

We consider a discrete-time environment with optimizing investors in financial markets and firms hiring workers in a frictional labor market. In this environment, optimality con-

ditions are described by two types of forward-looking restrictions. In the financial market, these restrictions have the form of Euler equations for the valuation of returns on alternative assets. In the labor market, optimal choice of hiring implies an intertemporal condition that equalizes the cost of hiring the marginal worker with the present discounted value of profits generated by employing this worker.

Firms in the production sector hire workers in a Diamond–Mortensen–Pissarides labor market. Firms post vacancies at a per-period cost κ, and the vacancies are filled at an equilibrium rate q_t. Optimal hiring choice implies that the value to the firm of hiring a marginal worker must be equal to the expected cost of hiring the marginal worker, κ/q_t. This implies the Euler equation

$$\frac{\kappa}{q_t} = E_t \left[s_{t+1} \left(\pi_{t+1} + (1 - \delta_{t+1}) \frac{\kappa}{q_{t+1}} \right) \right] \tag{1}$$

where π_{t+1} is the firm's profit flow from the marginal worker, s_{t+1} the firm's stochastic discount factor, and δ_{t+1} the probability of match separation. In recessions, vacancy filling rates q_t are high and hiring workers is cheap, which must be rationalized by low future expected discounted profit flows. While we focus here on the labor market, an analogous equation (and hence our analysis) holds for any Q-theory type optimality condition where κ/q_t represents the upfront cost of investment and δ_{t+1} the depreciation rate.

Dividing equation (1) by κ/q_t and using the definition of covariance, we obtain the usual valuation equation[1]

$$1 = E_t \left[s_{t+1} \right] E_t \left[\underbrace{\frac{\pi_{t+1} + (1 - \delta_{t+1}) \kappa/q_{t+1}}{\kappa/q_t}}_{R_{t+1}^h} \right] + \underbrace{Cov_t \left[s_{t+1}, \frac{\pi_{t+1} + (1 - \delta_{t+1}) \kappa/q_{t+1}}{\kappa/q_t} \right]}_{-\Gamma_t} \tag{2}$$

where R_{t+1}^h is the return on hiring the worker in period t, $E_t \left[s_{t+1} \right] \doteq 1/R_t^f$ is the reciprocal risk-free rate, and the covariance term is the negative of the risk-premium, Γ_t. This equation states that expected returns $E_t \left[R_{t+1}^h \right]$ in excess of the risk-free rate (such that the first term on the right-hand side is larger than one) reflect compensation for risk in the form of risk premia Γ_t, which arise due to the fact that states with high payoffs π_{t+1} comove negatively with states with high high marginal rates of substitution s_{t+1}.

Many models in the labor-macro literature imply negligible risk premia ($\Gamma_t \approx 0$). Since the risk-free rate is relatively stable over the business cycle, large fluctuations in vacancy filling rates q_t must be compensated by correspondingly large fluctuations in expected profit

[1]Details of all calculations from this section of the paper are provided in Appendix A.

flow $E_t\left[\pi_{t+1}\right]$ such that $E_t\left[R_{t+1}^h\right] \approx R_t^f$ also remains stable. More recently, a number of papers started focusing on the risk premium term Γ_t as well. According to this view, high current vacancy filling rates q_t during recessions are consistent with high expected returns $E_t\left[R_{t+1}^h\right]$ because they reflect high risk premia Γ_t during recessions, which restores equation (2).

In order to assess the quantitative feasibility of this channel, it is useful to construct a log-linear approximation of equation (2). In order to focus on the quantitatively relevant objects, we neglect fluctuations in the risk-free rate and set $E_t\left[s_{t+1}\right] \doteq \beta$, as well as $\delta_{t+1} = \delta$. We also assume that the vacancy filling rate q_t can be approximated by an AR(1) process with autoregression coefficient ρ_q. Then we obtain the approximation

$$\left(1 - \beta\left(1 - \delta\right) + \overline{\Gamma}\right) E_t\left[\widehat{\pi}_{t+1}\right] - \overline{\Gamma}\widehat{\Gamma}_t = -\left(1 - \beta\left(1 - \delta\right)\rho_q + \overline{\Gamma}\right)\widehat{q}_t. \tag{3}$$

In this equation, $\overline{\Gamma}$ represents the (risk-adjusted) steady state risk premium, and variables with hats are percentage deviations from this steady state. In absence of risk premia ($\overline{\Gamma} = \widehat{\Gamma}_t = 0$), we can relate the unconditional volatilities of $\widehat{q}_t$ and $E_t\left[\widehat{\pi}_{t+1}\right]$ as

$$\sigma\left(E_t\left[\widehat{\pi}_{t+1}\right]\right) = \frac{1 - \beta\left(1 - \delta\right)\rho_q}{1 - \beta\left(1 - \delta\right)}\sigma\left(\widehat{q}_t\right).$$

In the quarterly data on vacancy filling rates, $\sigma\left(\widehat{q}_t\right) = 0.223$ and $\rho_q = 0.9$. We can further calibrate the quarterly steady state separation rate $\delta = 0.1$ and the time-preference parameter $\beta = 0.9975$ to obtain the required volatility of expected profits that rationalizes the hiring equation $\sigma\left(E_t\left[\widehat{\pi}_{t+1}\right]\right) = 0.419$, which is more than twenty times the volatility of output. This is another manifestation of the unemployment volatility puzzle.

To which extent can fluctuations in risk premia help resolve this puzzle? Equation (3) allows us to construct a lower bound on the required volatility of risk premia as a function of volatilities of the the vacancy filling rate and expected profit flow

$$\sigma\left(\widehat{\Gamma}_t\right) \geq \frac{1}{\overline{\Gamma}}\left[\left(1 - \beta\left(1 - \delta\right)\rho_q + \overline{\Gamma}\right)\sigma\left(\widehat{q}_t\right) - \left(1 - \beta\left(1 - \delta\right) + \overline{\Gamma}\right)\sigma\left(E_t\left[\widehat{\pi}_{t+1}\right]\right)\right]. \tag{4}$$

where the equality is achieved when $E_t\left[\widehat{\pi}_{t+1}\right]$ and $\widehat{\Gamma}_t$ are perfectly negatively correlated.

Figure 1 displays the volatility of the risk premium $\overline{\Gamma}\sigma\left(\widehat{\Gamma}_t\right)$ in annualized percentage points when computed using quarterly and annual data, for alternative levels of the volatility of expected profits, $\sigma\left(E_t\left[\widehat{\pi}_{t+1}\right]\right)$. Empirical estimates of time-variation in risk premia are hard to compute but we use the volatility of the lower bound on the equity premium on the S&P 500 index computed in Martin (2017) as an illustrative example. Martin (2017) reports
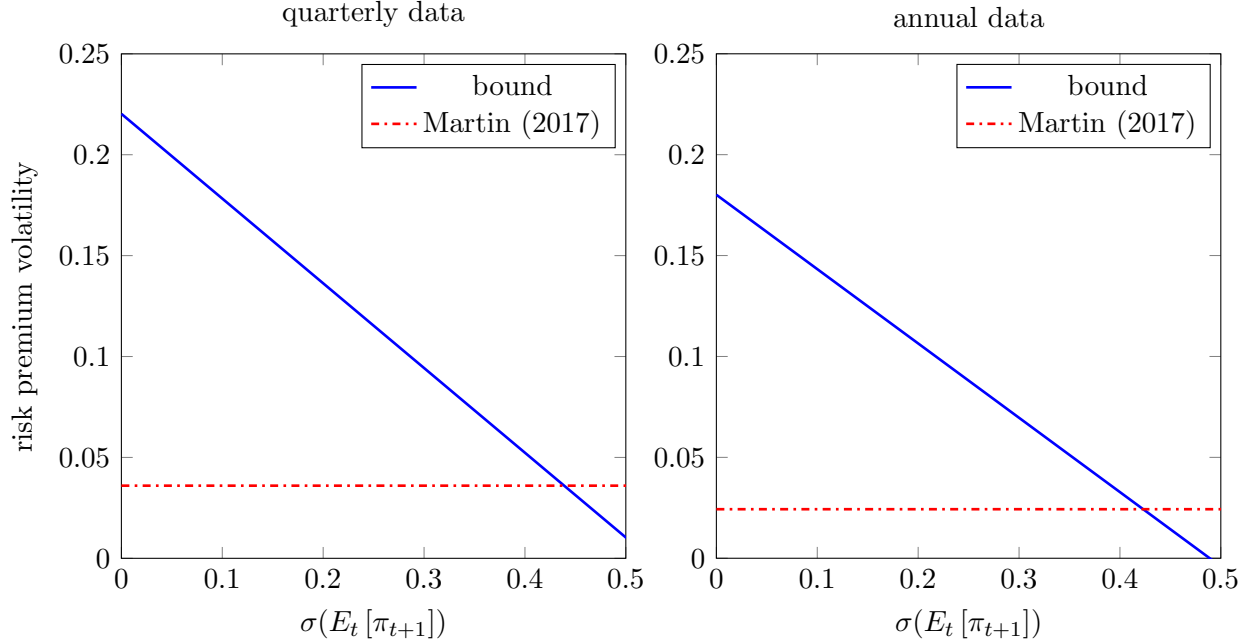
Figure 1: Lower bound on the volatility of the risk premium. Blue solid lines represent the lower bounds of the volatility of the annualized risk premium needed to satisfy the hiring equation, computed as $\overline{\Gamma}\sigma\left(\widehat{\Gamma}_t\right)$ using (4). Red dash-dotted lines are estimates of the volatility of the annualized risk premium from Martin (2017). The average risk premium $\overline{\Gamma}$ is calibrated to the 4.96% and 4.64% at the quarterly and annual frequency, respectively, as in Martin (2017).

this volatility to be 3.6% and 2.43% on the quarterly and annual frequency, respectively. These values are reported as the dash-dotted red lines in the figure.

The figure shows that in order to rationalize the fluctuations in vacancy filling rates in a model with plausibly calibrated time-variation in the risk premium, the required volatility of expected cash flows continues to be very high. In other words, fluctuations in the risk premium can play a dominant role in explaining labor market fluctuations only if they are substantially larger than those observed in the stock market, in the order of at least 10–15 annualized percentage points.

3   Theoretical framework

The above calculations studied the role of the volatility of risk premia, which intertwine the prices of risk embedded in the stochastic discount factor with risk exposures in the cash flows from the marginal worker. In this section, we separate the contributions of the stochastic discount factor and the cash flow process. We start with a stochastic discount process disciplined using financial market data, and then devise a bound that only involves

moments of the cash flow process, such that any cash flow process that does not satisfy the bound also cannot satisfy the hiring equation (1) for the given stochastic discount factor. This will allow us to study the tradeoff that alternative models of the stochastic discount factor generate between fluctuations in expected cash flows and in risk premia as a source of explanation of time-variation in vacancy filling rates.

3.1 Restrictions

Rewriting equation (1) as

$$\frac{\kappa}{q_t} = E_t \left[s_{t+1} \pi_{t+1} \right] + E_t \left[s_{t+1} \left(1 - \delta_{t+1} \right) \frac{\kappa}{q_{t+1}} \right], \tag{5}$$

we can separate out the role of hiring cost from the contribution of the profit flow. We assume that we have available data on vacancy filling rates q_t and separation rates δ_t, and we determined a model of the stochastic discount factor s_{t+1} using outside information. Our focus is to derive restrictions on the firm's profit flow from the marginal worker such that equation (5) holds. Writing the expected profit flow as

$$E_t \left[s_{t+1} \pi_{t+1} \right] = E_t \left[s_{t+1} \right] E_t \left[\pi_{t+1} \right] + Cov_t \left(s_{t+1}, \pi_{t+1} \right) \tag{6}$$

uncovers three sources of variation. When agents are risk-neutral, as in much of the search and matching literature, we have $s_{t+1} = \beta$ and observed differences in vacancy filling rates must be rationalized through fluctuations in firm's expected profits $E_t \left[\pi_{t+1} \right]$. In a more general environment, fluctuations in the conditional mean of the stochastic discount factor $E_t \left[s_{t+1} \right]$ and time-variation in the covariance $Cov_t \left(s_{t+1}, \pi_{t+1} \right)$ contribute to the variation in expected discounted profit flow as well.

Our goal is to provide a stochastic discount factor that is consistent with evidence on cross-sectional differences and time-variation in expected returns, and use it to infer characteristics of a class of processes π_t that are consistent with optimal hiring choice (5). Denoting

$$g_{t+1} = \frac{\kappa}{q_t} - s_{t+1} \left(1 - \delta_{t+1} \right) \frac{\kappa}{q_{t+1}} \tag{7}$$

we have

$$0 = E_t \left[s_{t+1} \pi_{t+1} - g_{t+1} \right]$$

In order to empirically implement the conditional restriction (5), we instrument the equation

as in Hansen and Singleton (1982) with a vector of variables z_t^π, and get

$$0 = E\left[z_t^\pi \left(s_{t+1}\pi_{t+1} - g_{t+1}\right)\right]. \tag{8}$$

The vector of instruments z_t^π contains business cycle variables that serve as predictors of future state of the labor market. Equation (8) states that the Euler equation errors $s_{t+1}\pi_{t+1} - g_{t+1}$ are not systematically related to these predictors.

3.2 Objective

There will typically be many processes π_t consistent with the set of restrictions (8), and both theoretical modeling as well as empirical measurement of π_t are challenging. Theory dictates that the Euler equation (5) holds for the profit flow associated with the *marginal* worker but this profit flow is not directly obtainable from data. The theoretical literature then frequently sidesteps this issue by introducing environments where the average and marginal workers are identical.

Rather than explicitly modeling the profit flow, we aim at asking what are the minimal requirements that every profit flow process consistent with (5) must satisfy. Since the literature has traditionally struggled to generate profit processes that are sufficiently volatile to rationalize fluctuations in the vacancy filling rates, we are looking for the lower bound on dispersion of the profit process for (8) to hold.

Since we can write (6) as

$$E_t\left[s_{t+1}\pi_{t+1}\right] = E_t\left[s_{t+1}\right]E_t\left[\pi_{t+1}\right] + \rho_t\left(s_{t+1}, \pi_{t+1}\right)\sigma_t\left(s_{t+1}\right)\sigma_t\left(\pi_{t+1}\right) \tag{9}$$

where ρ_t is the conditional correlation and σ_t the conditional standard deviation, the two key moments that can offset fluctuations in q_t are $E_t\left[\pi_{t+1}\right]$ and $\sigma_t\left(\pi_{t+1}\right)$. This motivates the objective

$$L_\alpha = \alpha Var\left[E_t\left[\pi_{t+1}\right]\right] + (1-\alpha)E\left[Var_t\left[\pi_{t+1}\right]\right] \tag{10}$$

As a function of α, this objective generates a lower bound on the combinations of variance of the conditional mean and average conditional variance of the profit process such that it is consistent with the restrictions above. When $\alpha = \frac{1}{2}$, then the objective is equal to $\frac{1}{2}Var\left[\pi_{t+1}\right]$. The objective L_α can therefore be thought of as a generalization of the Law of Total Variance.

When $\alpha \searrow 0$, the objective emphasizes the minimization of average conditional variance $E\left[Var_t\left[\pi_{t+1}\right]\right]$. Then, in order to rationalize (5), the process π_t has to become highly predictable and exhibit large fluctuations in the conditional mean $E_t\left[\pi_{t+1}\right]$. This is the only

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way how to resolve the unemployment volatility puzzle when $s_{t+1} = \beta$.

On the other hand, when $\alpha \nearrow 1$, the objective L_α stresses the minimization of the variation in $E_t [\pi_{t+1}]$. From equation (6), we infer that the resolution of the unemployment volatility puzzle must arise from sufficiently large business cycle fluctuations in $Cov_t (s_{t+1}, \pi_{t+1})$. These can emerge from sufficiently large fluctuations in the conditional volatility of the stochastic discount factor but if these are not large enough, conditional volatility of the profit process will have to fluctuate as well.

4   Bounds

For a given model of a stochastic discount factor and data on separation rates and vacancy filling probabilities, the problem is

$$\min_{\pi_{t+1}} L_\alpha \quad \text{subject to} \quad 0 = E \left[z_t^\pi \left(s_{t+1} \pi_{t+1} - g_{t+1} \right) \right]. \tag{11}$$

When $\alpha = \frac{1}{2}$, the problem reduces to finding the minimum variance profit process, and the solution corresponds to the lowest point on the Hansen and Jagannathan (1991) bound.

Proposition 4.1. *The solution to problem (11) satisfies*

$$\pi_{t+1} - E_t [\pi_{t+1}] = \frac{1}{2(1-\alpha)} (s_{t+1} - E_t [s_{t+1}]) (z_t^\pi)' \lambda^\pi \tag{12}$$

$$E_t [\pi_{t+1}] - \bar{\pi} = \frac{1}{2\alpha} (z_t^\pi E_t [s_{t+1}] - E [z_t^\pi s_{t+1}])' \lambda^\pi \tag{13}$$

where the vector of loadings λ^π is given by

$$\lambda^\pi = (V_\alpha)^{-1} \left(E [z_t^\pi g_{t+1}] - E [z_t^\pi s_{t+1}] \bar{\pi} \right)$$

with

$$V_\alpha = \frac{1}{2\alpha} Var [z_t^\pi E_t [s_{t+1}]] + \frac{1}{2(1-\alpha)} E [Var_t [z_t^\pi s_{t+1}]]$$

$$\bar{\pi} = \frac{(E [z_t^\pi s_{t+1}])' V_\alpha^{-1} E [z_t^\pi g_{t+1}]}{(E [z_t^\pi s_{t+1}])' V_\alpha^{-1} E [z_t^\pi s_{t+1}]}. \tag{14}$$

Proof. See Appendix B. □

For each choice of α, Proposition 4.1 yields a profit flow process that is consistent with restrictions in (11) implied by the hiring equation. These alternative profit flows differ in

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how their two conditional moments, $E_t[\pi_{t+1}]$ and $Var_t[\pi_{t+1}]$, contribute to rationalizing the labor market dynamics.

When $\alpha \nearrow 1$, innovations in (12) are large, and the profit process exhibits large conditional variance $Var_t[\pi_{t+1}]$. Fluctuations in vacancy filling rates q_t are then rationalized through time-variation in risk premia, i.e., through movements in the covariance term in the discounted profit flow (6). On the other hand, when $\alpha \searrow 0$, innovations in implies large fluctuations in the conditional mean $E_t[\pi_{t+1}]$ in (13), which leads to large fluctuations in the first-term on the right-hand side of (6).

Innovations to the profit process are conditionally linear in innovations to the stochastic discount factor, while deviations of the conditional mean of the profit process from unconditional mean are conditionally linear in the corresponding deviations of the stochastic discount factor.

The form of the process g_{t+1} in (7) implies that the average level of firm's profit will be proportional to the vacancy cost κ, which can be directly inferred from the solution (14). In the absence of a widely accepted value for this parameter, we normalize the results by the average profit $E[\pi_{t+1}]$. Including information on the vacancy cost parameter κ together with average profitability $\bar{\pi}$, instead of optimizing over $\bar{\pi}$ in (14), would further tighten the bound. The constructed bound is conservative in the sense that the profit flow process is only required to satisfy the set of instrumented restrictions (11), rather than the hiring Euler equation (1) in each time-t state. We will later discuss the role of the choice of the instruments z_t^π in more detail.

4.1   Interpretation

It is useful to highlight the differences of the constructed weighted variance bound on firm's profit relative to the familiar Hansen and Jagannathan (1991) and other similar bounds.

Proposition 4.1 derives the weighted variance bound on firm's profit for a *given* model of the stochastic discount factor. The construction can be implemented empirically for any particular model of the stochastic discount factor, as long as we can provide an observable counterpart to the path s_{t+1}. For instance, the realized path of an Epstein–Zin stochastic discount factor can be constructed using realizations of consumption growth and a proxy for the return on aggregate wealth. In what follows we consider alternative methods for the construction of empirically relevant stochastic discount factors.

The bound on the moments of the profit flow is to be understood as a joint test of the moments of the profit flow and the model of the stochastic discount factor. One interpretation of the bound is to study the tradeoffs between the two moments of the profit flow in (10)

offered by alternative stochastic discount factors. This is useful for the evaluation of the role risk premia play in different models as an explanation of fluctuations in vacancy posting rates. On the other hand, if the constructed bound is to be interpreted truly as the lower bound on the variability of the profit flow that is consistent with the (correct) hiring Euler equation, we need to invoke the following assumption.

Assumption 4.2. *The stochastic discount factor s_{t+1} prices all risks relevant for the hiring decision implied by the hiring Euler equation (1).*

To see the role of this assumption, consider an alternative stochastic discount factor

$$s_{t+1}^{\varepsilon} = s_{t+1} + \varepsilon_{t+1}$$

where ε_{t+1} is a mean-zero perturbation orthogonal to the vector of priced returns but correlated with the profit flow. Then s_{t+1}^{ε} prices the same returns as s_{t+1} but (6) now implies

$$E_t \left[s_{t+1}^{\varepsilon} \pi_{t+1} \right] = E_t \left[s_{t+1} \right] E_t \left[\pi_{t+1} \right] + Cov_t \left(s_{t+1}, \pi_{t+1} \right) + Cov_t \left(\varepsilon_{t+1}, \pi_{t+1} \right)$$

Fluctuations in $Cov_t \left(\varepsilon_{t+1}, \pi_{t+1} \right)$ can now amplify the time-variation in expected discounted profits, thus reducing the need to impose a large dispersion of the profit flow.

4.2   Implementation

In order to provide an empirical implementation of the profit flow process characterized in Proposition 4.1, we need to construct conditional moments $E_t \left[s_{t+1} \right]$ and $E_t \left[g_{t+1} \right]$. The former is the reciprocal of the risk-free rate, which we directly extract from data. In order to construct the latter, we assume that the economy follows a Markov process

$$X_{t+1} = \phi \left(X_t, W_{t+1} \right) \tag{15}$$

with state vector X_t and innovations W_{t+1}, and that the vacancy filling rate q_t and the separation rate δ_t are functions of the Markov state

$$q_t = q \left(X_t \right), \qquad \delta_t = \delta \left(X_t \right).$$

For instance, (15) can be a linear VAR, with q_t and δ_t included as elements of the state vector. We specify the choice of the Markov process in the empirical implementation below.

5   Models of stochastic discount factors

The bound constructed in Proposition 4.1 gives us considerable freedom in terms of the models of stochastic discount factors that we can entertain in the analysis. Here, we outline two approaches that we implement in the next sections. The first is based on a structural model of an Epstein and Zin (1989, 1991) stochastic discount factor with stochastic volatility, derived in Bansal et al. (2014). The second approach is a non-parametric construction of a minimum-variance stochastic discount factor that lies on the Hansen and Jagannathan (1991) bound.

5.1   A structural model with recursive preferences

The representative investor endowed with recursive preferences ranks consumption streams C_t using the continuation value recursion

$$V_t = \left[(1 - \beta) \, C_t^{1-1/\psi} + \beta E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}},$$

where β represents the time preference parameter, γ the relative risk aversion, and ψ the intertemporal elasticity of substitution. We show in Appendix D that the logarithm of the implied one-period stochastic discount factor can be written as a function of the one-period consumption growth rate $\Delta c_{t+1} = \log C_{t+1} - \log C_t$ and the logarithm of the return on aggregate wealth r_{t+1}^w:

$$\log s_{t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) \, r_{t+1}^w \tag{16}$$

where $\theta \doteq (1 - \gamma) / (1 - 1/\psi)$. When $\gamma = 1/\psi$, then the last term in the stochastic discount factor is zero, and the preferences reduce to standard constant relative risk aversion preference with risk aversion γ.

We are interested in an empirical implementation of the stochastic discount factor that features conditional heteroskedasticity. Following Bansal et al. (2014), we specify a linear model for the underlying state

$$X_{t+1} = \phi_x X_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim N \left(0, \Omega_t \right). \tag{17}$$

We include consumption growth as one element of the state vector,

$$\Delta c_{t+1} = \iota_c' X_{t+1}$$

where ι_c is an $n \times 1$ indicator vector. Similarly, we include in X_t a univariate measure of conditional expected variance that drives time-variation in the covariance matrix Ω_t, such that

$$\sigma_t^2 = \iota_\sigma' X_t. \tag{18}$$

Bansal et al. (2014) use realized monthly variation in monthly industrial production growth as a measure of *realized* variance, $RV_{t+1} = \iota_v' X_{t+1}$. The measure of *expected* variance is then given by

$$\sigma_t^2 = E_t\left[RV_{t+1}\right] = \iota_v' \phi_x X_t$$

and hence $\iota_\sigma' = \iota_v' \phi_x$. We also alternatively consider measures of conditional expected variance constructed by Jurado et al. (2015), in which case ι_σ is the indicator vector that selects the particular measure of expected variance included in the vector X_t.

The stochastic discount factor (16) therefore inherits time-variation in conditional volatility from the dynamics of consumption growth and return on aggregate wealth. In order to deal with lack of direct measurement of aggregate wealth, we again proceed as in Bansal et al. (2014) and Lustig and Van Nieuwerburgh (2008) and assume that aggregate wealth portfolio consists of financial wealth and human capital. By combining observable measures of the return on financial wealth, labor income growth and consumption growth in the budget constraint of the representative investor, we can infer a restriction on the dynamics of human capital component of wealth and hence infer implied returns on aggregate wealth. Details are provided in Appendix D.

5.2 Non-parametric construction

The stochastic discount factor introduced in Section 5.1 imposes structural restrictions on the relationship between the dynamics of consumption growth, measured stochastic volatility, and time-variation in risk-premia. As an alternative construction, we consider a non-parametric extraction of the stochastic discount factor that is consistent with pricing restrictions imposed on a set of traded returns. In particular, we assume that investors have unconstrained access to a vector of n securities with one-period real returns R_{t+1}^i, $i = 1, \ldots, n$. Investor optimization implies that the stochastic discount factor satisfies the vector of pricing restrictions

$$E_t\left[s_{t+1} R_{t+1}^i\right] = 1, \qquad i = 1, \ldots, n.$$

In order to implement these restrictions empirically, we proceed in an analogous fashion as in the case of implementing the hiring restriction (8) and use a vector of instruments z_t^s.

This implies the set of restrictions

$$0 = E\left[z_t^s\left(s_{t+1}R_{t+1}^i - 1\right)\right], \qquad i = 1,\ldots n. \tag{19}$$

There are typically many stochastic discount factors consistent with (19), and we choose the one with the lowest variance. This stochastic discount factor is obtained by solving

$$\min_{s_{t+1}} \frac{1}{2}E\left[(s_{t+1} - E\left[s_{t+1}\right])^2\right] \quad \text{subject to} \quad 0 = E\left[z_t^s\left(s_{t+1}R_{t+1}^i - 1\right)\right], \qquad i = 1,\ldots n. \tag{20}$$

The variance-minimizing stochastic discount factor lies at the lowest point of the Hansen–Jagannathan bound and the solution has a structure analogous to that for the profit process π_{t+1} in Proposition 4.1.[2] The solution to this optimization problem is summarized in the following proposition.

Proposition 5.1. *The minimum-variance stochastic discount factor that solves problem (20) is given by*[3]

$$s_{t+1} = \bar{s} + \left(z_t^s \otimes R_{t+1} - E\left[z_t^s \otimes R_{t+1}\right]\right)' \lambda^s$$

where the vector of loadings λ^s and the mean of the stochastic discount factor are

$$\lambda^s = \left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1} \left(E\left[z_t^s \otimes \mathbf{1}\right] - E\left[z_t^s \otimes R_{t+1}\right]\bar{s}\right)$$

$$\bar{s} = \frac{\left(E\left[z_t^s \otimes R_{t+1}\right]\right)'\left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1}E\left[z_t^s \otimes \mathbf{1}\right]}{\left(E\left[z_t^s \otimes R_{t+1}\right]\right)'\left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1}E\left[z_t^s \otimes R_{t+1}\right]}.$$

Proof. See Appendix B. □

The stochastic discount factor is therefore a conditionally linear function of returns, and variation in instruments serves as fluctuations in the its conditional mean and conditional volatility. In order to obtain conditional moments of the stochastic discount factor needed to construct the profit bound, we project the constructed path of s_{t+1} on the vector of predictors (17).

---

[2]Problem 20 ignores the usual no-arbitrage restriction that the stochastic discount factor has to remain strictly positive. See Hansen and Jagannathan (1991) and Hansen et al. (1995) for analysis with restrictions that assure positivity of the solution. We abstract from this restriction for reasons of analytical tractability.

[3]Here, $\otimes$ is the Kronecker product. For two column vectors $x = \left(x^i\right)_{i=1}^k$ and $y = \left(y^j\right)_{j=1}^n$, $x \otimes y$ is a column vector of length $k \times n$ with $(x \otimes y)^{(i-1)n+j} = x^i y^j$. For instance, we can write the set of restrictions (19) in concise form as $0 = E\left[z_t^s \otimes (s_{t+1}R_{t+1} - \mathbf{1})\right]$ where $\mathbf{1}$ is a column vector of ones of length n.

6 Data

We use quarterly macroeconomic and financial data for the period 1953Q1–2017Q4. The state vector X_t includes several macroeconomic and financial variables, mostly following Bansal et al. (2014),

$$X_t = \left[1, \Delta c_t, \Delta y_t, RV_t, q_t, R_t^f, R_t^m, pd_t\right],$$

where 1 is a constant, Δc_t consumption growth, Δy_t income growth, R_t^f the risk-free rate, R_t^m market return, pd_t logarithm of the price-dividend ratio, RV_t a measure of realized variance, and q_t vacancy-filling rate.

The consumption and income growth are constructed from the BEA series for real personal consumption expenditures per capita and real disposable personal income per capita. The realized variance RV_t is the sum of squared monthly growth rates of the real per capita industrial production over the last 12 months,

$$RV_t = \sum_{m=0}^{11} \left(\frac{id_{t-m/3} - id_{t-(m+1)/3}}{id_{t-(m+1)/3}}\right)^2.$$

The vacancy filling probability is the key labor market variable in our setting. Vacancy data have been provided to us by Nicolas Petrosky-Nadeau, who combined several data sources to create a consistent time-series of vacancies for the 1929–2016 period; the details are explained in Petrosky-Nadeau and Zhang (2013). We follow Shimer (2012) in constructing the job finding rate f_t and job separation rate δ_t using data on civilian employment level, unemployment level and number of civilians unemployed for less than 5 weeks. We use data on unemployment rate u_t, vacancy rate v_t and job finding rate f_t to construct the vacancy filling rate as $q_t = f_t/\theta_t$, where $\theta_t = v_t/u_t$ is the market tightness. We note that the job finding and job separation rates correspond to movements between employment and unemployment.

The risk free rate R_t^f is the one-month T-bill rate and R_t^m the value-weighted U.S. market return. Both series are downloaded from Kenneth French's website and net of realized inflation. To construct the price dividend ratio, we download monthly returns on the value-weighted stock market index, including dividends $vwrid_t$ and excluding dividends, $vwred_t$. Denoting P_t and D_t the price and dividend flow, respectively, we have

$$vwrid_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1, \quad vwred_{t+1} = \frac{P_{t+1}}{P_t} - 1.$$

We use these two formulas to obtain a series of prices and dividends. Starting from $P_0 = 1$, we recursively update $P_t = P_{t-1}\left(1 + vwred_{t+1}\right), D_t = P_{t-1}\left(vwrid_t - vwred_t\right)$. Our quarterly
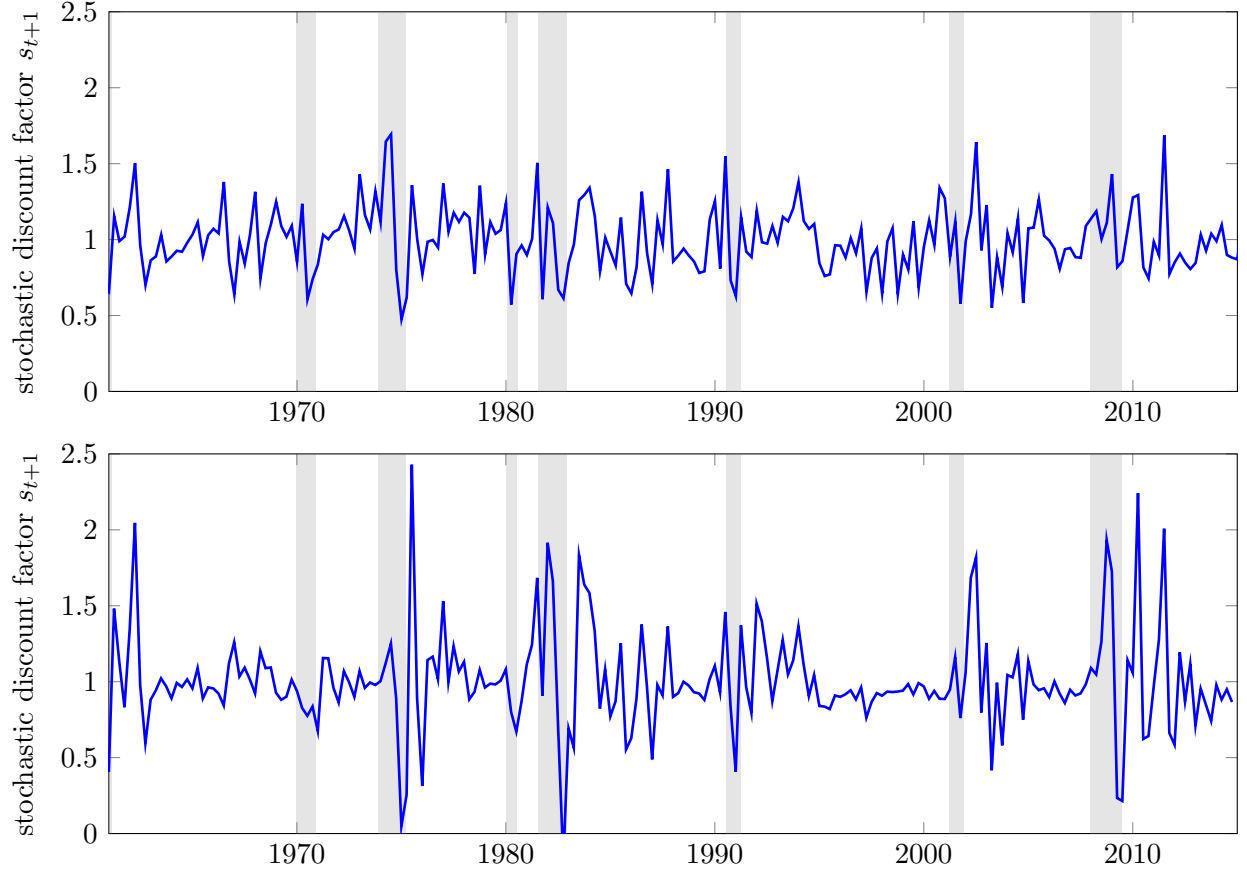
Figure 2: Extracted path of quarterly realizations of the stochastic discount factors. *Top panel:* Recursive-preference stochastic discount factor. *Bottom panel:* Minimum variance stochastic discount factor. NBER recessions shaded.

measure of the price dividend ratio is the price P_t at the end of the quarter divided by the sum of dividends over the last 12 months.

We also construct a pair of empirical proxies for the profit flow. First, we construct real profits per worker using after-tax corporate profits from NIPA, divided by the GDP deflator and employment. We detrend the time series by real potential GDP per worker.

We also measure real per capita profits in Compustat. We use operating income before depreciation (OIBDPQ), deflated by the GDP deflator and divided by employment as measured in Compustat. Employment in Compustat is available only at annual frequency, and we linearly interpolate the data to obtain a quarterly data series. We again detrend by dividing the time series by real potential GDP per worker.
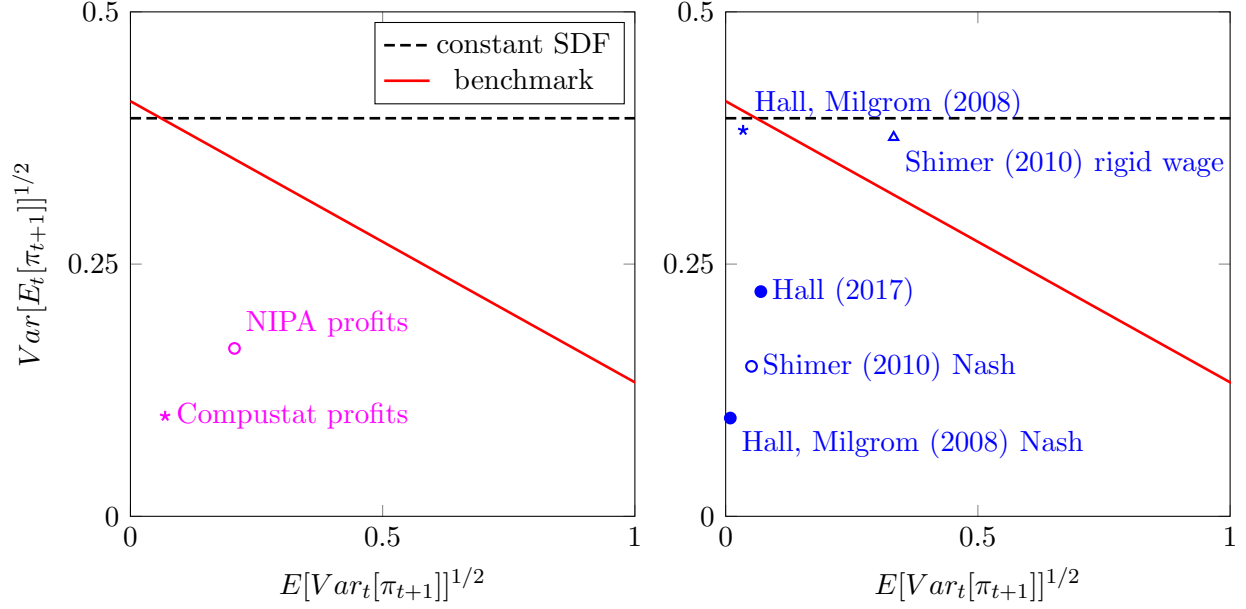
Figure 3: Normalized weighted variance bound for the profit process. The horizontal axis represents the square root of the average conditional variance of the profit process, while the vertical axis the square root of the variance of its conditional expectation. Both quantities are normalized by the average profit $E[\pi_{t+1}]$. The black dashed line is the bound constructed for a constant stochastic discount factor. The solid red line is the bound constructed using BKSY(2014) stochastic discount factor. The magenta dots represent empirical proxies for the moments. Blue dots are model-implied moments calculated from models used in the labor literature.

7 Results

Figure 2 depicts paths of the two stochastic discount factors outlined in Section 5. The top panel shows the structural stochastic discount factor based on Epstein–Zin preferences and stochastic volatility. The bottom panel represents the minimum variance stochastic discount factor that uses R_t^m and R_t^f as vector of priced returns, and the state vector X_t as the vector of instruments z_t^s in (19). Both stochastic discount factors exhibit substantial countercyclical fluctuations in conditional variance, with negligible movements in conditional mean, both of which are needed to jointly rationalize countercyclical risk premia and the relatively stable real risk-free rate.

7.1 Weighted variance bound

We can now implement problem (11) using observable data for vacancy filling rates, separation rates and the path of the stochastic discount factor. For alternative values of $\alpha \in (0, 1)$, we obtain a locus of points in the space of the average conditional variance $E[Var_t[\pi_{t+1}]]$ and variance of the conditional expectation $Var[E_t[\pi_{t+1}]]$ that form a bound above which

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all profit processes satisfying restrictions (8) must lie. The roots of these two moments are depicted on the horizontal and vertical axes in Figure 3, respectively.

The hiring Euler equation (1) is homogeneous degree one in (κ, π_{t+1}). In the absence of an agreed upon value of the vacancy posting cost κ, we impose a normalization and plot the roots of the average conditional variance $E\left[Var_t\left[\pi_{t+1}\right]\right]$ and variance of the conditional expectation $Var\left[E_t\left[\pi_{t+1}\right]\right]$ normalized by the mean profit $E\left[\pi_{t+1}\right]$.

The left panel in Figure 3 shows the bounds for two alternative stochastic discount factors. In both cases, the vector of instruments z_t^π used in the minimization problem (11) only involves the constant and the vacancy filling rate q_t.

The black dashed line represents the risk-neutral case $s_{t+1} \equiv \beta$. This bound is horizontal. Since the covariance term in (6) is zero, the only way how to rationalize fluctuations in vacancy filling rates is to generate sufficiently large fluctuations in expected profit flow $E_t\left[s_{t+1}\right]$, while conditional volatility $\sigma_t\left[\pi_{t+1}\right]$ is irrelevant. For the given processes for vacancy filling rate q_t and separation rate δ_t, the given risk neutral stochastic discount factor $s_{t+1} \equiv \beta$, and uncertainty driven by the Markov process X_t in (17), any profit flow process that is to satisfy the hiring Euler equation (1) must exhibit sufficiently large fluctuations in the conditional mean (sufficiently large $Var\left[E_t\left[\pi_{t+1}\right]\right]$) to lie above the black bound.

The solid red line represents the profit variability bound constructed using the Bansal et al. (2014) model for the stochastic discount factor. The bound is downward sloping, implying that the bound can be satisfied using a profit process with high volatility of the conditional mean (i.e., a process that is highly predictable, the usual channel in the search literature), or a process with high conditional volatility, the new channel operating through time-varying risk premia on hiring a worker, or a combination of both, as we move along the bound.

The steeper the bound, the larger is the set of profit flow processes that are consistent with the restrictions in (8). The last term in expression (9) implies that a more volatile stochastic discount factor with more countercyclical volatility will generate large fluctuations in risk premia without the need for excessive volatility of the profit flow. The average volatility of the profit flow is depicted on the horizontal axis in Figure 3, and stochastic discount factors with more countercyclical volatility will typically lead to steeper bounds.

7.2   Empirical and model implied measures of the profit flow

We now study alternative empirical proxies for the profit flow as well as theoretical processes implied by different models used in the labor market literature, and confront them with the bound we inferred in the previous section. These results are also depicted in Figure 3.

First, we examine two empirical measures of average aggregate profits per worker, computed from NIPA and Compustat profits, respectively, that can be used as proxies for the theoretical profit from the marginal worker. For both proxies of the profits, we compute $E[Var_t[\pi_{t+1}]]$ and $Var[E_t[\pi_{t+1}]]$ using a projection on the vector of instruments used in equations (8) and (19). We again normalize both objects by the mean value of the profit flow, $E[\pi_{t+1}]$. Magenta dots in the left panel of Figure 3 represent these data.

These profit flow measures lie significantly below both bounds. This implies that introducing a stochastic discount factor that generates time-varying risk premia consistent with data from financial markets has only limited capacity in rationalizing fluctuations in vacancy filling rates in the labor market. To put it differently, models calibrated to be consistent with variability of these time series would not be able to explain the unemployment volatility puzzle.

It is important to stress the discrepancy between these measures of profit per worker and the theoretical object we are interested in. The theoretical model predicts that π_{t+1} corresponds to the profit flow from the marginal worker. Constructed empirical proxies, however, represents average profits per worker. Hence a failure of this empirical proxy to meet the bound can potentially be attributable to the discrepancy between average and marginal profits from a worker. In absence of empirical measures of marginal profits, one potential avenue is to use the distance between the data points and the bound as a constraint on the wedge implied by a theoretical model that distinguishes between average and marginal worker profit.

In the right panel of Figure 3, we depict profit flow processes obtained in theoretical models constructed by Hall and Milgrom (2008) (with Nash and alternative offer bargaining, respectively), Shimer (2010) (with Nash bargaining and rigid wages, respectively), and Hall (2017) (with a model of the stochastic discount factor that rationalizes business cycle movements in price-dividend ratios). These models introduce specific mechanisms that alter the baseline Diamond–Mortensen–Pissarides search and matching model to rationalize fluctuations in vacancy filling rates.

For each of these models, we then construct $E[Var_t[\pi_{t+1}]]$ and $Var[E_t[\pi_{t+1}]]$ directly using the structure of models. The details are explained in the Appendix C. We normalize both objects by the mean value of the profit flow, $E[\pi_{t+1}]$. Each blue dot in the right panel of Figure 3 represents one of these models.

Most of these profit flows exhibit very small conditional volatility, and most of the variation comes in the business cycle fluctuations of the conditional mean. These profit flows were designed to rationalize the labor market dynamics in models with essentially risk-neutral firms, and hence the only way how to meet the hiring Euler equation is through
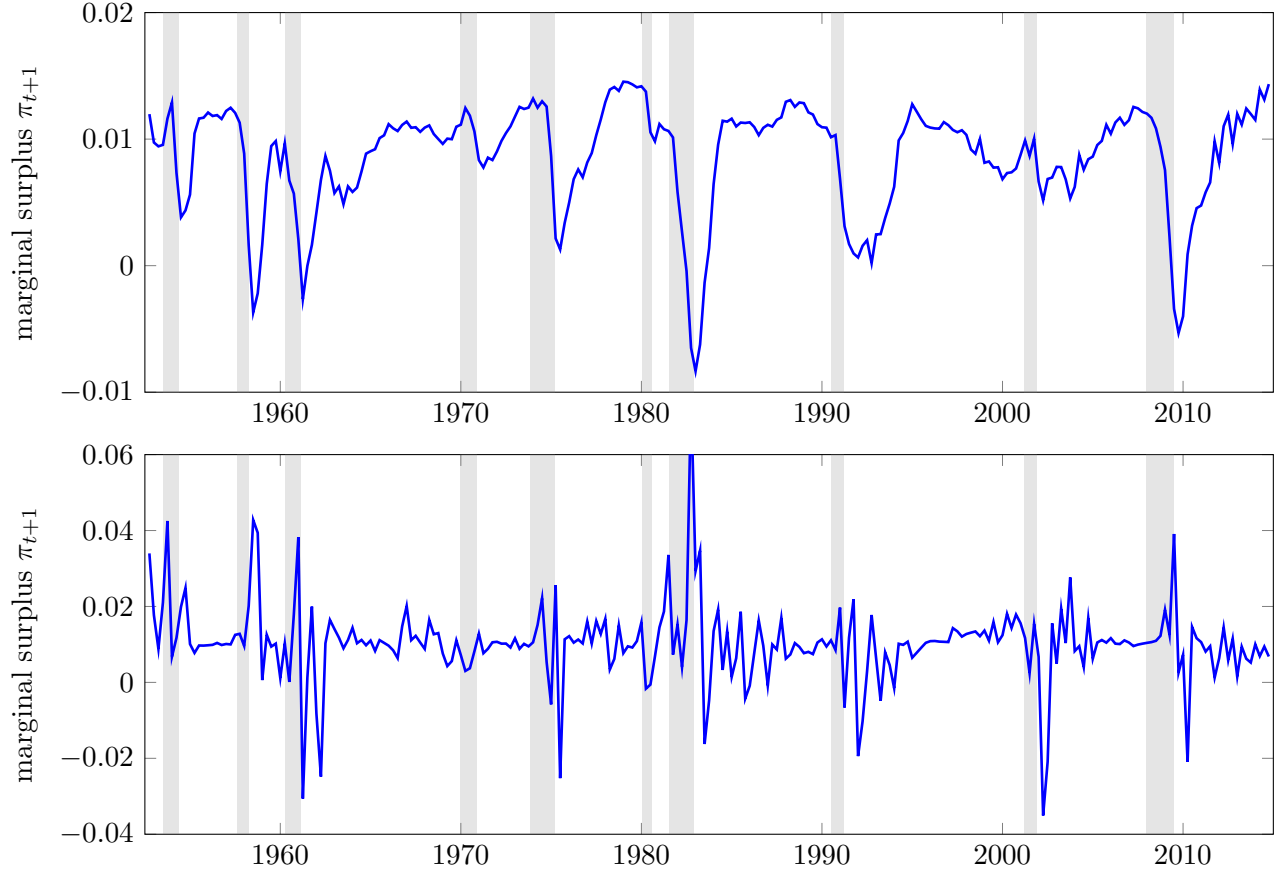
Figure 4: Alternative profit flow paths satisfying the weighted variance bounds. *Top panel*: $\alpha = 0.001$: high weight on minimizing conditional variance. *Bottom panel*: $\alpha = 0.999$, high weight on minimizing variance of conditional expectation. NBER recessions shaded.

fluctuations in expected profits. This is the essence of the alternative offer bargaining and rigid wage mechanisms introduced in Hall and Milgrom (2008) and Shimer (2010), respectively. Interestingly, these two specifications essentially meet our bound for $s_{t+1} \equiv \beta$, while the Nash bargaining specifications do not, indicating that the bound is sufficiently tight to meaningfully discriminate profit flow processes that do not rationalize observed fluctuations in vacancy filling rates.

Figure 4 displays two alternative extracted paths of firms' profit flow that satisfy the weighted variance bound represented by the red line in Figure 3. The top panel in Figure 4 shows the path for $\alpha = 0.001$, which corresponds to the top left end of the bound. The trajectory now exhibits low conditional volatility and a high predictability over the business cycle. On the other hand, the bottom panel depicts the path for weight $\alpha = 0.999$, which corresponds to the bottom right end of the bound. The path is visibly heteroskedastic, with no clear predictable pattern for its conditional mean. While these two processes have very

different stochastic properties, they both satisfy the set of restrictions in (8). The bound thus provides guidance for alternative ways in which modeled or measured cash flow processes can be consistent with the dynamics of hiring in the labor market.

7.3 Robustness and the role of instruments

The optimization problem for the bound (11) uses a vector of instruments z_t^π. Here we discuss how the choice of instruments affects results.

The restriction in the optimization problem is that the errors in the hiring Euler equation cannot systematically move with the instrumenting variables. Hence, variables which capture the business cycle well are the natural choice of instruments. Note that with one instrument, a suitably chosen constant profit flow $\pi_{t+1} = \bar{\pi}$ satisfies the restriction. Since the value of the objective L_α is zero for any value of α in this case, a constant profit flow is the solution to (11). This is not an interesting solution. Therefore, we need at least two instruments. Increasing the number of instruments further adds additional restrictions on the profit flow and as a result, the minimum will be reached at higher values, which shifts the bound outward.

Our benchmark calculation includes two variables in z_t^π, a constant and the vacancy filling probability q_t. We depict the resulting bound again as the solid red line in Figure 5. As we have seen, even with a single (nonconstant) instrument q_t the bound is sufficiently tight to discriminate against models with low fluctuations in the profit flow.

Adding instruments tightens the bound. The dotted line represents the bound when all variables from the state vector X_t are included as instruments. Excluding q_t from the list of instruments, the bound shifts substantially downward to the dash-dotted line.

Which instruments to add to z_t^π? Macroeconomic models typically strive to capture comovement between a small set of variables of interest, which then lend themselves as instruments.

The right panel of Figure 5 shows robustness to several alternative specifications. In the first one, we keep the conditional mean of the stochastic discount factor constant, which results in a constant risk-free rate. This leaves the bound essentially unchanged (dashed red line). In the second alternative specification, we exclude δ_t from the set of state variables and impose that it is constant over time and equal to its mean value over the studied period. The separation rate is countercyclical, which makes hiring a worker in a recession less valuable due to lower expected duration of the match. Turning off this channel by making the separation rate constant means that we need more variable profit flow. Indeed, the bound shifts to the right, as depicted by the dotted red line. Finally, we decrease the risk aversion parameter in
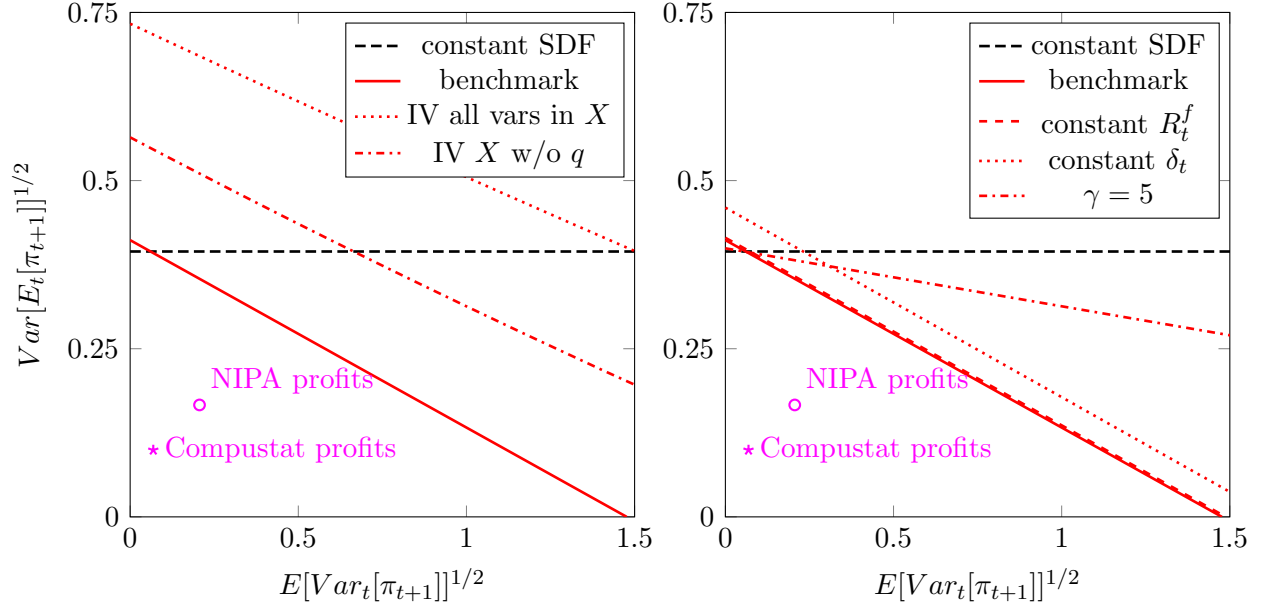
Figure 5: Normalized weighted variance bound for the profit process. The horizontal axis represents the square root of the average conditional variance of the profit process, while the vertical axis the square root of the variance of its conditional expectation. Both quantities are normalized by the average profit $E[\pi_{t+1}]$. Alternative lines represent bounds constructed for choices of instrumenting variables z_t (left panel) or different stochastic discount factors (right panel). Left panel: the solid red line is the benchmark bound, the dotted red line uses all state variables X as instruments in (11), the dash dotted red line uses all state variables X expect for vacancy filling rate q as instruments. Right panel: the dashed red line corresponds to a SDF with constant conditional mean, the dotted red line corresponds to an SDF with constant job separation rate, the dash dotted red line corresponds to a SDF with risk-aversion parameter 5.

our baseline specification from $\gamma = 12$ to $\gamma = 5$. The stochastic discount factor now has a lower conditional volatility, and therefore the conditional volatility of the profit flow has to increase. The bound becomes flatter, as depicted by the dash dotted line.

7.4   Alternative stochastic discount factors

Figure 6 includes bounds for the profit flow process constructed using different models of the stochastic discount factor. In the left panel, the stochastic discount factor is constructed as in Proposition 5.1, using the market return R_t^m and the risk-free rate R_t^f as priced returns, and conditioning using X_t as the vector of instruments. The bound is very close to that constructed using the structural model of the stochastic discount factor.

   In the right panel, we construct the bound using the theoretical model of the stochastic discount factor introduced in Hall (2017). In that model, the state space consists of a grid of five states, and hence dummy variables for each of these states serve as conditioning
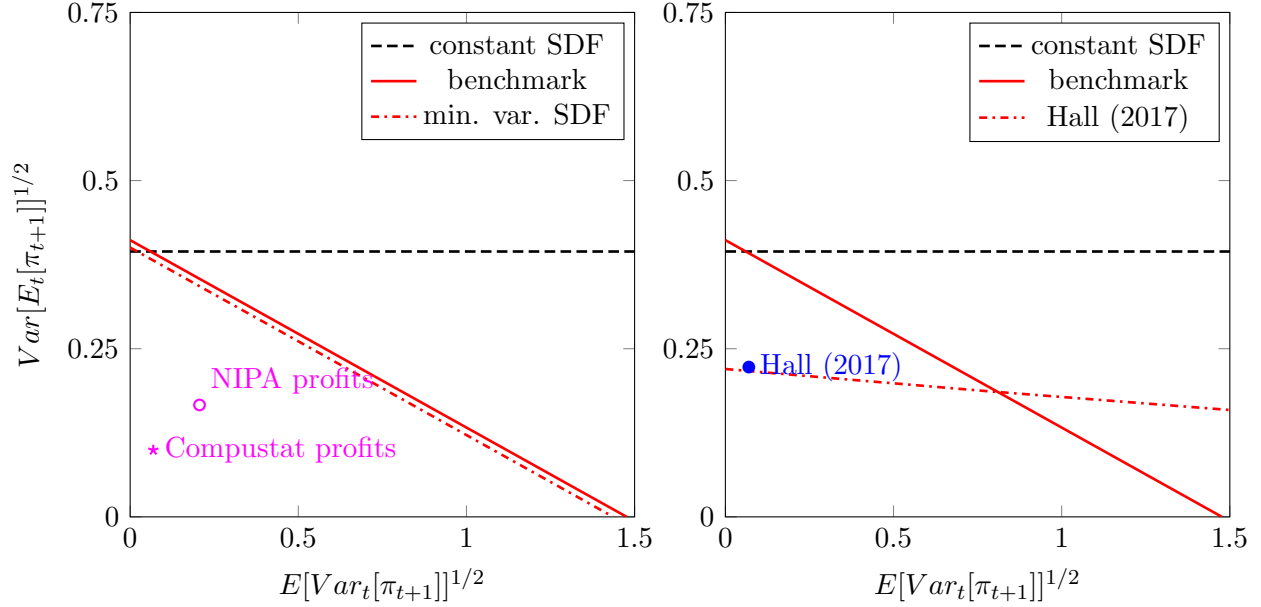
Figure 6: Normalized weighted variance bounds for alternative models of the stochastic discount factor. The dash-dotted line in the left panel represents the bound constructed using the minimum variance stochastic discount factor. In right right panel, it corresponds to the stochastic discount factor from Hall (2017).

variables. This makes the vector of instrumented restrictions (8) equivalent to (1).

Interestingly, this bound is almost flat, and also substantially below the bound constructed using $s_{t+1} \equiv \beta$. The first property implies that fluctuations in risk premia play a negligible role in Hall (2017). The key to understanding the second property is the fact that the model of the stochastic discount factor in Hall (2017) generates a strongly procyclical conditional mean of the stochastic discount factor (i.e., a strongly countercyclical risk-free rate). In other words, the covariance term in (6) is quantitatively small, so a strongly procyclical first term on the right-hand side of (6) is needed to rationalize fluctuations in vacancy filling rates. Empirically, the risk-free rate is smooth, so we require a high $Var\left[E_t\left[\pi_{t+1}\right]\right]$, which is mirrored in the tight dashed line in Figure 6. However, the strong procyclicality of $E_t\left[s_{t+1}\right]$ in Hall (2017) implies that a lower variation $Var\left[E_t\left[\pi_{t+1}\right]\right]$ is needed to explain labor market fluctuations.

Finally, it is worth noting that the profit flow process generated in Hall (2017)'s model (blue dot) lies almost exactly on the bound constructed using the model implied stochastic discount factor. This is not surprising, as the model was designed to essentially fit vacancy filling rates state by state. Were the blue dot below the dash-dotted line, it would imply that the model does not exhibit sufficient dispersion in the profit flow to rationalize the instrumented version of the hiring equation.

7.5 A broader interpretation of profit flow

Our focus lies in the study of the discounted profit flow $E_t[s_{t+1}\pi_{t+1}]$. So far, we have interpreted π_{t+1} as the net cash flow the firm earns from hiring the marginal worker, discounted by the stochastic discount factor inferred from financial market data, and assumed that firms have the same unconstrained access to financing as investors in the financial market. However, if firms face financial constraints, then the corresponding discounted profit flow is given by

$$E_t[s_{t+1}\lambda_{t+1}\pi_{t+1}] \doteq E_t[s_{t+1}\widetilde{\pi}_{t+1}]$$

where $\widetilde{\pi}_{t+1}$ is the profit flow adjusted by the shadow prices of the borrowing constraints in individual future states relative to today, λ_{t+1}. These shadow prices emerge, for instance, in situations when firms need to borrow to finance the cost of hiring or to pre-pay wages. In such situations, an empirical proxy for profits constructed using NIPA or Compustat data would provide a biased measure of $\widetilde{\pi}_{t+1}$, and λ_{t+1} could explain, at least to some extent, the distance between the magenta points in Figure 3 and the weighted variance bounds. To answer this question, one needs to construct an empirical or theoretical measure of the fluctuations in these shadow prices λ_{t+1}.

This interpretation also provides a possible reconciliation between our bound and the results in the Hall (2017) model. If we write the discounted profit flow as $E_t[\widetilde{s}_{t+1}\pi_{t+1}]$ where $\widetilde{s}_{t+1} = s_{t+1}\lambda_{t+1}$, then $\widetilde{s}_{t+1}$ would not be the stochastic discount factor pricing assets in unconstrained markets but rather a constraint-adjusted stochastic discount factor that takes into account the role of pricing impact of constrained borrowing. We leave the exploration of this topic for future work.

8 Predicted unemployment dynamics

We now ask a different question: Are fluctuations in the value of the job large enough to explain unemployment fluctuations?

In particular, we use the stochastic discount factor and alternative profit flow processes to compute the value of the job, J_t. We then use it to predict the path of the unemployment rate.

We first consider a counterfactual profit flow that lies exactly on the bound that we have constructed. Given that this process has been constructed to be consistent with the fluctuations in the vacancy filling rate, it has the potential to generate enough volatility in the unemployment rate. However, it is not guaranteed that it will be consistent with other moments of the unemployment dynamics, for example its persistence, since this is
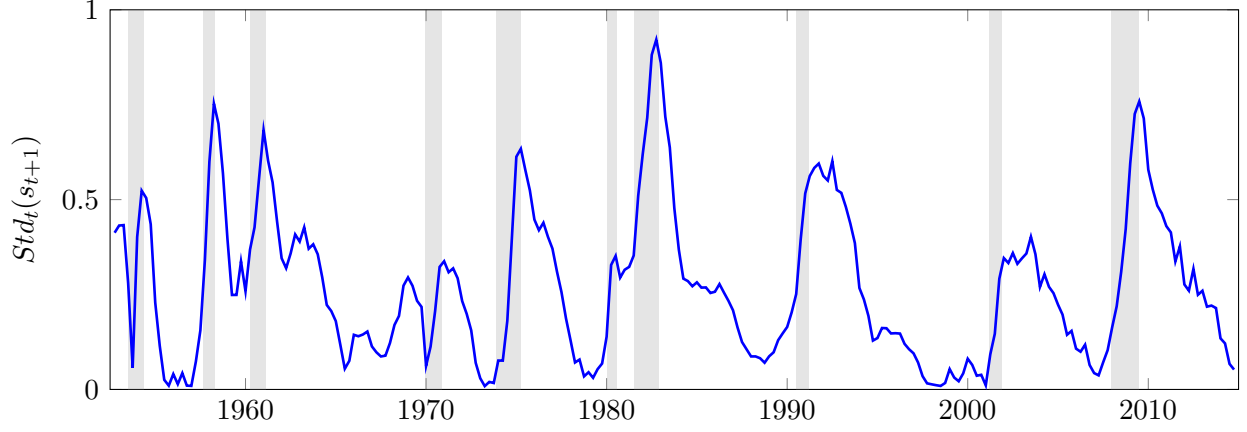
Figure 7: SDF volatility NBER recessions shaded.

left unconstrained in our calculation. We therefore investigate whether a profit flow process consistent with the bound can also translate into a plausible path of the unemployment rate.

We proceed as follows. We first use time series for s_{t+1}, π_{t+1} and δ_{t+1} to construct the value of the job J_t as the present discounted value of the profit flow stream.[4] We then use the hiring Euler equation, the law of motion for unemployment and a Cobb-Douglas matching function to construct a time series for market tightness and the unemployment rate:

$$
\begin{aligned}
\frac{\kappa}{q\left(\theta_t\right)} &= J_t \\
u_{t+1} &= f\left(\theta_t\right) u_t + \delta_t \left(1 - u_t\right) \\
q\left(\theta_t\right) &= B\theta_t^{-\eta}, \quad f\left(\theta_t\right) = B\theta_t^{1-\eta}
\end{aligned}
$$

In particular, for given value of J_t, the Euler equation determines $q\left(\theta_t\right)$. We then use the matching function to recover θ_t and $f\left(\theta_t\right)$, and the law of motion for unemployment to determine calculate u_{t+1}.

We choose the elasticity of the matching $\eta = 0.72$ following Shimer (2005), and normalize B to match the mean unemployment rate in the simulated path to the one measured in the data.

Figure 8 plots the time series for $J\left(X_t\right)$ constructed using the two extracted time series for firms' profit flow that satisfy the weighted variance bound for $\alpha = 0.001$ and $\alpha = 0.999$, respectively. In both cases, these values fall dramatically during recessions, although for different reasons. In the top panel, $J\left(X_t\right)$ falls because the conditional mean $E_t\left[\pi_{t+1}\right]$ falls

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[4]This calculation utilizes formulas on conditional expectations in exponential-quadratic framework given in Appendix E.
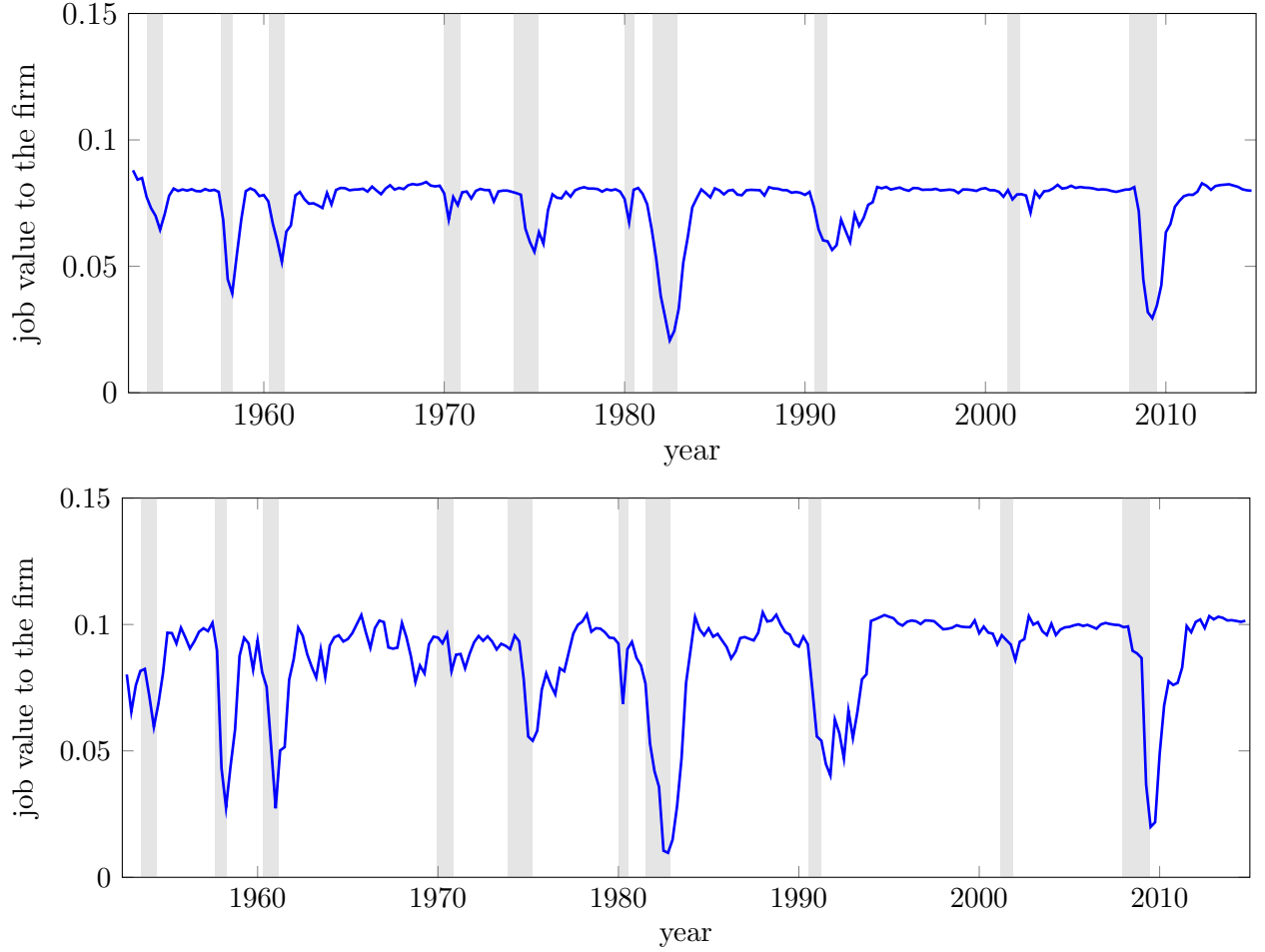
Figure 8: Value of the job using surplus from bounding exercise, Top: $\alpha = 0.001$: high weight on minimizing the expected conditional variance. Bottom: $\alpha = 0.999$: high weight on minimizing variance of conditional expectation.

substantially. In the bottom panel, $J(X_t)$ falls because the conditional covariance between s_{t+1} and π_{t+1} becomes more negative.

Figure 9 shows the calculated paths for the unemployment rate constructed using three different profit flow processes. The top two panels show results for profit flow processes satisfying the bound for $\alpha = 0.001$ and $\alpha = 0.999$, respectively. The bottom panel uses the empirical proxy for the profit flow using NIPA tables.

The profit flows which lie on the bound highly successful in delivering a path for the unemployment rate that lines up well with data, though they operate through different channels. The top one generates the time-variation in incentives to hire through time-varying covariance with the stochastic discount factor while the middle one generates the flucutations in the hiring rate through fluctuations in the expected profit flow. Recall that the empirical proxy does not satisfy the bound and indeed this exericise confirms that it

	std	autocorr.	corr. with data
actual	0.130	0.906	–
predicted; π_t for $\alpha = 0.001$	0.113	0.838	0.753
predicted; π_t for $\alpha = 0.999$	0.153	0.857	0.730
predicted; NIPA π_t	0.064	0.710	0.633

Table 1: Summary of the predicted unemployment rate. The first row shows the standard deviation and autocorrelation of the actual log unemployment rate. The other rows show these statistics for predicted unemployment rates using different profit flows: the second and third row use profit flow lying on the bound for $\alpha = 0.001$ and $\alpha = 0.999$, respectively. The fourth row shows uses profit flow from NIPA. The last column shows correlation of the predicted and actual unemployment path.

does not feature enough variation to explain the unemployment dynamics.

Table 1 shows several moments of the predicted unemployment paths and compares it to actual unemployment rate. It only confirms the conclusion from Figure 9.

The conclusion from this exercise is that the bound is meaningful: profit flows lying on the bound are consistent with actual unemployment dynamics while those lying inside the bound are not. Moreover, this also demonstrates how one can relate the distance of a particular profit flow process from the bound to how much variation of the actual unemployment rate is explained by that particular profit flow.

9   Directions for future research

In this paper, we connect the pricing of returns in the stock market with the pricing of cash flows accrued by the firm from worker-firm matches. We start by constructing a non-parametric lower bound on two moments of the profit flow the firm receives from the marginal worker that must be satisfied in order for the hiring Euler equation to hold. This weighted variance bound, while conservative, is able to discriminate among theoretical models used in the literature as well as among empirical proxies for the marginal profit flow. The bound is constructed conditional on a model of a stochastic discount factor that prices financial assets instrumented by a vector of variables that capture business-cycle variation in risk premia.

The properties of the profit flow and stochastic discount factor consistent with the bound and returns on financial assets lead us to construct a parametric model from which we infer fluctuations in the value of the worker to a firm. When constructed using inferred profit processes consistent with the weighted variance bound, these fluctuations in the value of the worker are able to match the business-cycle volatility of the unemployment rate.

Once we restrict the model of the stochastic discount factor to be consistent with the

Epstein–Zin recursive preference specification with a time-varying price of risk, and inform the profit flow process using an empirical proxy, the ability of the model to generate empirically observed fluctuations in the unemployment rates decreases, although it remains substantial. We argue that the remaining wedge can be attributable to several factors, including the discrepancy between the profit from the average and the marginal worker, as well as shadow prices on financial constraints the firms are facing, relative to investors in financial markets. A further study of the role of these constraints is left for future work.

Similarly, it would be interesting to study the implications of this weighted variance bound for cross-sectional firm-level and industry-level data. If we interpret the wedge between the bound and the moments measured using observed cash flows as the contribution of financial constraints faced by different firms, then our framework allows to non-parametrically study the distribution of these wedges across firms and industries, allowing us to measure the cross-sectional heterogeneity in the impact of borrowing constraints for the unemployment dynamics.
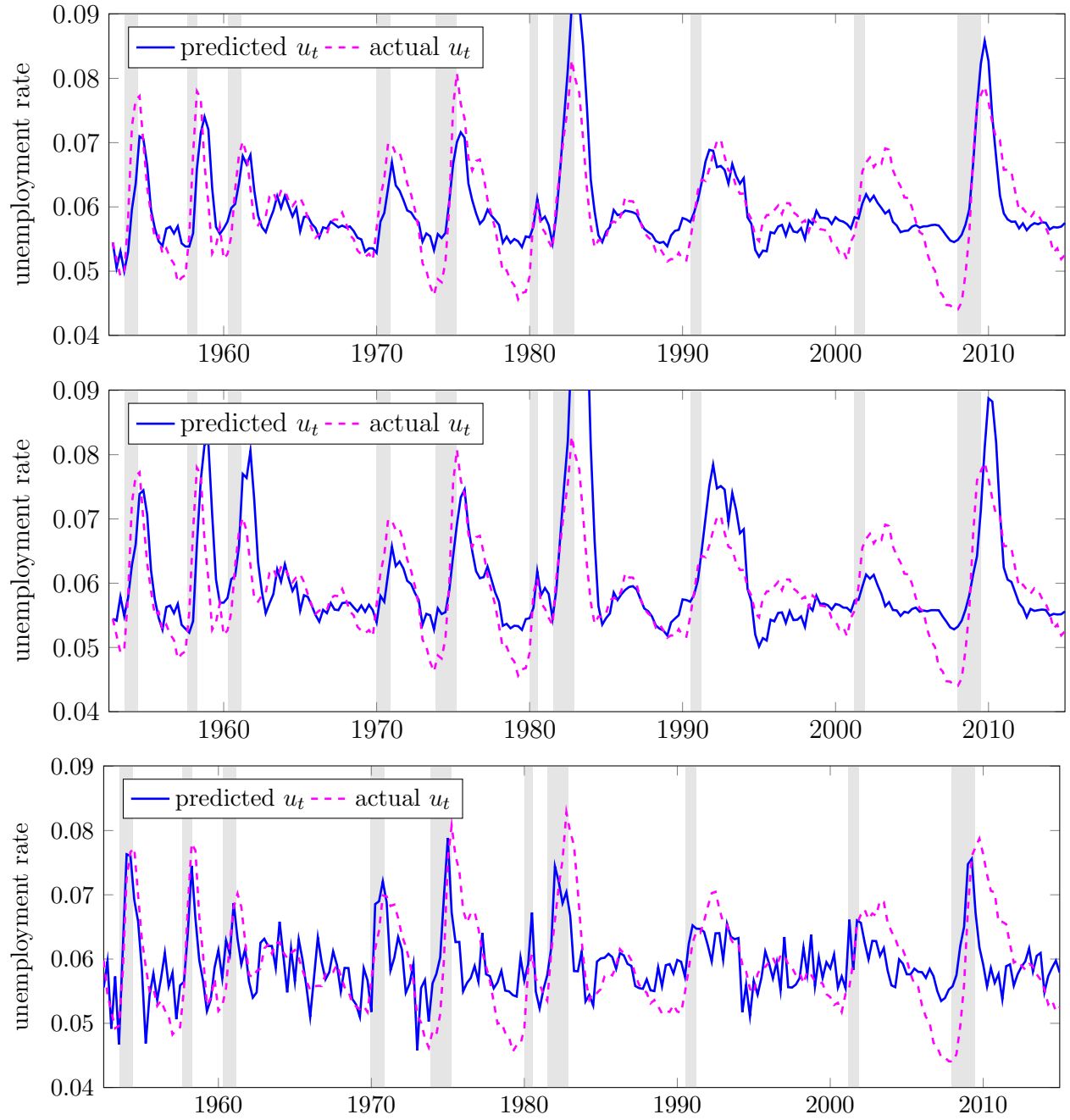
Figure 9: Predicted unemployment rate constructed using J_t generated using alternative profit flow processes. *Top panel*: profit flow lying on the bound for $\alpha = 0.001$, high weight on minimizing the expected conditional variance. *Middle panel*: profit flow lying on the bound for $\alpha = 0.999$, high weight on minimizing variance of conditional expectation. *Bottom panel*: profit flow constructed using NIPA tables.

References

Bansal, Ravi, Dana Kiku, Ivan Shaliastovich, and Amir Yaron. 2014. Volatility, the Macroeconomy, and Asset Prices. *Journal of Finance* 69 (6):2471–2511.

Belo, Frederico, Xiaoji Lin, and Santiago Bazdresch. 2014. Labor Hiring, Investment and Stock Return Predictability in the Cross Section. *Journal of Political Economy* 122 (1):129–177.

Borovička, Jaroslav and Lars Peter Hansen. 2014. Examining Macroeconomic Models through the Lens of Asset Pricing. *Journal of Econometrics* 183 (1):67–90.

Campbell, John Y. and Robert J. Shiller. 1988. The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors. *Review of Financial Studies* 1:195–228.

Christiano, Lawrence J., Martin S. Eichenbaum, and Mathias Trabandt. 2016. Unemployment and Business Cycles. *Econometrica* 84 (4):1523–1569.

Donangelo, Andres, Francois Gourio, and Miguel Palacios. 2016. The Cross-Section of Labor Leverage and Equity Returns. May.

Epstein, Larry G. and Stanley E. Zin. 1989. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica* 57 (4):937–969.

———. 1991. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis. *The Journal of Political Economy* 99 (2):263–286.

Favilukis, Hack and Xiaoji Lin. 2016. Wage Rigidity: A Quantitative Solution to Several Asset Pricing Puzzles. *Review of Financial Studies* 29 (1):148–192.

Favilukis, Jack, Xiaoji Lin, and Xiaofei Zhao. 2015. The Elephant in the Room: The Impact of Labor Obligations on Credit Risk.

Hagedorn, Marcus and Iourii Manovskii. 2008. The Cyclical Behavior of Equilibrium Unemployment and Vacancies Revisited. *American Economic Review* 98 (4):1692–1706.

Hall, Robert E. 2017. High Discounts and High Unemployment. *American Economic Review* 107 (2):305–330.

Hall, Robert E. and Paul R. Milgrom. 2008. The Limited Influence of Unemployment on the Wage Bargain. *American Economic Review* 98 (4):1653–1674.

Hansen, Lars Peter and Ravi Jagannathan. 1991. Implications of Security Market Data for Models of Dynamic Economies. *Journal of Political Economy* 99 (2):225–262.

Hansen, Lars Peter and Kenneth J. Singleton. 1982. Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models. *Econometrica* 50 (5):1269–1286.

Hansen, Lars Peter, John C. Heaton, and Erzo G. J. Luttmer. 1995. Econometrics Evaluation of Asset Pricing Models. *Review of Financial Studies* 8 (2):237–274.

Jurado, Kyle, Sydney C. Ludvigson, and Serena Ng. 2015. Measuring Uncertainty. *American Economic Review* 105 (3):1177–1216.

Kehoe, Patrick, Virgiliu Midrigan, and Elena Pastorino. 2015. Discount Rates, Learning by Doing and Employment Fluctuations.

Kilic, Mete and Jessica A. Wachter. 2015. Risk, Unemployment, and the Stock Market: A Rare-Events-Based Explanation of Labor Market Volatility.

Kuehn, Lars-Alexander, Mikhail Simutin, and Jessie Jiaxu Wang. 2014. A Labor Capital Asset Pricing Model.

Lustig, Hanno and Stijn Van Nieuwerburgh. 2008. The Returns on Human Capital: Good News on Wall Street is Bad News on Main Street. *Review of Financial Studies* 21 (5):2097–2137.

Martin, Ian. 2017. What Is the Expected Return on the Market. *Quarterly Journal of Economics* 132 (1):467–433.

Petrosky-Nadeau, Nicolas and Lu Zhang. 2013. Unemployment Crises. NBER Working paper W19207.

Petrosky-Nadeau, Nicolas, Lu Zhang, and Lars-Alexander Kuehn. 2015. Endogenous Disasters.

Shiller, Robert J. 1981. Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends. *American Economic Review* 71 (3):421–436.

Shimer, Robert. 2005. The Cyclical Behavior of Equilibrium Unemployment and Vacancies. *American Economic Review* 95 (1):25–49.

———. 2010. *Labor Markets and Business Cycles*. Princeton University Press.

———. 2012. Reassessing the Ins and Outs of Unemployment. *Review of Economic Dynamics* 15 (2):127–148.

Appendix

A Log-linearization of the hiring equation

Assume that the risk-free rate and separation rate are constant, $E_t[s_{t+1}] = \beta$ and $\delta_t = \delta$. Let the bar and hat variables denote the steady state values and log deviations from the steady state, respectively. Then log-linearize equation (2):

$$1 = \beta\pi\frac{\bar{q}}{\kappa}E_t\left[(1 + \widehat{\pi}_{t+1} + \widehat{q}_t)\right] + (1-\delta)\beta E_t\left[1 + \widehat{q}_t - \widehat{q}_{t+1}\right] - \overline{\Gamma}\left(1 + \widehat{\Gamma}_t\right).$$

In the steady state,

$$1 = \beta\pi\frac{\bar{q}}{\kappa} + (1-\delta)\beta - \overline{\Gamma}.$$

Further assume that $\hat{q}_t$ follows an AR(1) process with autocorrelation ρ_q. Using this and the steady-state relationship, we obtain

$$0 = \beta\pi\frac{\bar{q}}{\kappa}E_t\left[\widehat{\pi}_{t+1} + \widehat{q}_t\right] + \beta(1-\delta)(1-\rho_q)\widehat{q}_t - \overline{\Gamma}\widehat{\Gamma}_t.$$

Finally, substitute $\beta\pi\bar{q}/\kappa$ out using the steady state relationship to find (3):

$$\left(1 - \beta(1-\delta) + \overline{\Gamma}\right)E_t\left[\widehat{\pi}_{t+1}\right] - \overline{\Gamma}\widehat{\Gamma}_t = -\left(1 - \beta(1-\delta)\rho_q + \overline{\Gamma}\right)\widehat{q}_t. \tag{21}$$

When the risk premium is zero, $\overline{\Gamma} = \widehat{\Gamma}_t = 0$, then the above relationship simplifies to

$$E_t\left[\widehat{\pi}_{t+1}\right] = -\frac{1 - \beta(1-\delta)\rho_q}{1 - \beta(1-\delta)}\widehat{q}_t,$$

which implies

$$\sigma\left(E_t\left[\widehat{\pi}_{t+1}\right]\right) = \frac{1 - \beta(1-\delta)\rho_q}{1 - \beta(1-\delta)}\sigma\left(\widehat{q}_t\right).$$

We now consider the case with risk premia. Take the variance of both sides of (21),

$$
\begin{aligned}
\left(1 - \beta(1-\delta)\rho_q + \overline{\Gamma}\right)^2\sigma^2\left(\widehat{q}_t\right) &= \left(1 - \beta(1-\delta) + \overline{\Gamma}\right)^2\sigma^2\left(E_t\left[\widehat{\pi}_{t+1}\right]\right) + \overline{\Gamma}^2\sigma^2\left(\widehat{\Gamma}_t\right) \\
&\quad -2\left(1 - \beta(1-\delta) + \overline{\Gamma}\right)\overline{\Gamma}Cov\left(E_t\left[\widehat{\pi}_{t+1}\right], \widehat{\Gamma}_t\right) \\
&= \left(1 - \beta(1-\delta) + \overline{\Gamma}\right)^2\sigma^2\left(E_t\left[\widehat{\pi}_{t+1}\right]\right) + \overline{\Gamma}^2\sigma^2\left(\widehat{\Gamma}_t\right) \\
&\quad -2\left(1 - \beta(1-\delta) + \overline{\Gamma}\right)\overline{\Gamma}\sigma\left(\widehat{q}_t\right)\sigma\left(\widehat{\Gamma}_t\right)\rho\left(E_t\left[\widehat{\pi}_{t+1}\right], \widehat{\Gamma}_t\right).
\end{aligned}
$$

We want to find, for the given variance of conditional expected profit flow $\sigma^2\left(E_t\left[\widehat{\pi}_{t+1}\right]\right)$, the minimum variance of the risk premium $\sigma^2\left(\widehat{\Gamma}_t\right)$ needed to generate the observed variance of the vacancy-filling rate $\sigma^2\left(\widehat{q}_t\right)$. The minimum is achieved when the correlation between the expected profit flow

and the risk premium, $\rho\left(E_t\left[\widehat{\pi}_{t+1}\right], \widehat{\Gamma}_t\right)$, is equal to -1. In particular, we get

$$
\begin{aligned}
\left(1-\beta\left(1-\delta\right)\rho_q+\overline{\Gamma}\right)^2\sigma^2\left(\widehat{q}_t\right) \;\leq\; & \left(1-\beta\left(1-\delta\right)+\overline{\Gamma}\right)^2\sigma^2\left(E_t\left[\widehat{\pi}_{t+1}\right]\right)+\overline{\Gamma}^2\sigma^2\left(\widehat{\Gamma}_t\right) \\
& +2\left(1-\beta\left(1-\delta\right)+\overline{\Gamma}\right)\overline{\Gamma}\sigma\left(\widehat{q}_t\right)\sigma\left(\widehat{\Gamma}_t\right) \\
=\; & \left[\left(1-\beta\left(1-\delta\right)+\overline{\Gamma}\right)\sigma\left(E_t\left[\widehat{\pi}_{t+1}\right]\right)+\overline{\Gamma}\sigma\left(\widehat{\Gamma}_t\right)\right]^2
\end{aligned}
$$

Consequently, we obtain the desired inequality (4):

$$
\overline{\Gamma}\sigma\left(\widehat{\Gamma}_t\right) \geq \left(1-\beta\left(1-\delta\right)\rho_q+\overline{\Gamma}\right)\sigma\left(\widehat{q}_t\right)-\left(1-\beta\left(1-\delta\right)+\overline{\Gamma}\right)\sigma\left(E_t\left[\widehat{\pi}_{t+1}\right]\right).
$$

B   Proofs

Proof of Proposition 4.1. Let λ^π be the Lagrange multiplier on the constraint in (11). The Lagrangean for the problem is

$$
\begin{aligned}
\mathcal{L}_\alpha \;=\; & \alpha Var\left[E_t\left[\pi_{t+1}\right]\right]+\left(1-\alpha\right)E\left[Var_t\left[\pi_{t+1}\right]\right]-\left(E\left[z_t^\pi\left(s_{t+1}\pi_{t+1}-g_{t+1}\right)\right]\right)'\lambda^\pi \\
\;=\; & \alpha E\left[\left(E_t\left[\pi_{t+1}\right]-E\left[\pi_{t+1}\right]\right)^2\right]+\left(1-\alpha\right)E\left[E_t\left[\left(\pi_{t+1}-E_t\left[\pi_{t+1}\right]\right)^2\right]\right] \\
& -\left(E\left[z_t^\pi\left(s_{t+1}\pi_{t+1}-g_{t+1}\right)\right]\right)'\lambda^\pi
\end{aligned}
$$

The first-order condition with respect to π_{t+1} yields

$$
0=2\alpha\left(E_t\left[\pi_{t+1}\right]-E\left[\pi_{t+1}\right]\right)+2\left(1-a\right)\left(\pi_{t+1}-E_t\left[\pi_{t+1}\right]\right)-\left(z_t^\pi s_{t+1}\right)'\lambda^\pi. \tag{22}
$$

Conditioning down this equation to the time-t information set and further to the unconditional distribution yields

$$
\begin{aligned}
0 \;&=\; 2\alpha\left(E_t\left[\pi_{t+1}\right]-E\left[\pi_{t+1}\right]\right)-\left(E_t\left[z_t^\pi s_{t+1}\right]\right)'\lambda^\pi \\
0 \;&=\; \left(E\left[z_t^\pi s_{t+1}\right]\right)'\lambda^\pi.
\end{aligned}
$$

Substituting these back into (22), we obtain

$$
0=2\left(1-a\right)\left(\pi_{t+1}-E_t\left[\pi_{t+1}\right]\right)-\left(z_t^\pi s_{t+1}-E_t\left[z_t^\pi s_{t+1}\right]\right)'\lambda^\pi
$$

and hence

$$
\begin{aligned}
\pi_{t+1}-E_t\left[\pi_{t+1}\right] \;&=\; \frac{1}{2\left(1-\alpha\right)}\left(z_t^\pi s_{t+1}-E_t\left[z_t^\pi s_{t+1}\right]\right)'\lambda^\pi \\
E_t\left[\pi_{t+1}\right]-E\left[\pi_{t+1}\right] \;&=\; \frac{1}{2\alpha}\left(E_t\left[z_t^\pi s_{t+1}\right]-E\left[z_t^\pi s_{t+1}\right]\right)'\lambda^\pi.
\end{aligned}
$$

33

Plugging these restrictions back into the constraint in (11) yields

$$
\begin{aligned}
E\left[z_t^\pi g_{t+1}\right] &= E\left[z_t^\pi s_{t+1}\pi_{t+1}\right] = E\left[z_t^\pi s_{t+1}\right]E\left[\pi_{t+1}\right] + E\left[z_t^\pi s_{t+1}\left(\pi_{t+1} - E\left[\pi_{t+1}\right]\right)\right] \\
&= E\left[z_t^\pi s_{t+1}\right]E\left[\pi_{t+1}\right] + \frac{1}{2\alpha}E\left[z_t^\pi s_{t+1}\left(E_t\left[z_t^\pi s_{t+1}\right] - E\left[z_t^\pi s_{t+1}\right]\right)'\right]\lambda^\pi \\
&\quad + \frac{1}{2(1-\alpha)}E\left[z_t^\pi s_{t+1}\left(z_t^\pi s_{t+1} - E_t\left[z_t^\pi s_{t+1}\right]\right)'\right]\lambda^\pi \\
&= E\left[z_t^\pi s_{t+1}\right]E\left[\pi_{t+1}\right] + \frac{1}{2\alpha}Var\left[E_t\left[z_t^\pi s_{t+1}\right]\right]\lambda^\pi + \frac{1}{2(1-\alpha)}E\left[Var_t\left[z_t^\pi s_{t+1}\right]\right]\lambda^\pi.
\end{aligned}
$$

We can define the weighted variance matrix

$$
V_\alpha \doteq \frac{1}{2\alpha}Var\left[E_t\left[z_t^\pi s_{t+1}\right]\right] + \frac{1}{2(1-\alpha)}E\left[Var_t\left[z_t^\pi s_{t+1}\right]\right]
$$

and solve for the Lagrange multiplier

$$
\lambda^\pi = (V_\alpha)^{-1}\left(E\left[z_t^\pi g_{t+1}\right] - E\left[z_t^\pi s_{t+1}\right]E\left[\pi_{t+1}\right]\right).
$$

Using the fact that $0 = \left(E\left[z_t^\pi s_{t+1}\right]\right)'\nu$, we can premultiply this equation by $\left(E\left[z_t^\pi s_{t+1}\right]\right)'$ and solve for $E\left[\pi_{t+1}\right]$:

$$
E\left[\pi_{t+1}\right] \doteq \bar\pi = \frac{\left(E\left[z_t^\pi s_{t+1}\right]\right)'(V_\alpha)^{-1}E\left[z_t^\pi g_{t+1}\right]}{\left(E\left[z_t^\pi s_{t+1}\right]\right)'(V_\alpha)^{-1}E\left[z_t^\pi s_{t+1}\right]}.
$$

These expressions replicate the statement of the proposition. □

Proof of Proposition 5.1. The construction of the minimum-variance stochastic discount factor as a solution to (20) is standard and can be viewed as a special case of the calculations in the proof of Proposition 4.1 above, when setting $\alpha = 1/2$ and stacking the set of instrumented pricing restrictions (19) using Kronecker notation as:

$$
0 = E\left[z_t^s \otimes (s_{t+1}R_{t+1} - \mathbf{1})\right]. \tag{23}
$$

The Lagrangean now is

$$
\mathcal{L} = \frac{1}{2}E\left[\left(s_{t+1} - E\left[s_{t+1}\right]\right)^2\right] - \left(E\left[z_t^s \otimes (s_{t+1}R_{t+1} - \mathbf{1})\right]\right)'\lambda^s
$$

and the implied first-order condition

$$
0 = s_{t+1} - E\left[s_{t+1}\right] - \left(z_t^s \otimes R_{t+1}\right)'\lambda^s
$$

which also implies that $\left(E\left[z_t^s \otimes R_{t+1}\right]\right)'\lambda^s = 0$ and hence

$$
s_{t+1} = E\left[s_{t+1}\right] + \left(z_t^s \otimes R_{t+1} - E\left[z_t^s \otimes R_{t+1}\right]\right)'\lambda^s
$$

Substituting this into the constraint (23) yields

$$
\begin{aligned}
E\left[z_t^s \otimes \mathbf{1}\right] &= E\left[z_t^s \otimes R_{t+1}\right] E\left[s_{t+1}\right] + E\left[\left(z_t^s \otimes R_{t+1}\right)\left(z_t^s \otimes R_{t+1} - E\left[z_t^s \otimes R_{t+1}\right]\right)'\right] \lambda^s \\
&= E\left[z_t^s \otimes R_{t+1}\right] E\left[s_{t+1}\right] + Var\left[z_t^s \otimes R_{t+1}\right] \lambda^s
\end{aligned}
$$

and consequently

$$
\lambda^s = \left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1}\left(E\left[z_t^s \otimes \mathbf{1}\right] - E\left[z_t^s \otimes R_{t+1}\right] E\left[s_{t+1}\right]\right).
$$

Premultiplying by $\left(E\left[z_t^s \otimes R_{t+1}\right]\right)'$, we obtain

$$
\left(E\left[z_t^s \otimes R_{t+1}\right]\right)' \lambda^s = 0 = \left(E\left[z_t^s \otimes R_{t+1}\right]\right)'\left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1}\left(E\left[z_t^s \otimes \mathbf{1}\right] - E\left[z_t^s \otimes R_{t+1}\right] E\left[s_{t+1}\right]\right)
$$

and finally

$$
E\left[s_{t+1}\right] \doteq \bar{s} = \frac{\left(E\left[z_t^s \otimes R_{t+1}\right]\right)'\left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1}\left(E\left[z_t^s \otimes \mathbf{1}\right]\right)}{\left(E\left[z_t^s \otimes R_{t+1}\right]\right)'\left(Var\left[z_t^s \otimes R_{t+1}\right]\right)^{-1} E\left[z_t^s \otimes R_{t+1}\right]}
$$

which completes the proof. $\qquad\square$

C   Bounds in search models

We construct $E\left[Var_t[\pi_{t+1}]\right]$ and $Var\left[E_t\left[\pi_{t+1}\right]\right]$ using profit flow process π_t generated by several search models. We explain here in detail how it is constructed.

C.1   Shimer (2010)

We start with Shimer (2010). Shimer solves his model using log-linearization around the steady state which implies that the law of motion for the state variable as well as the policy function are linear functions of the state. Let's $\hat{x}_t$ be a vector of state variables expressed in log deviations from the steady state. Then

$$
\begin{aligned}
\hat{x}_{t+1} &= A\hat{x}_t + Dv_{t+1} \\
\hat{\pi}_t &= B\hat{x}_t
\end{aligned}
$$

where v_{t+1} is a standard normal shock and $\hat{\pi}_t$ is the model implied profit flow. Matrices A, B, D are known, this is the solution of the model. We are interested in calculating the conditional moments. Let's start with establishing unconditional mean and variance moments of $\hat{x}_t$:

$$
\begin{aligned}
E\left[\hat{x}_t\right] &= 0 \\
\Sigma &= Var\left[\hat{x}_t\right] = E\left[\hat{x}_t \hat{x}_t'\right] = A\Sigma A' + DD'
\end{aligned}
$$

Given that innovation v_{t+1} is normal, the distribution of $\hat{x}_t$ is normal with mean zero and variance Σ.

Let's now compute conditional moments of π_t. The conditional mean

$$
\begin{aligned}
E_t\left[\pi_{t+1}\right] &= E_t\left[\bar{\pi}\exp\left(\hat{\pi}_{t+1}\right)\right] = \bar{\pi}E_t\left[\exp\left(B\hat{x}_{t+1}\right)\right] \\
&= \bar{\pi}E\left[\exp\left(BA\hat{x}_t + BDv_{t+1}\right)|\hat{x}_t\right] \\
&= \bar{\pi}\exp\left(BA\hat{x}_t + \frac{1}{2}BD\left(BD\right)'\right)
\end{aligned}
$$

The last equality uses the mean of the log normal variable. It is then straightforward to derive that

$$
\begin{aligned}
E_t\left[\pi_{t+1}^2\right] &= E_t\left[\bar{\pi}^2\exp\left(2\hat{\pi}_{t+1}\right)\right] = \bar{\pi}^2E\left[\exp\left(2B\left(A\hat{x}_t + Dv_{t+1}\right)\right)|\hat{x}_t\right] \\
&= \bar{\pi}^2\exp\left(2BA\hat{x}_t + 2\left(BD\right)\left(BD\right)'\right), \\
Var_t\left[\pi_{t+1}\right] &= E_t\left[\pi_{t+1}^2\right] - E_t\left[\pi_{t+1}\right]^2 \\
&= \bar{\pi}^2\exp\left(2BA\hat{m}_t\right)\exp\left(BD\left(BD\right)'\right)\left(\exp\left(BD\left(BD\right)'\right) - 1\right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
E\left[Var_t\left[\pi_{t+1}\right]\right] &= \bar{\pi}^2E\left[\exp\left(2BA\hat{m}_t\right)\exp\left(BD\left(BD\right)'\right)\left(\exp\left(BD\left(BD\right)'\right) - 1\right)\right] \\
&= \bar{\pi}^2\exp\left(2\left(BA\right)\Sigma\left(BA\right)' + BD\left(BD\right)'\right)\left(\exp\left(BD\left(BD\right)'\right) - 1\right) \\
Var\left[E_t\left[\pi_{t+1}\right]\right] &= Var\left(\bar{\pi}\exp\left(BA\hat{m}_t + \frac{1}{2}BD\left(BD\right)'\right)\right) \\
&= \bar{\pi}^2\left(\exp\left(BA\Sigma A'B'\right) - 1\right)\exp\left(B\Sigma B'\right)
\end{aligned}
$$

where we utilized formulas for the variance of a log normal variable.

Finally, observe that

$$
E\left[\pi_t\right] = \bar{\pi}\exp\left(\frac{1}{2}B\Sigma B'\right)
$$

and therefore the normalized moments are

$$
\begin{aligned}
\frac{E\left[Var_t\left[\pi_{t+1}\right]\right]}{E\left[\pi_t\right]^2} &= \exp\left(\left(BA\right)\Sigma\left(BA\right)'\right)\left(\exp\left(BD\left(BD\right)'\right) - 1\right) \\
\frac{Var\left[E_t\left[\pi_{t+1}\right]\right]}{E\left[\pi_t\right]^2} &= \exp\left(BA\Sigma A'B'\right) - 1
\end{aligned}
$$

The matrices A, D are taken directly from Shimer (2010), since these emerge as solution to the model. The matrix B has to be constructed by log-linearizing the marginal product of labor minus the wage.

C.2 Hall and Milgrom (2008)

Business cycle fluctuations in the model are driven by a productivity shock which follows a 5-state Markov process with a transition matrix P. The authors provide values for wage and productivity in each state of the five states on their website as the solution of hte model. It is then straightforward to calculate the appropriate moments. Let π be the 5x1 vector of profit flow values $\pi_i = s_i - w_i$; these are known as it is the solution of the model. Let $\mathbf{p}^*$ be the stationary distribution asociated with P. Then

$$
\begin{aligned}
E_t[\pi] &= P\pi, \quad E_t[\pi^2] = P\pi^2, \quad E[\pi] = \pi'\mathbf{p}^* \\
Var_t[\pi] &= P\pi^2 - (P\pi)^2 \\
E[Var_t[\pi]] &= \left(P\pi^2 - (P\pi)^2\right)'\mathbf{p}^* \\
Var[E_t[\pi]] &= \left((\pi - P\pi)^2\right)'\mathbf{p}^*
\end{aligned}
$$

D Recursive preferences

The derivation in this section closely follows Bansal et al. (2014). Assuming that aggregate wealth W_t is traded, the return $R_{t+1}^w = \exp\left(r_{t+1}^w\right)$ in (16) must satisfy the valuation restriction $1 = E_t\left[s_{t+1}R_{t+1}^w\right]$. Under joint log-normality, this restriction can be rewritten as

$$
E_t[\Delta c_{t+1}] = \psi \log \beta + \psi E_t[r_{w,t+1}] - \frac{\psi - 1}{\gamma - 1}V_t \tag{24}
$$

where $V_t \doteq \frac{1}{2}Var_t\left[\log s_{t+1} + r_{t+1}^w\right]$ is a measure of conditional variance in the model. The return on aggregate wealth must also obey the usual budget constraint $W_{t+1} = (W_t - C_t)R_{t+1}^w$. Denoting $w_t = \log(W_t/C_t)$ the logarithm of the wealth consumption ratio and log-linearizing around its steady state value $\bar{w}$, we obtain

$$
r_{t+1}^w = \kappa_0 + w_{t+1} + \Delta c_{t+1} - \frac{1}{\kappa_1}w_t \tag{25}
$$

where κ_0 and κ_1 are log-linearization constants, in particular $\kappa_1 = (\exp(\bar{w}) - 1)/\exp(\bar{w})$.

Solving this equation forward and imposing a transversality condition, we obtain

$$
w_t = \sum_{j=0}^{\infty} \kappa_1^j \left(\Delta c_{t+j+1} - r_{t+j+1}^w + \kappa_0\right),
$$

or, in terms of innovations,

$$
c_{t+1} - E_t[c_{t+1}] = (E_{t+1} - E_t)\left[\sum_{j=0}^{\infty}\kappa_1^j r_{t+j+1}^w\right] - (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty}\kappa_1^j \Delta c_{t+j+1}\right]. \tag{26}
$$

To simplify notation, it is convenient to define

$$N_{C,t+1} \doteq c_{t+1} - E_t[c_{t+1}], \qquad N_{ECF,t} \doteq (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty} \kappa_1^j \Delta c_{t+j+1}\right]$$

$$N_{CF,t} \doteq (E_{t+1} - E_t)\left[\sum_{j=0}^{\infty} \kappa_1^j \Delta c_{t+j+1}\right], \qquad N_{V,t+1} \doteq (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty} \kappa_1^j V_{t+j}\right] \qquad (27)$$

$$N_{R,t+1} \doteq r_{t+1}^w - E_t[r_{t+1}^w], \qquad N_{DR,t+1} \doteq (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty} \kappa_1^j r_{t+j+1}^w\right].$$

Then we can rewrite (26) as $N_{CF,t+1} = N_{R,t+1} + N_{DR,t+1}$. This relationship states that since the wealth-consumption ratio is assumed to be stationary in the present-value budget constraint, the discounted long-run response of consumption, $N_{CF,t+1}$, must be equal to the sum of responses of current and future discounted returns on wealth, $N_{R,t+1} + N_{DR,t+1}$. Using the asset pricing relationship (24) to substitute out future consumption growth from (26), we obtain

$$N_{C,t+1} = N_{R,t+1} + (1-\psi) N_{DR,t+1} + \frac{\psi - 1}{\gamma - 1} N_{V,t+1}. \qquad (28)$$

In order to operationalize the model of stochastic volatility, Bansal et al. (2014) assume a one-factor linear structure for conditional volatility, imposed by (18). First, we can express V_t as a function of innovations as

$$V_t = \frac{1}{2} Var_t\left[(1-\gamma) N_{CF,t+1} + N_{V,t+1}\right]$$

where we used the specification of the stochastic discount factor (16). Second, assuming that innovations to the long-run conditional variance, $N_{V,t+1}$, are homoskedastic and orthogonal to innovations to cash flows, we can write

$$V_t = c + \frac{1}{2}(1-\gamma)^2 Var_t\left[N_{CF,t+1}\right].$$

where the constant term $c = \frac{1}{2} Var[N_{V,t+1}]$ is unimportant. Given the one-factor structure of conditional variance, fluctuations in conditional variance of all variables will be proportional, and hence

$$V_t = c + \frac{1}{2}\chi(1-\gamma)^2 Var_t\left[\Delta c_{t+1}\right].$$

The empirical strategy used in Bansal et al. (2014) is to normalize the measure of realized variance included in the VAR so that its average is equal to the unconditional variance of consumption growth, and then determine the constant χ as the ratio of the unconditional variance of long-run and short-run consumption innovations,

$$\chi = Var[N_{CF,t+1}]/Var[N_{C,t+1}]. \qquad (29)$$

Then we get

$$V_t = c + \frac{1}{2}\chi(1-\gamma)^2 E_t[RV_{t+1}] = c + \frac{1}{2}\chi(1-\gamma)^2 \iota_v' \phi_x X_t.$$

To complete the model, we need to empirically identify the return on aggregate wealth, r_{t+1}^w. The

idea is to assume that aggregate wealth consists of two components, financial wealth represented by the stock market with return r_{t+1}^d, and human capital with cash flows given by the presented discounted value of labor income, and return r_{t+1}^y. While r_{t+1}^y is unobservable, we assume that the expected return obeys

$$E_t\left[r_{t+1}^y\right] = a + b'X_t$$

where the vector b is inferred from relationship (28). In particular, we assume, as in Bansal et al. (2014), that the vector X_t consists of

$$X_t = \left(\Delta c_t, \Delta y_t, r_t^d, w_t, RV_t\right)' \tag{30}$$

where Δy_t is the growth rate of real labor income and r_t^d is the real market return. Using the VAR structure (17), we can express the innovations to economic variance and to current and future market returns as

$$N_{V,t+1} = \frac{1}{2}\chi\left(1-\gamma\right)^2\iota_v'Q\varepsilon_{t+1}, \quad N_{R,t+1}^d = \iota_r'\varepsilon_{t+1}, \quad N_{DR,t+1}^d = \iota_r'Q\varepsilon_{t+1}$$

where $Q \doteq \kappa_1\phi_x\left(I-\kappa_1\phi_x\right)^{-1}$ represents the cumulative long-run response matirx to innovation ε_{t+1}. Further, we can use definitions (27) to express the innovation to future returns to human capital as

$$N_{DR,t+1}^y = b'\phi_x^{-1}Q\varepsilon_{t+1}, \quad N_{CF,t+1}^y = \iota_y'\left(I+Q\right)\varepsilon_{t+1}.$$

We can then use the equation that expresses the return to human capital, analogous to (25),

$$r_{t+1}^y = \kappa_0 + w_{t+1}^y + \Delta y_{t+1} - \frac{1}{\kappa_1}w_t^y$$

to infer that

$$N_{R,t+1}^y = N_{CF,t+1}^y - N_{DR,t+1}^y = \iota_y'\left(I+Q\right)\varepsilon_{t+1} - b'\phi_x^{-1}Q\varepsilon_{t+1}.$$

Finally, Bansal et al. (2014) approximate the return on aggregate wealth as a weighted average of the market return and return on human capital, with fixed weight ω:

$$r_t^w = \left(1-\omega\right)r_t^d + \omega r_t^y$$

which implies

$$N_{R,t+1} = \left(1-\omega\right)N_{R,t+1}^d + \omega N_{R,t+1}^y, \quad N_{DR,t+1} = \left(1-\omega\right)N_{DR,t+1}^d + \omega N_{DR,t+1}^y. \tag{31}$$

Equation (28) then implies that

$$
\begin{aligned}
\iota_c' &= \left(1-\omega\right)\iota_r' + \omega\left[\iota_y'\left(I+Q\right) - b'\phi_x^{-1}Q\right] + \left(1-\psi\right)\left[\left(1-\omega\right)\iota_r'Q + \omega b'\phi_x^{-1}Q\right] + \frac{\psi-1}{\gamma-1}\frac{1}{2}\chi\left(1-\gamma\right)^2\iota_v'Q \\
&= q_0 + b'q_1
\end{aligned}
$$

with

$$q_0 = (1-\omega)\,\iota_r' + \omega\iota_y'\,(I+Q) + (1-\psi)\,(1-\omega)\,\iota_r'Q + \frac{\psi-1}{\gamma-1}\frac{1}{2}\chi\,(1-\gamma)^2\,\iota_v'Q \qquad (32)$$

$$q_1 = -\omega\phi_x^{-1}Q + (1-\psi)\,\omega\phi_x^{-1}Q$$

and hence $b' = (\iota_c' - q_0)\,q_1^{-1}$. Finally, notice that the expression for b depends on χ, which is defined in (29), and depend on the ratio of unconditional variances of $N_{CF,t}$ and $N_{C,t}$. The former can be computed from $N_{CF,t+1} = N_{R,t+1} + N_{DR,t+1}$ and (31). Iterating on b and χ yields the desired fixed point.

From the above calculations, we can construct the stochastic discount factor (16) as

$$
\begin{aligned}
\log s_{t+1} - E_t\,[\log s_{t+1}] &= -\frac{\theta}{\psi}\,(\Delta c_{t+1} - E_t\,[\Delta c_{t+1}]) + (\theta-1)\,[r_{w,t+1} - E_t\,[r_{w,t+1}]] \\
&= -\frac{\theta}{\psi}N_{C,t+1} + (\theta-1)\,N_{R,t+1} \\
&= \left(-\frac{\theta}{\psi} + \theta - 1\right) N_{R,t+1} - \frac{\theta}{\psi}\,(1-\psi)\,N_{DR,t+1} - \frac{\theta}{\psi}\frac{\psi-1}{\gamma-1}N_{V,t+1} \\
&= -\gamma N_{CF,t+1} + N_{DR,t+1} + N_{V,t+1} = \iota_s'\varepsilon_{t+1}
\end{aligned}
$$

where

$$\iota_s' = -\gamma\iota_c'\,(I+Q) + (1-\omega)\,\iota_r'Q + \omega b'\phi_x^{-1}Q + \frac{1}{2}\chi\,(1-\gamma)^2\,\iota_v'Q$$

and

$$
\begin{aligned}
E_t\,[\log s_{t+1}] &= \theta\log\beta - \frac{\theta}{\psi}E_t\,[\Delta c_{t+1}] + (\theta-1)\,E_t\,[r_{t+1}^w] \\
&= \theta\log\beta - \frac{\theta}{\psi}\iota_c'\phi_x X_t + (\theta-1)\,E_t\left[(1-\omega)\,r_{t+1}^d + \omega r_{t+1}^y\right] \\
&= \theta\log\beta - \frac{\theta}{\psi}\iota_c'\phi_x X_t + (\theta-1)\,(1-\omega)\,\iota_r'\phi_x X_t + (\theta-1)\,\omega\,(a + b'X_t) \\
&= \theta\log\beta + (\theta-1)\,\omega a + \left[-\frac{\theta}{\psi}\iota_c'\phi_x + (\theta-1)\,(1-\omega)\,\iota_r'\phi_x + (\theta-1)\,\omega b'\right] X_t
\end{aligned}
$$

where a is not determined but we can set it to calibrate the average one-period real interest rate. In particular,

$$
\begin{aligned}
\log E_t\,[s_{t+1}] &= \theta\log\beta + (\theta-1)\,\omega a + \left[-\frac{\theta}{\psi}\iota_c'\phi_x + (\theta-1)\,(1-\omega)\,\iota_r'\phi_x + (\theta-1)\,\omega b'\right] X_t \\
&\quad + \frac{1}{2}\iota_s'Var_t\,[\varepsilon_{t+1}]\,\iota_s
\end{aligned}
$$

Consequently, knowing that $Var\left[\varepsilon_{t+1}\right] = E\left[Var_t\left[\varepsilon_{t+1}\right]\right]$ (since $E_t\left[\varepsilon_{t+1}\right] = 0$), we have

$$
\begin{aligned}
E\left[\log E_t\left[s_{t+1}\right]\right] &= \theta\log\beta + (\theta - 1)\omega a + \left[-\frac{\theta}{\psi}\iota_c'\phi_x + (\theta - 1)(1 - \omega)\iota_r'\phi_x + (\theta - 1)\omega b'\right]E\left[X_t\right] \\
&\quad + \frac{1}{2}\iota_s'Var\left[\varepsilon_{t+1}\right]\iota_s
\end{aligned}
$$

Since we know that

$$
E\left[\log E_t\left[s_{t+1}\right]\right] = -E\left[\log R_t^f\right]
$$

and we can infer $E\left[X_t\right]$ and $Var\left[\varepsilon_{t+1}\right]$ from the data, we obtain a. Further, in the one-factor volatility model

$$
Var_t\left[\varepsilon_{t+1}\right] = \frac{\iota_c'Var_t\left[\varepsilon_{t+1}\right]\iota_c}{\iota_c'Var\left[\varepsilon_{t+1}\right]\iota_c}Var\left[\varepsilon_{t+1}\right] = \frac{E_t\left[RV_{t+1}\right]}{E\left[RV_{t+1}\right]}Var\left[\varepsilon_{t+1}\right] = \frac{\iota_\sigma'X_t}{\iota_\sigma'E\left[X_t\right]}Var\left[\varepsilon_{t+1}\right]
$$

and hence we can construct the conditional expectation $\log E_t\left[s_{t+1}\right]$.

D.1   Alternative measures of conditional variance

Instead of using a measure of realized variance in (30), we also consider measures of conditional variance from Jurado et al. (2015). In this case,

$$
X_t = \left(\Delta c_t, \Delta y_t, r_t^d, w_t, \sigma_t^2\right)'
$$

and

$$
N_{V,t+1} = \frac{1}{2}\chi(1 - \gamma)^2\iota_\sigma'\phi_x^{-1}Q\varepsilon_{t+1}
$$

where ι_σ selects the variable σ_t^2 from X_t. Consequently, (32) becomes

$$
q_0 = (1 - \omega)\iota_r' + \omega\iota_y'(I + Q) + (1 - \psi)(1 - \omega)\iota_r'Q + \frac{\psi - 1}{\gamma - 1}\frac{1}{2}\chi(1 - \gamma)^2\iota_\sigma'\phi_x^{-1}Q
$$

while the rest of the calculation to obtain the vector b remains unchanged.

D.2   Market model specification

Bansal et al. (2014) also consider an alternative model where aggregate wealth is associated with the value of the stock market.

E   Conditional expectations in exponential-quadratic framework

In this appendix, we present formulas for conditional expectations in a general form of the exponential-quadratic model used in the main text. The results are derived in Borovička and Hansen (2014) where this model arises as an approximation for a class of dynamic stochastic general equilibrium models constructed using a second-order series expansion.

Let $X = (X_1', X_2')'$ be an $(n_1+n_2) \times 1$ vector of states, $W \sim N(0, I)$ a $k \times 1$ vector of independent Gaussian shocks, and $\mathcal{F}_t$ the filtration generated by $(X_0, W_1, \ldots, W_t)$. We will show that given the law of motion

$$
\begin{aligned}
X_{1,t+1} &= \Theta_{10} + \Theta_{11} X_{1,t} + \Lambda_{10} W_{t+1} \\
X_{2,t+1} &= \Theta_{20} + \Theta_{21} X_{1,t} + \Theta_{22} X_{2,t} + \Theta_{23} \left(X_{1,t} \otimes X_{1,t} \right) + \\
&\quad + \Lambda_{20} W_{t+1} + \Lambda_{21} \left(X_{1,t} \otimes W_{t+1} \right) + \Lambda_{22} \left(W_{t+1} \otimes W_{t+1} \right)
\end{aligned}
\tag{33}
$$

and a process $M_t = \exp(Y_t)$ whose additive increment is given by

$$
\begin{aligned}
Y_{t+1} - Y_t &= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3 \left(X_{1,t} \otimes X_{1,t} \right) + \\
&\quad + \Psi_0 W_{t+1} + \Psi_1 \left(X_{1,t} \otimes W_{t+1} \right) + \Psi_2 \left(W_{t+1} \otimes W_{t+1} \right),
\end{aligned}
\tag{34}
$$

we can write the conditional expectation of M as

$$
\log E \left[M_t \mid \mathcal{F}_0 \right] = \left(\bar{\Gamma}_0 \right)_t + \left(\bar{\Gamma}_1 \right)_t X_{1,0} + \left(\bar{\Gamma}_2 \right)_t X_{2,0} + \left(\bar{\Gamma}_3 \right)_t \left(X_{1,0} \otimes X_{1,0} \right)
\tag{35}
$$

where $\left(\bar{\Gamma}_i \right)_t$ are constant coefficients to be determined.

The dynamics given by (33)–(34) embed as a special case the VAR specification of the Markov dynamics. In this case the state vector X_t is represented by $X_{1,t}$ in (33), and the vector $X_{2,t}$ is empty.

E.1 Definitions

To simplify work with Kronecker products, we define two operators vec and $\text{mat}_{m,n}$. For an $m \times n$ matrix H, $\text{vec}(H)$ produces a column vector of length mn created by stacking the columns of H:

$$
h_{(j-1)m+i} = \left[\text{vec}(H) \right]_{(j-1)m+i} = H_{ij}.
$$

For a vector (column or row) h of length mn, $\text{mat}_{m,n}(h)$ produces an $m \times n$ matrix H created by 'columnizing' the vector:

$$
H_{ij} = \left[\text{mat}_{m,n}(h) \right]_{ij} = h_{(j-1)m+i}.
$$

We drop the m, n subindex if the dimensions of the resulting matrix are obvious from the context. For a square matrix A, define the sym operator as

$$
\text{sym}(A) = \frac{1}{2} \left(A + A' \right).
$$

Apart from the standard operations with Kronecker products, notice that the following is true. For a row vector $H_{1 \times nk}$ and column vectors $X_{n \times 1}$ and $W_{n \times 1}$

$$
H \left(X \otimes W \right) = X' \left[\text{mat}_{k,n}(H) \right]' W
$$

and for a matrix $A_{n\times k}$, we have

$$X'AW = \left(\text{vec}A'\right)'\left(X \otimes W\right).$$

Also, for $A_{n\times n}$, $X_{n\times 1}$, $K_{k\times 1}$, we have

$$
\begin{aligned}
(AX) \otimes K &= (A \otimes K) X \\
K \otimes (AX) &= (K \otimes A) X.
\end{aligned}
$$

Finally, for column vectors $X_{n\times 1}$ and $W_{k\times 1}$,

$$(AX) \otimes (BW) = (A \otimes B)(X \otimes W)$$

and

$$(BW) \otimes (AX) = [B \otimes A_{\bullet j}]_{j=1}^{n}(X \otimes W)$$

where

$$[B \otimes A_{\bullet j}]_{j=1}^{n} = [B \otimes A_{\bullet 1} \ B \otimes A_{\bullet 2} \ \ldots \ B \otimes A_{\bullet n}].$$

E.2   Conditional expectations

Notice that a complete-the squares argument implies that, for a $1 \times k$ vector A, a $1 \times k^2$ vector B, and a scalar function $f(w)$,

$$
\begin{aligned}
E\left[\exp\left(B\left(W_{t+1} \otimes W_{t+1}\right) + AW_{t+1}\right) f\left(W_{t+1}\right) \mid \mathcal{F}_t\right] = \qquad\qquad (36) \\
= E\left[\exp\left(\frac{1}{2}W'_{t+1}\left(\text{mat}_{k,k}\left(2B\right)\right)W_{t+1} + AW_{t+1}\right) f\left(W_{t+1}\right) \mid \mathcal{F}_t\right] \\
= \left|I_k - \text{sym}\left[\text{mat}_{k,k}\left(2B\right)\right]\right|^{-1/2} \exp\left(\frac{1}{2}A\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2B\right)\right]\right)^{-1} A'\right) \tilde{E}\left[f\left(W_{t+1}\right) \mid \mathcal{F}_t\right]
\end{aligned}
$$

where $\tilde{\cdot}$ is a measure under which

$$W_{t+1} \sim N\left(\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2B\right)\right]\right)^{-1} A', \left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2B\right)\right]\right)^{-1}\right).$$

We start by utilizing formula (36) to compute

$$
\begin{aligned}
\bar{Y}\left(X_t\right) &= \log E\left[\exp\left(Y_{t+1} - Y_t\right) \mid \mathcal{F}_t\right] = \\
&= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3\left(X_{1,t} \otimes X_{1,t}\right) + \\
&\quad + \log E\left[\exp\left(\left[\Psi_0 + X'_{1t}\left[\text{mat}_{k,n}\left(\Psi_1\right)\right]'\right] W_{t+1} + \frac{1}{2}W'_{t+1}\left[\text{mat}_{k,k}\left(\Psi_2\right)\right] W_{t+1}\right) \mid \mathcal{F}_t\right] \\
&= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3\left(X_{1,t} \otimes X_{1,t}\right) - \\
&\quad - \frac{1}{2}\log\left|I_k - \text{sym}\left[\text{mat}_{k,k}\left(2\Psi_2\right)\right]\right| + \frac{1}{2}\mu'\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\mu
\end{aligned}
$$

with μ defined as

$$\mu = \Psi_0' + [\mathrm{mat}_{k,n}(\Psi_1)] X_{1,t}.$$

Reorganizing terms, we obtain

$$\bar{Y}(X_t) = \bar{\Gamma}_0 + \bar{\Gamma}_1 X_{1,t} + \bar{\Gamma}_2 X_{2,t} + \bar{\Gamma}_3 (X_{1,t} \otimes X_{1,t})$$

where

$$
\begin{aligned}
\bar{\Gamma}_0 &= \Gamma_0 - \frac{1}{2}\log|I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}(2\Psi_2)\right]| + \frac{1}{2}\Psi_0\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}(2\Psi_2)\right]\right)^{-1}\Psi_0' \\
\bar{\Gamma}_1 &= \Gamma_1 + \Psi_0\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}(2\Psi_2)\right]\right)^{-1}\left[\mathrm{mat}_{k,n}(\Psi_1)\right] \qquad\qquad (37)\\
\bar{\Gamma}_2 &= \Gamma_2 \\
\bar{\Gamma}_3 &= \Gamma_3 + \frac{1}{2}\mathrm{vec}\left[\left[\mathrm{mat}_{k,n}(\Psi_1)\right]'\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}(2\Psi_2)\right]\right)^{-1}\left[\mathrm{mat}_{k,n}(\Psi_1)\right]\right]'.
\end{aligned}
$$

For the set of parameters $\mathcal{P} = (\Gamma_0,\ldots,\Gamma_3,\Psi_0,\ldots,\Psi_2)$, equations (37) define a mapping

$$\bar{\mathcal{P}} = \bar{\mathcal{E}}(\mathcal{P}),$$

with all $\bar{\Psi}_j = 0$. We now substitute the law of motion for X_1 and X_2 to produce $\bar{Y}(X_t) = \tilde{Y}(X_{t-1}, W_t)$. It is just a matter of algebraic operations to determine that

$$
\begin{aligned}
\tilde{Y}(X_{t-1}, W_t) &= \log E\left[\exp\left(Y_{t+1} - Y_t\right) \mid \mathcal{F}_t\right] = \\
&= \tilde{\Gamma}_0 + \tilde{\Gamma}_1 X_{1,t-1} + \tilde{\Gamma}_2 X_{2,t-1} + \tilde{\Gamma}_3(X_{1,t-1} \otimes X_{1,t-1}) \\
&\quad + \tilde{\Psi}_0 W_t + \tilde{\Psi}_1(X_{1,t-1} \otimes W_t) + \tilde{\Psi}_2(W_t \otimes W_t)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\Gamma}_0 &= \bar{\Gamma}_0 + \bar{\Gamma}_1\Theta_{10} + \bar{\Gamma}_2\Theta_{20} + \bar{\Gamma}_3(\Theta_{10} \otimes \Theta_{10}) \qquad\qquad (38)\\
\tilde{\Gamma}_1 &= \bar{\Gamma}_1\Theta_{11} + \bar{\Gamma}_2\Theta_{21} + \bar{\Gamma}_3(\Theta_{10} \otimes \Theta_{11} + \Theta_{11} \otimes \Theta_{10}) \\
\tilde{\Gamma}_2 &= \bar{\Gamma}_2\Theta_{22} \\
\tilde{\Gamma}_3 &= \bar{\Gamma}_2\Theta_{23} + \bar{\Gamma}_3(\Theta_{11} \otimes \Theta_{11}) \\
\tilde{\Psi}_0 &= \bar{\Gamma}_1\Lambda_{10} + \bar{\Gamma}_2\Lambda_{20} + \bar{\Gamma}_3(\Theta_{10} \otimes \Lambda_{10} + \Lambda_{10} \otimes \Theta_{10}) \\
\tilde{\Psi}_1 &= \bar{\Gamma}_2\Lambda_{21} + \bar{\Gamma}_3\left(\Theta_{11} \otimes \Lambda_{10} + \left[\Lambda_{10} \otimes (\Theta_{11})_{\bullet j}\right]_{j=1}^n\right) \\
\tilde{\Psi}_2 &= \bar{\Gamma}_2\Lambda_{22} + \bar{\Gamma}_3(\Lambda_{10} \otimes \Lambda_{10}).
\end{aligned}
$$

This set of equations defines the mapping

$$\tilde{\mathcal{P}} = \tilde{\mathcal{E}}(\bar{\mathcal{P}}).$$

E.3   Iterative formulas

We can write the conditional expectation in (35) recursively as

$$\log E\left[M_t \mid \mathcal{F}_0\right] = \log E\left[\exp\left(Y_1 - Y_0\right) E\left[\frac{M_t}{M_1} \mid \mathcal{F}_1\right] \mid \mathcal{F}_0\right].$$

Given the mappings $\bar{\mathcal{E}}$ and $\tilde{\mathcal{E}}$, we can therefore express the coefficients $\bar{\mathcal{P}}$ in (35) using the recursion

$$\bar{\mathcal{P}}_t = \bar{\mathcal{E}}\left(\mathcal{P} + \tilde{\mathcal{E}}\left(\bar{\mathcal{P}}_{t-1}\right)\right)$$

where the addition is by coefficients and all coefficients in $\bar{\mathcal{P}}_0$ are zero matrices.