

Sentiment and speculation in a market with heterogeneous beliefs

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Abstract

We present a model featuring risk-averse investors with heterogeneous beliefs. Individuals who are correct in hindsight, whether through luck or judgment, become relatively wealthy. As a result, market sentiment is bullish following good news and bearish following bad news. Sentiment drives up volatility, and hence also risk premia. In a continuous-time Brownian limit, moderate investors trade against market sentiment in the hope of capturing a variance risk premium created by the presence of extremists. In a Poisson limit that features sudden arrivals of information, CDS rates spike following bad news and decline during quiet times.

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In the short run, the market is a voting machine, but in the long run it is a weighing machine.

—Benjamin Graham.

In this paper, we study the effect of heterogeneity in beliefs on asset prices. We work with a frictionless dynamically complete market populated by a continuum of risk-averse agents who differ in their beliefs about the probability of good news.

As a result, agents position themselves differently in the market. Optimistic investors make leveraged bets on the market; pessimists go short. If the market rallies, the wealth distribution shifts in favor of the optimists, whose beliefs become overrepresented in prices. If there is bad news, money flows to pessimists and prices more strongly reflect their pessimism going forward. At any point in time, one can define a representative agent who chooses to invest fully in the risky asset, with no borrowing or lending—our analog of Benjamin Graham’s “Mr. Market”—but the identity of the representative agent changes every period, with his or her beliefs becoming more optimistic following good news and more pessimistic following bad news. Thus market sentiment shifts constantly despite the stability of individual beliefs.

All agents understand the importance of sentiment and take it into account in the risk premia that they demand, as they correctly foresee that either good or bad news will be amplified by a shift in sentiment. The idea that sentiment itself is a source of systematic price risk appears in [De Long et al. \(1990\)](#), but in our model sentiment emerges endogenously rather than being modelled as random noise. The presence of sentiment induces speculation: agents take temporary positions, at prices they do not perceive as justified by fundamentals, in anticipation of adjusting their positions in the future.

We start in discrete time, providing a general pricing formula for arbitrary, exogenously specified, terminal payoffs. We find the wealth distribution, prices, and agents’ investment decisions at every point in time, together with their subjective perceptions of expected returns, volatilities, and Sharpe ratios; and other quantities of interest, such as aggregate volume, leverage, and the level of the VIX index.

In our model, speculation can act in either direction, driving prices up in some states and down in others. This feature is emphasized by [Keynes \(1936, Chapter 12\)](#); in [Harrison and Kreps \(1978\)](#), by contrast, speculation drives prices above fundamental value due to the presence of short sales constraints. We provide conditions that dictate whether heterogeneity drives prices up or down relative to the homogeneous benchmark. For a broad class of assets, including the discrete-time analog of the lognormal case

in which asset payoffs are exponential in the number of up-moves, heterogeneity drives prices down and risk premia up.

For most of the paper, we focus our attention on heterogeneity in beliefs by working in the limit in which investors have dogmatic priors, as is broadly consistent with the findings of Giglio et al. (2019) and Meeuwis et al. (2019). Although individual investors do not learn in this limit, the market exhibits “the wisdom of the crowd,” in that the redistribution of wealth between agents over time causes the market to behave as if it is learning as a whole. That said, our most general formulation allows the agents to learn over time by updating their heterogeneous priors according to Bayes’ rule. Following good news, not only do optimists become relatively wealthier, as described above, but also every individual updates his or her beliefs in an optimistic direction. Formalizing this intuition, we show a precise sense in which learning tends to amplify the effect of heterogeneity in beliefs.

We explore the key properties of the model in a series of examples. The first makes the point that extreme states are much more important than they are in a homogeneous-belief economy. A risky bond matures in 50 days, and will default, paying \$30 rather than the par value of \$100, only in the “bottom” state of the world—that is, only if there are 50 consecutive pieces of bad news. Investors’ beliefs about the probability, h , of an up-move are uniformly distributed between 0 and 1. Initially, the representative investor is the median agent, $h = 0.5$, who thinks the default probability is less than 10^{-15} . And yet we show that the bond trades at what might seem (given that the riskless interest rate is zero) the remarkably low price of \$95.63. Moreover, almost half the agents—all agents with beliefs h below 0.48—initially go short at this price, though most will reverse their position within two periods of bad news. The low price arises because all agents understand that if there is bad news next period, pessimists’ trades will have been profitable: their views will become overrepresented in the market, so the bond’s price will decline sharply *in the short run*. Only agents with $h < 0.006$ stay short to the bitter end.

In our second example, we modify the first by considering an asset with a *high* payoff in the “top” state of the world. The presence of sentiment inflates the price of such an asset relative to the homogeneous benchmark, and there are several interesting differences in the dynamics relative to the risky bond case. First, sentiment becomes increasingly important as time passes: if repeated good news arrives, the asset becomes more and more bubbly. By contrast, sentiment has most impact early in the life of the risky bond. Second, the risk premium perceived by the median investor is initially

positive; becomes increasingly negative as the bubble grows; but then starts to rise and ultimately turns positive again at the height of the bubble, just before the terminal date. As a result, the median investor reverses position *twice* during the lifetime of the bubble. Finally, implied volatility, as measured by the VIX index, rises as the bubble grows, whereas the reverse happens in the risky bond example.

In our third example, we construct a stark situation in which there is *no* volatility if beliefs are homogeneous: the asset is (until the final period) totally riskless. When beliefs are heterogeneous, however, our investors' short-termism induces them to speculate on sentiment.¹ Moreover, the resulting volatility is socially costly, in the sense that average realized utility is lower than it would be if investors were prevented from speculating.

The fourth and fifth examples are continuous time limits that model information as arriving continuously over time in small pieces (formally, as driven by a Brownian motion), or as arriving infrequently in lumps (formally, as driven by a Poisson process).

In the *Brownian limit*, the risky asset has lognormal terminal payoffs. Sentiment drives up true (\mathbb{P}) and implied (\mathbb{Q}) volatility, particularly in the short run, and hence also risk premia; both types of volatility are lower at long horizons due to the moderating influence of the terminal date at which pricing is dictated by fundamentals. “In the short run, the market is a voting machine but in the long run it is a weighing machine.”

Extremists speculate increasingly aggressively as the market moves in their favor, whereas moderate investors trade in contrarian fashion and capture a variance risk premium created by the presence of the extremists. Among moderates, there is a particularly interesting *gloomy investor*, who is somewhat more pessimistic than the median investor and who perceives the lowest maximum attainable Sharpe ratio of all investors. Despite believing that the risky asset earns zero instantaneous risk premium, he thinks that a sizeable Sharpe ratio can be attained by exploiting what he views as irrational exuberance on the up side and irrational pessimism on the down side. The gloomy investor can therefore be thought of as supplying liquidity to the extremists.

Each investor has a target price—the ideal outcome for that investor, given his or her beliefs and hence trading strategy—that can usefully be compared to what the investor expects to happen. An extremist is happy if the market moves even more than he or she

¹This echoes an observation of Keynes (1936, Chapter 12), who writes, “It might have been supposed that competition between expert professionals . . . would correct the vagaries of the ignorant individual left to himself. It happens, however, that the energies and skill of the professional investor and speculator are mainly occupied otherwise. . . . They are concerned, not with what an investment is really worth to a man who buys it ‘for keeps’, but with what the market will value it at, under the influence of mass psychology, three months or a year hence.”

expected. The gloomy investor, in contrast, hopes to be proved right: in a sense that we make precise, the best outcome for him is the one that he expects.

In the second continuous time example—the *Poisson limit*—news arrives infrequently. The jumps that occur at such times represent bad news, perhaps driven by credit or catastrophe risk. Optimistic investors sell insurance against jumps to pessimists: as long as things are quiet, wealth flows smoothly from pessimists to optimists, but at the time of a jump there is a sudden shift in the pessimists’ favor. Optimists are in the position in which derivative traders inside major financial institutions have traditionally found themselves: short volatility, making money in quiet times but occasionally subject to severe losses at times of market turmoil. As a result, even though all individuals perceive constant jump arrival rates, the market-implied (i.e., risk-neutral) jump arrival rate—which can be interpreted as a CDS rate—declines smoothly in the absence of jumps, but spikes sharply after a jump occurs. Similar patterns have been documented empirically in catastrophe insurance pricing by [Froot and O’Connell \(1999\)](#) and [Born and Viscusi \(2006\)](#), among others.

Related literature. Our paper intersects with several strands of the large literature on the effects of disagreement in financial markets. The closest antecedent of—and the inspiration for—our paper is [Geanakoplos \(2010\)](#), whose paper studies disagreement among risk-neutral investors (as do [Harrison and Kreps, 1978](#); [Scheinkman and Xiong, 2003](#)). Risk-neutrality simplifies the analysis in some respects but complicates it in others. For example, short sales must be restricted for equilibrium to exist. This is natural in some settings, but not if one thinks of the risky asset as representing, say, a broad stock market index; and the resulting kinked indirect utility functions are not very tractable. Moreover, the aggressive trading behavior of risk-neutral investors leads to extreme predictions: every time there is a down-move in the Geanakoplos model, all agents who are invested in the risky asset go bankrupt.

In a variation on the Geanakoplos model, [Simsek \(2013\)](#) emphasizes the importance of the *type* of disagreement: for example, an agent might be considered relatively optimistic either because she perceives a high chance of some good outcome, or because she perceives a low chance of a bad outcome. In our binomial setting there is no distinction between these alternatives, as an agent who perceives a high chance of “up” must also perceive a low chance of “down”, but Simsek allows for more than two—in fact, for a continuum of—possible outcomes. (In the other direction, we have a continuum of investor types whereas Simsek has two.) The mechanisms in the two papers are complementary: Simsek’s model features just one period, so his agents (who are risk-neutral)

do not speculate in our sense.

Other strands of the literature have focussed on the role of disagreement in the amplification of volatility and trading volume (Basak, 2005; Banerjee and Kremer, 2010; Atmaz and Basak, 2018), in the evolution of the wealth distribution (Zapatero, 1998; Jouini and Napp, 2007; Bhamra and Uppal, 2014), in amplifying the importance of extremely unlikely states (Kogan et al., 2006), and in the pricing of options (Buraschi and Jiltsov, 2006). Other papers generate similar asset-pricing effects by allowing for heterogeneity in risk aversion rather than beliefs (Dumas, 1989; Chan and Kogan, 2002), though of course they do not account for the direct evidence from surveys that individuals have heterogeneous beliefs (Shiller, 1987; Ben-David et al., 2013).

A related literature addresses the question of which agents will survive into the infinite future (Sandroni, 2000; Jouini and Napp, 2007; Borovička, 2020). Our paper does not directly bear on this question, as we fix a finite terminal horizon. But as the truth lies in the support of every agent’s prior in our extended model with learning, all agents would in principle survive to infinity (Blume and Easley, 2006).

Most of the prior literature restricts to the diffusion setting (of the papers mentioned, Dumas, 1989; Zapatero, 1998; Chan and Kogan, 2002; Scheinkman and Xiong, 2003; Basak, 2005; Buraschi and Jiltsov, 2006; Kogan et al., 2006; Jouini and Napp, 2007; Dumas et al., 2009; Cvitanić et al., 2011; Atmaz and Basak, 2018; Borovička, 2020); while Banerjee and Kremer (2010) work with a CARA–Normal model, and Geanakoplos (2010) and Simsek (2013) with one- or two-period models. (A notable exception is Chen et al. (2012), who present a model with heterogeneity in beliefs about disaster risk.)

Our model is extremely tractable, which allows us to study all these issues analytically—together with new results on the implied volatility surface, the variance risk premium, individual investors’ trading strategies and attitudes to speculation and so forth—in a simple framework that allows for learning and for general terminal payoffs. This tractability is due in part to our use of log utility, which we view as a reasonable benchmark given the results of Martin (2017), Kremens and Martin (2019), and Martin and Wagner (2019), and which implies (even in a non-diffusion setting) that the representative investor’s perceived risk premium is equal to risk-neutral variance so that our model generates empirically plausible first and second moments of returns. It also reflects the fact that we work with a continuum of beliefs, like Geanakoplos (2010) and Atmaz and Basak (2018) but unlike the two-type models of, for example, Harrison and Kreps (1978); Scheinkman and Xiong (2003); Basak (2005); Buraschi and Jiltsov (2006); Kogan et al. (2006); Dumas et al. (2009); Banerjee and Kremer (2010); Simsek (2013); Bhamra and

Uppal (2014); Borovička (2020). Aside from the evident desirability of having a realistic belief distribution, the identities of the representative investor and of the investor who chooses to sit out of the market entirely then become smoothly varying equilibrium objects that are determined endogenously in an intuitive and tractable way.

1 The model

We work in discrete time, $t = 0, \dots, T$. Uncertainty evolves on a binomial tree, so that whatever the current state of the world, there are two possible successor states next period: “up” and “down.” There is a risky asset, whose payoffs at the terminal date T are specified exogenously. These payoffs must be strictly positive so that expected utility is finite at every node, but they can otherwise be arbitrary. We will assume that the binomial tree is recombining—i.e., that the terminal payoffs depend on the number of total up- and down-moves rather than on the path by which the terminal node is reached—but our approach generalizes to the non-recombining case. In our examples, payoffs will be higher at nodes closer to the “top” of the tree, so we will think of an up-move as good news, and of agents with higher values of h as being more optimistic.

We normalize the net interest rate to zero. This implies that any variation in expected returns, across agents or over time, reflects variation in risk premia.

There is a unit mass of agents indexed by $h \in (0, 1)$. Each agent has log utility over terminal wealth, zero time-preference rate, and is initially endowed with one unit of the risky asset, which we will think of as representing “the market,” so the risky asset is in unit supply and the riskless asset is in zero net supply. Agent h believes that the probability of an up-move is h ; we often refer to h as the agent’s *belief*, for short. By working with the open interval $(0, 1)$, as opposed to the closed interval $[0, 1]$, we ensure that the investors agree on *what events can possibly happen* (more formally, their beliefs are absolutely continuous with respect to each other). These assumptions imply that no investor will allow his or her wealth to go to zero in any state of the world.

The mass of agents with belief h is $f(h)$. We allow $f(h)$ to be an arbitrary probability density function (pdf) throughout Section 1.1 and in our main pricing Result 1. But for much of the paper, we find it convenient to specify that the cross-section of beliefs obeys a beta distribution, so that the pdf is

$$f(h) = \frac{h^{\alpha-1}(1-h)^{\beta-1}}{B(\alpha, \beta)}, \quad (1)$$

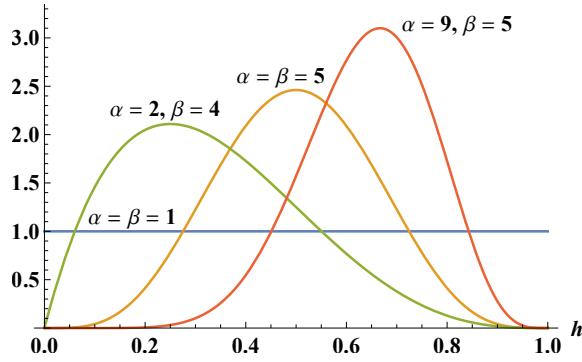


Figure 1: The distribution of beliefs for various choices of α and β .

where $\alpha > 0$ and $\beta > 0$ are parameters and $B(\alpha, \beta) = \int_{h=0}^1 h^{\alpha-1}(1-h)^{\beta-1} dh$ is the beta function.² The beta distribution is the conjugate prior for the binomial distribution, which makes the analysis tractable. The flexibility and tractability of the family of beta distributions is particularly important in the continuous-time limits considered in Sections 2.4 and 2.5.

Figure 1 illustrates beta distributions for a range of choices of α and β . If $\alpha = \beta$ then the distribution of beliefs is symmetric with mean $1/2$. In particular, if $\alpha = \beta = 1$ then $f(h) = 1$, so that beliefs are uniformly distributed over $(0, 1)$; this is a useful case to keep in mind as one works through the algebra. More generally, the case $\alpha \neq \beta$ allows for asymmetric distributions with mean $\alpha/(\alpha + \beta)$ and variance $\alpha\beta/[(\alpha + \beta)^2(\alpha + \beta + 1)]$. Thus the distribution shifts toward 1 if $\alpha > \beta$ and toward 0 if $\alpha < \beta$, and there is little disagreement when α and β are large: if, say, $\alpha = 90$ and $\beta = 10$ then beliefs are concentrated around a mean of 0.9, with standard deviation 0.03.

1.1 Equilibrium

As agents have log utility over terminal wealth, they behave myopically; we can therefore consider each period in isolation. We start by taking next-period prices at the up- and down-nodes as given, and use these prices to determine the equilibrium price at the current node. This logic will ultimately allow us to solve the model by backward induction, and to express the price at time 0 in terms of the exogenous terminal payoffs.³

²The beta function is related to the gamma function by $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$. If α and β are integers, then $B(\alpha, \beta) = (\alpha - 1)!(\beta - 1)!/(\alpha + \beta - 1)!$.

³A referee pointed out to us that we could also exploit dynamic completeness to solve our model as a static Arrow–Debreu equilibrium. We lay out this approach, which gives a shorter proof of Result 1,

Suppose, then, that the price of the risky asset will be either p_d or p_u next period. Our problem, for now, is to determine the equilibrium price, p , at the current node; we assume that $p_d \neq p_u$ so that this pricing problem is nontrivial. (If $p_d = p_u$ then the asset is riskless so $p = p_d = p_u$.) Suppose also that agent h has wealth w_h at the current node. If he chooses to hold x_h units of the asset, then his wealth next period is $w_h - x_h p + x_h p_u$ in the up-state and $w_h - x_h p + x_h p_d$ in the down-state. So the portfolio problem is

$$\max_{x_h} h \log [w_h - x_h p + x_h p_u] + (1 - h) \log [w_h - x_h p + x_h p_d].$$

The agent's first-order condition is therefore

$$x_h = w_h \left(\frac{h}{p - p_d} - \frac{1 - h}{p_u - p} \right). \quad (2)$$

The sign of x_h is that of $p - p_u$ for $h = 0$ and that of $p - p_d$ for $h = 1$. These must have opposite signs to avoid an arbitrage opportunity, so at every node there are some agents who are short and others who are long. The most optimistic agent⁴ levers up as much as possible without risking default. From the perspective of an extreme optimist, p_d can be thought of as the liquidation value: when it is large, the optimist can get more leverage. For, the first-order condition (2) implies that as $h \rightarrow 1$, agent h holds $w_h/(p - p_d)$ units of the risky asset. This is the largest possible position that does not risk default: to acquire it, the agent must borrow $w_h p/(p - p_d) - w_h = w_h p_d/(p - p_d)$. If the unthinkable (to this most optimistic agent!) occurs and the down state materialises, the agent's holdings are worth $w_h p_d/(p - p_d)$, which is precisely what the agent owes to his creditors. Correspondingly, the most pessimistic agent takes on the largest short position possible that does not risk default if the good state occurs.

It will often be convenient to think in terms of the risk-neutral probability of an up-move, h^* , defined by the property that the price can be interpreted as a risk-neutral expected payoff, $p = h^* p_u + (1 - h^*) p_d$. (There is no discounting, as the riskless rate is zero.) Hence

$$h^* = \frac{p - p_d}{p_u - p_d}.$$

in the Online Appendix. We take the approach in the body of the paper for expositional reasons, as it allows us to introduce quantities that will be important for understanding the dynamics of the model.

⁴This is an abuse of terminology: there is no 'most optimistic agent' since h lies in the *open* set $(0, 1)$. More formally, this discussion relates to the behavior of agents in the limit as $h \rightarrow 1$. An agent with $h = 1$ would want to take arbitrarily large levered positions in the risky asset, so there is a behavioral discontinuity at $h = 1$ (and similarly at $h = 0$).

In these terms, the first-order condition (2) becomes

$$x_h = \frac{w_h}{p_u - p_d} \frac{h - h^*}{h^*(1 - h^*)},$$

for example. An agent whose h equals h^* will have zero position in the risky asset: by the defining property of the risk-neutral probability, such an agent perceives that the risky asset has zero expected excess return.

Agent h 's wealth next period is therefore $w_h + x_h(p_u - p) = w_h \frac{h}{h^*}$ in the up-state, and $w_h - x_h(p - p_d) = w_h \frac{1-h}{1-h^*}$ in the down-state. In either case, all agents' returns on wealth are linear in their beliefs. Moreover, this relationship applies at every node. It follows that person h 's wealth at the current node is $\lambda_{\text{path}} h^m (1 - h)^n$, where λ_{path} is a constant that is independent of h but which can depend on the path travelled to the current node, which we have assumed has m up and n down steps.

As aggregate wealth is equal to the value of the risky asset—which is in unit supply—we must have

$$\int_0^1 \lambda_{\text{path}} h^m (1 - h)^n f(h) dh = p.$$

This enables us to solve for the value of λ_{path} :

$$\lambda_{\text{path}} = \frac{p}{\int_0^1 h^m (1 - h)^n f(h) dh}$$

Substituting back, agent h 's wealth equals

$$w_h = \frac{h^m (1 - h)^n p}{\int_0^1 h^m (1 - h)^n f(h) dh}.$$

This is maximized by $h \equiv m/(m + n)$: the agent whose beliefs turned out to be most accurate ex post ends up richest.

The wealth distribution—that is, the fraction of aggregate wealth held by type- h agents—satisfies

$$\frac{w_h f(h)}{p} = \frac{h^m (1 - h)^n f(h)}{\int_0^1 h^m (1 - h)^n f(h) dh}. \quad (3)$$

The *wealth-weighted cross-sectional average belief*, H , therefore equals

$$H = \int_0^1 h \frac{w_h f(h)}{p} dh = \frac{\int_0^1 h^{m+1} (1 - h)^n f(h) dh}{\int_0^1 h^m (1 - h)^n f(h) dh} \quad (4)$$

at time t , after m up-moves and n down-moves.⁵

These expressions take a particularly convenient form if $f(h) = h^{\alpha-1}(1-h)^{\beta-1}/B(\alpha, \beta)$ is the density function of the Beta(α, β) distribution. In that case equation (3) implies that the wealth distribution is also a beta distribution with parameters $\alpha + m$ and $\beta + n$, so that

$$\frac{w_h f(h)}{p} = \frac{h^{\alpha+m-1}(1-h)^{\beta+n-1}}{B(\alpha+m, \beta+n)}. \quad (5)$$

In this case the integrals in equation (4) can be evaluated, giving

$$H = \frac{m + \alpha}{t + \alpha + \beta}.$$

Example.—Let us now revisit Figure 1. For the sake of argument, suppose that $f(h)$ describes a beta distribution with $\alpha = \beta = 1$ so that investor beliefs $h \in (0, 1)$ are uniformly distributed. If, by time 4, there have been $m = 1$ up-moves and $n = 3$ down-moves, then equation (5) implies that the new wealth distribution follows the line labelled $\alpha = 2, \beta = 4$. (Investors with h close to 0 or to 1 have been almost wiped out by their aggressive trades; the best performers are moderate pessimists with $h = 1/4$, whose beliefs happen to have been vindicated ex post.) At time 8, following three more up-moves and one down-move, the new wealth distribution is indicated by the line labelled $\alpha = \beta = 5$. And if by time 12 there have been a further four up-moves then the wealth distribution is given by the line labelled $\alpha = 9, \beta = 5$. These shifts in the wealth distribution are central to our model: they reflect the fact that money flows, over time, toward investors whose beliefs appear correct in hindsight.

Now we solve for the equilibrium price using the first-order condition described in (2). The price p adjusts to clear the market, so that aggregate demand for the asset by

⁵If there is a fixed true probability of an up-move, h_{true} , then investors with $h \neq h_{\text{true}}$ will eventually become irrelevant. Let us refer to the share of wealth (3) held by type- h agents as $\Omega(h, m, n)$. As the elapsed number of periods, t , tends to infinity, $m/t \rightarrow h_{\text{true}}$ and $n/t \rightarrow 1 - h_{\text{true}}$ almost surely, by the strong law of large numbers, and the asymptotic rate of exponential decay in the type- h wealth share is

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \log \Omega(h, h_{\text{true}}t, (1 - h_{\text{true}})t) = h_{\text{true}} \log \frac{h_{\text{true}}}{h} + (1 - h_{\text{true}}) \log \frac{1 - h_{\text{true}}}{1 - h}.$$

This holds for any belief distribution $f(h)$. The decay rate is strictly positive when $h \neq h_{\text{true}}$, so the wealth share of any incorrect investor declines exponentially fast. But investors who are roughly correct will retain a substantial share of wealth for many periods. For example, the half-life for investor $h = 0.5$ —that is, $\frac{\log 2}{\text{decay rate}}$, the time required for the investor's wealth to halve—is more than 34 periods for all values of h_{true} between 0.4 and 0.6. See the Online Appendix for a proof and more discussion.

the agents equals the unit aggregate supply:

$$\int_0^1 x_h f(h) dh = \frac{p [H(p_u - p) - (1 - H)(p - p_d)]}{(p_u - p)(p - p_d)} = 1.$$

This has a unique solution with respect to p :

$$p = \frac{p_d p_u}{H p_d + (1 - H) p_u}. \quad (6)$$

In equilibrium, therefore, the risk-neutral probability of an up-move is

$$h^* = \frac{H p_d}{H p_d + (1 - H) p_u}. \quad (7)$$

It follows that

$$\frac{p_u}{p} = \frac{H}{h^*} \quad \text{and} \quad \frac{p_d}{p} = \frac{1 - H}{1 - h^*}. \quad (8)$$

Hence h^* is smaller than H if $p_u > p_d$ and larger than H if $p_u < p_d$: in either case, risk-neutral beliefs are more pessimistic than the wealth-weighted average belief.

The share of wealth an agent of type h invests in the risky asset is

$$\frac{x_h p}{w_h} = \frac{h - h^*}{H - h^*}, \quad (9)$$

using equations (2) and (8). We can use this equation to calculate the leverage of investor h , which we define as the ratio of funds borrowed to wealth:

$$\frac{x_h p - w_h}{w_h} = \frac{h - H}{H - h^*}.$$

The agent with $h = H$ can be thought of as *the representative agent*: by equation (9), this is the agent who chooses to invest her wealth fully in the market, with no borrowing or lending. Similarly, the investor with $h = h^*$ chooses to hold his or her wealth fully in the bond. Pessimistic investors with $h < h^*$ choose to short the risky asset; moderate investors with $h^* < h < H$ hold a balanced portfolio with long positions in both the bond and the risky asset; and optimistic investors with $h > H$ take on leverage, shorting the bond to go long the risky asset.

In a homogeneous economy in which all agents agree on the up-probability, $h = \bar{H}$,

it is easy to check that

$$h^* = \frac{\bar{H}p_d}{\bar{H}p_d + (1 - \bar{H})p_u}.$$

Comparing this expression with equation (7), we see that for *short-run* pricing purposes our heterogeneous economy looks the same as a homogeneous economy featuring a representative agent with belief H . But as the identity, H , of the representative agent changes over time in our model, the similarity will disappear when we study the pricing of multi-period claims.

To understand the expression for the wealth share (9), note that agent h perceives an expected excess return

$$\frac{hp_u + (1 - h)p_d}{p} - 1 = \frac{(h - h^*)(H - h^*)}{h^*(1 - h^*)} \quad (10)$$

and that the risk-neutral variance of the asset return is

$$h^* \left(\frac{p_u}{p} \right)^2 + (1 - h^*) \left(\frac{p_d}{p} \right)^2 - 1 = \frac{(H - h^*)^2}{h^*(1 - h^*)}. \quad (11)$$

In particular, these expressions imply that the representative agent's perceived risk premium equals risk-neutral variance. More generally, any individual's risky share (9) equals his or her perceived risk premium (10) divided by risk-neutral variance (11). Note, further, that by combining equation (10) with the definition (4) of H , the representative agent's perceived risk premium equals the wealth-weighted cross-sectional average risk premium, which is therefore positive. This object is distinct from the equally-weighted cross-sectional average risk premium, which in the beta case (1) is given by equation (10) with h replaced by $\alpha/(\alpha + \beta)$, and which may be negative.⁶

We can calculate the level of the VIX index on similar lines.⁷ To do so, we use the model-free relationship between VIX and risk-neutral entropy (see, e.g., [Martin, 2017](#)) $\text{VIX}_{t \rightarrow t+1}^2 = 2(\log \mathbb{E}_t^* R_{t \rightarrow t+1} - \mathbb{E}_t^* \log R_{t \rightarrow t+1})$, where $R_{t \rightarrow t+1}$ is the gross return on the risky asset from t to $t + 1$. As the net riskless rate is zero, we have $\mathbb{E}_t^* R_{t \rightarrow t+1} = 1$;

⁶For comparison, [Atmaz and Basak \(2018\)](#) study wealth-weighted average beliefs but not equally-weighted average beliefs in their theoretical work, while [Ben-David et al. \(2013\)](#) use equal-weighted measures in their empirical work.

⁷By definition, $\text{VIX}_{t \rightarrow t+1}^2 \equiv 2R_{f,t} \left(\int_0^{F_t} \frac{1}{K^2} \text{put}_t(K) dK + \int_{F_t}^{\infty} \frac{1}{K^2} \text{call}_t(K) dK \right)$, where $R_{f,t}$ is the gross one-period interest rate, F_t is the one-period-ahead forward price of the risky asset, and $\text{put}_t(K)$ and $\text{call}_t(K)$ are time t prices of one-period put and call options with strike K .

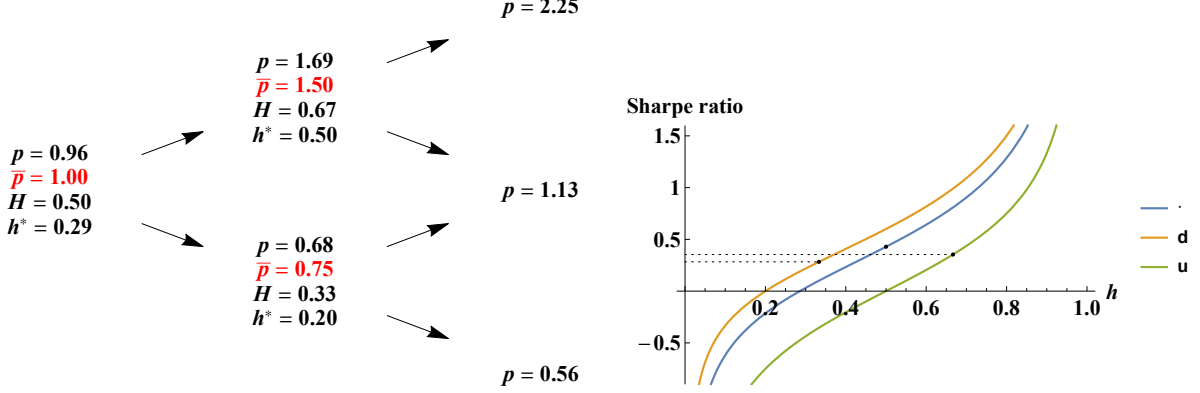


Figure 2: Left: \bar{p} denotes the price in a homogeneous economy with $\bar{H} = 1/2$; p is the price in a heterogeneous economy with $\alpha = \beta = 1$; and h^* and H indicate the risk-neutral probability of an up-move and the identity of the representative agent in the heterogeneous economy. Right: The Sharpe ratio perceived by different agents in the initial state (\cdot), down state (d), and up state (u).

together with the equilibrium relationship (8), this implies that

$$\text{VIX}_{t \rightarrow t+1}^2 = 2 \left[h^* \log \frac{h^*}{H} + (1 - h^*) \log \frac{1 - h^*}{1 - H} \right]. \quad (12)$$

Thus the VIX index equals twice the relative entropy (or Kullback–Leibler divergence) of the beliefs of the representative agent with respect to the beliefs of the agent who is out of the market. When VIX is high, the two agents have very different beliefs.

The left panel of Figure 2 gives a numerical example with uniformly distributed beliefs and $T = 2$. Sentiment in the heterogeneous belief economy is initially the same as it would be in a homogeneous economy— $H = 1/2$ at the initial node—but the price is lower because of sentiment risk. If bad news arrives, money flows to pessimists, the representative agent and risk-neutral beliefs become more pessimistic, and the price declines further than it would in a homogeneous economy.

The right panel plots the Sharpe ratios perceived by different investors in each of the possible states. As person h 's subjectively perceived variance of the asset's return is

$$h \left(\frac{p_u}{p} \right)^2 + (1 - h) \left(\frac{p_d}{p} \right)^2 - \left(\frac{hp_u + (1 - h)p_d}{p} \right)^2 = \frac{h(1 - h)(H - h^*)^2}{h^{*2}(1 - h^*)^2},$$

his or her perceived Sharpe ratio is

$$\frac{h - h^*}{\sqrt{h(1 - h)}},$$

which is increasing in h for all h^* . Extremists perceive extreme Sharpe ratios, reflecting their perception that true volatility is close to zero. This might seem surprising, given the heuristic that second moments of returns are relatively easy to measure empirically, and hence relatively difficult to disagree upon. Indeed this is to some extent an artefact of the two-period setting of the present example. In the Brownian limit of Section 2.4, all agents disagree about expected returns, but agree on the volatility of returns.

The figure also shows that all investors believe that Sharpe ratios are high in bad times and low in good times. Thus the model does not generate extrapolative beliefs (as studied empirically by Greenwood and Shleifer (2014) and theoretically by Barberis et al. (2015)) on the part of individual investors. But the *representative* investor (whose identity in each state is indicated by dots in the right panel) is more optimistic, and perceives a higher Sharpe ratio, in good times than in bad times. Our model generates extrapolative behavior in a dollar-weighted sense: “Mr. Market” disagrees with *every* individual investor about the behavior of Sharpe ratios in good and bad states.

1.2 The general case

From now on we will keep track of the current node by writing subscripts to indicate the number of up-moves to date and total time elapsed. Thus, for example, $p_{m,t}$ is the price at time t after m up-moves and $n = t - m$ down-moves, and $H_{m,t}$ and $h_{m,t}^*$ represent the identities of the representative investor and of the investor who is fully invested in cash, respectively. Translating the notation of the last section into this new notation, we have $p = p_{m,t}$, $p_u = p_{m+1,t+1}$, $p_d = p_{m,t+1}$, $H = H_{m,t}$, and $h^* = h_{m,t}^*$.

Writing $z_{m,t} = 1/p_{m,t}$, equation (6) implies that the following recurrence relation holds at each node:

$$z_{m,t} = H_{m,t} z_{m+1,t+1} + (1 - H_{m,t}) z_{m,t+1}. \quad (13)$$

That is, the price at each node is the weighted harmonic mean of the next-period prices, with weights given by the beliefs of the currently representative agent. This leads to our main pricing result, whose proof is in the Appendix.

Result 1. *If the risky asset has terminal payoffs $p_{m,T}$ at time T (for $m = 0, \dots, T$),*

then its initial price is⁸

$$p_0 = \frac{1}{\sum_{m=0}^T \frac{c_m}{p_{m,T}}}, \quad \text{where} \quad c_m = \binom{T}{m} \int_0^1 h^m (1-h)^{T-m} f(h) dh.$$

For reference, note that if the cross-section of beliefs obeys a beta distribution, as in equation (1), then c_m satisfies

$$c_m = \binom{T}{m} \frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}.$$

Conversely, if beliefs are homogeneous and all agents perceive that the probability of an up-move is \bar{H} , then $c_m = \binom{T}{m} \bar{H}^m (1 - \bar{H})^{T-m}$.

In our setting, pricing is the same as it would be if a single representative investor with appropriately chosen prior beliefs learned over time about the probability of an up-move. Although such a model is inconsistent with the evidence that individuals have different beliefs, the link reveals a sense in which the market exhibits “the wisdom of the crowd”, in that the redistribution of wealth between agents causes the market to behave as if it is learning as a whole.⁹

Result 2 (The wisdom of the crowd). *Pricing in the heterogeneous-agent economy is identical to pricing in an economy with a representative agent with log utility whose prior belief, as of time 0, about the probability of an up-move has distribution $f(h)$, and who updates his or her beliefs over time via Bayes’ rule.*

The next result characterizes the effect of belief heterogeneity on prices for a broad class of assets. In this result, and for the rest of the paper, we restrict attention to the beta family of belief distributions, so that $f(h)$ is the pdf given in equation (1).

Result 3. *If the risky asset has terminal payoffs such that $1/p_{m,T}$ is convex when viewed as a function of m , then the asset’s time 0 price decreases as beliefs become more heterogeneous (that is, as the cross-sectional variance of h increases with its mean held*

⁸It is of course also possible to represent the price as $p_0 = \sum_{m=0}^T q_m^* p_{m,T}$ for appropriately chosen risk-neutral probabilities that we provide in the appendix. In equilibrium, these risk-neutral probabilities are such that the formula in Result 1 holds.

⁹The existence of a representative investor in this sense is guaranteed by the results of Rubinstein (1976). Our result makes explicit what the beliefs of such an investor must be. See Blume and Easley (1993) and Blume and Easley (2010) for further discussion.

constant). In particular, it is sufficient, though not necessary, that $\log p_{m,T}$ be weakly concave for the asset’s price to be decreasing in the degree of belief heterogeneity.

Conversely, if $1/p_{m,T}$ is concave in m then the asset’s time 0 price increases as beliefs become more heterogeneous.

Result 3 implies that if the risky asset’s terminal payoff $p_{m,T}$ is concave in m , then its price declines as heterogeneity increases. But the same may be true even for assets with convex payoffs—for example, if the asset’s payoffs are exponential in m then the log payoff is linear, and hence weakly concave, in m . The examples of Sections 2.4 and 2.5 fall into this category. On the other hand,¹⁰ if the risky asset has highly convex payoffs—as might be the case for a “growth” asset with a large payoff in some extreme state of the world—then its price increases with heterogeneity. We explore the two cases further via concrete examples in Sections 2.1 and 2.2.

Notice that if there were only one period, then—with only two possible terminal payoffs— $1/p_{m,T}$ would be both convex and concave in m , and hence the asset’s price would be independent of the degree of heterogeneity. Thus the dynamic aspect of our model is critical for Result 3 to be interesting and nontrivial. It is therefore complementary to Simsek (2013, Theorems 4 and 5), who presents results with a superficially similar flavor in a static, one-period, model: whereas Result 3 characterizes the impact of heterogeneity in terms of the concavity or convexity of the asset’s payoffs, Simsek emphasizes the importance of the degree of skewness of beliefs in the minds of investors.

1.3 Bayesian learning

We can extend our model to allow the heterogeneous individuals to update their beliefs over time using Bayes’ rule. We continue to assume that each investor has a type $h \in (0, 1)$, and that types follow a beta distribution with parameters α and β , as in equation (1). Now, however, investor h ’s prior belief is that the probability of an up-move is $\tilde{h} \sim \text{Beta}(\zeta h, \zeta(1 - h))$. This prior has mean h and variance $h(1 - h)/(1 + \zeta)$, so is sharply peaked around h when the (positive) constant ζ is large. This structure allows us to calibrate the disagreement across individuals in the population, which is controlled by α and β , separately from the uncertainty in the mind of a fixed individual,

¹⁰The empirical evidence concerning the effect of belief dispersion on prices is mixed. Johnson (2004) considers levered firms with option-like payoffs and finds that price is increasing in belief dispersion (see also Chen et al., 2002; Yu, 2011) while Avramov et al. (2009), Banerjee (2011) and others find the opposite result.

which is controlled by ζ . In the limit as ζ tends to infinity, we recover the dogmatic limiting case considered in the rest of the paper.

Result 4 (Pricing with belief heterogeneity and learning). *If the risky asset has terminal payoffs $p_{m,T}$ at time T (for $m = 0, \dots, T$), then its initial price is*

$$p_0 = \frac{1}{\sum_{m=0}^T \frac{\tilde{c}_m}{p_{m,T}}}, \quad \text{where} \quad \tilde{c}_m = \binom{T}{m} \int_0^1 \frac{B(\zeta h + m, \zeta(1-h) + T - m)}{B(\zeta h, \zeta(1-h))} f(h) dh,$$

where $f(h)$ is the pdf of a beta distribution, as defined in equation (1).

If the risky asset has terminal payoffs such that $1/p_{m,T}$ is convex when viewed as a function of m , then for any level of belief heterogeneity the asset's time 0 price decreases as investors' prior uncertainty increases (i.e., as ζ decreases, with $\alpha/(\alpha+\beta)$ held constant so that the mean investor type is held constant). Conversely, if $1/p_{m,T}$ is concave in m then the asset's time 0 price increases as investors' prior uncertainty increases.

This result generalizes Results 1 and 2 in the case where $f(h)$ is the beta pdf. (To recover the former, let ζ tend to infinity; to recover the latter, set $\alpha = aN$, $\beta = bN$, and $\zeta = a + b$, and let N tend to infinity.) It shows that the effect of learning compounds the effect of sentiment, thereby putting Result 3 into a broader context. In the Online Appendix, we show how the initial price of the risky asset depicted in the left panel of Figure 2 varies when agents learn, for a range of values of $\zeta < \infty$; and we illustrate the effect of learning in Figure 3 of the next section. Elsewhere, we focus on the dogmatic limit case $\zeta \rightarrow \infty$.

2 Some examples

We now explore the properties of the model via a series of examples. The first three examples are highly stylized even by the standards of our highly stylized model: they are intended to bring out certain features of the model in a stark way. The final two examples are continuous-time limits that embed these features in a richer setting.

2.1 A risky bond

The dynamic that drives our model is particularly stark in the risky bond example outlined in the introduction. Suppose that the terminal payoff is 1 in all states apart

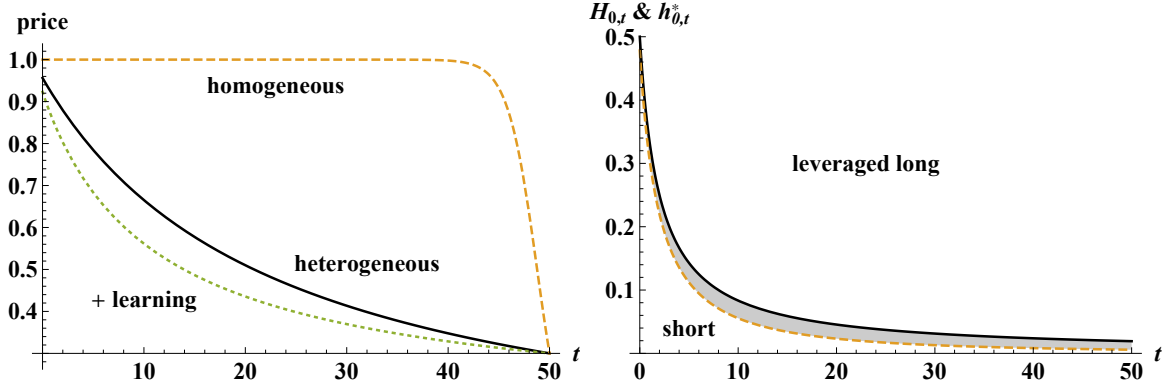


Figure 3: Left: The risky bond's price over time in the heterogeneous and homogeneous economies following consistently bad news. Right: The identity, at time t , following consistently bad news, of the representative agent, $H_{t,t}$; and of the investor who is fully invested in the riskless bond at time t , with zero position in the risky bond, $h_{0,t}^*$.

from the very bottom one, in which the payoff is ε ; the price of the asset is therefore 1 as soon as an up-move occurs. Writing p_t for the price at time t following t consecutive down-moves, Result 1 implies that¹¹

$$p_{0,t} = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} \frac{\Gamma(\beta+T)\Gamma(\alpha+\beta+t)}{\Gamma(\beta+t)\Gamma(\alpha+\beta+T)}}. \quad (14)$$

If the belief distribution is uniform, $\alpha = \beta = 1$, we can simplify further, to

$$p_{0,t} = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} \frac{1+t}{1+T}}. \quad (15)$$

We can calculate the risk-neutral probability of an up-move at time t , following 0 up- and t down-moves, $h_{0,t}^*$, by applying (8) with $p = p_{0,t}$ and $p_u = 1$ to find that

$$h_{0,t}^* = H_{0,t} p_{0,t} = \frac{\alpha p_{0,t}}{\alpha + \beta + t}.$$

Figure 3 illustrates these calculations with uniform beliefs ($\alpha = \beta = 1$), $T = 50$ periods to go, and a recovery value of $\varepsilon = 0.30$. The panels show how the price, risk-neutral probability, and the identity of the representative agent evolve if bad news arrives each period. (The left panel also shows how the price evolves if investors have

¹¹In this special case, we could argue directly: from equation (6), $p_{0,t} = \frac{\alpha p_{0,t+1} + (t+\beta)p_{0,t+1}}{\alpha p_{0,t+1} + t + \beta}$. Defining $y_t \equiv 1/p_{0,t} - 1$, this can be rearranged as $y_t = \frac{\beta+t}{\alpha+\beta+t} y_{t+1}$. Solving forward, imposing the terminal condition that $y_T = (1-\varepsilon)/\varepsilon$, and using the fact that $\Gamma(z+1)/\Gamma(z) = z$ for any $z > 0$, we have (14).

heterogeneous priors and learn about the probability of a down-move as in Section 1.3. We set $\zeta = 24$ so that the standard deviation of the median investor's prior belief about the probability of an up-move is 10%.)

For comparison, in a homogeneous economy with $H = 1/2$ the price and risk-neutral probability would be

$$p_{0,t} = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} 2^{-(T-t)}} \quad (16)$$

and $h_{0,t}^* = \frac{p_{0,t}}{2}$, respectively. Thus with homogeneous beliefs the bond price is approximately 1, and the risk-neutral probability of an up-move is approximately 1/2, until shortly before the bond's maturity.

From the perspective of time 0, the risk-neutral probability of default, δ^* , satisfies $p_{0,0} = 1 - \delta^* + \delta^* \varepsilon$, so $\delta^* = (1 - p_{0,0})/(1 - \varepsilon)$. In the homogeneous case, therefore, $\delta^* = 1/(1 + \varepsilon(2^T - 1)) = O(2^{-T})$, whereas in the heterogeneous case with uniform belief distribution we have $\delta^* = 1/(1 + \varepsilon T) = O(1/T)$. There is a qualitative difference between the homogeneous economy, in which default is exponentially unlikely, and the heterogeneous economy, in which default is only polynomially unlikely. More generally, it is straightforward to show that $\delta^* = O(T^{-\alpha})$ for any α and β , using Stirling's formula. And the result remains true if $\varepsilon > 1$, as in the bubbly asset example of the next section: the probability of the bubbly asset having a large payoff is exponentially small in the homogeneous economy but only polynomially small in the heterogeneous belief economy.

To understand pricing in the heterogeneous economy, it is helpful to think through the portfolio choices of individual investors. The median investor, $h = 0.5$, thinks the probability that the bond will default—i.e., that the price will follow the path shown in Figure 3 all the way to the end—is $2^{-50} < 10^{-15}$. Even so, he believes the price is right at time zero (in the sense that he *is* the representative agent) because of the short-run impact of sentiment. Meanwhile, a modestly pessimistic agent with $h = 0.25$ will choose to short the bond at the price of 0.9563—and will remain short at time $t = 1$ before reversing her position at $t = 2$ —despite believing that the bond's default probability is less than 10^{-6} . (Recall from equation (9) that $h_{0,t}^*$ is the belief of the agent who is neither long nor short the asset. More optimistic agents, $h > h_{0,t}^*$, are long, and more pessimistic agents, $h < h_{0,t}^*$, are short.) Following a few periods of bad news, almost all investors are long; but the most pessimistic investors have become extraordinarily wealthy.

The left panel of Figure 4 shows the holdings of the risky asset for a range of investors with different beliefs, along the trajectory in which bad news keeps on coming. The

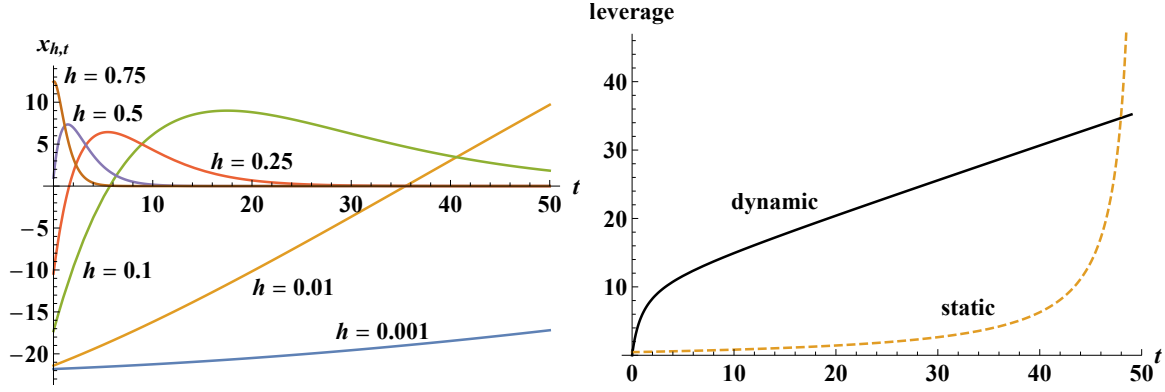


Figure 4: Left: The number of units of the risky bond held by different agents, $x_{h,t}$, plotted against time. Right: The evolution of leverage for the median investor under the optimal dynamic and static strategies. Both panels assume bad news arrives each period.

optimistic investor $h = 0.75$ starts out highly leveraged so rapidly loses almost all his money. The median investor, $h = 0.5$, initially invests fully in the risky bond without leverage. If bad news arrives, this investor takes on leverage in order to be able to increase the size of his position despite his losses; after about 10 periods, the median investor is almost completely wiped out. Moderately bearish investors start out short. For example, investor $h = 0.25$ starts out short about 10 units of the bond, despite believing that the probability it defaults is less than one in a million, but reverses her position after two down-moves. Investor $h = 0.01$, who thinks that there is more than a 60% chance of default, is initially extremely short but eventually reverses position as still more bearish investors come to dominate the market.

The right panel of Figure 4 shows how the median investor's leverage changes over time if he follows the optimal dynamic and static strategies. If forced to trade statically, his leverage ratio is initially 0.457. This seemingly modest number is dictated by the requirement that the investor avoid bankruptcy at the bottom node. If the median investor can trade dynamically, by contrast, the optimal strategy is, initially, to invest fully in the risky bond without leverage. Subsequently, however, optimal leverage rises fast. Thus the dynamic investor keeps his powder dry by investing cautiously at first but then aggressively exploiting further selloffs. We report further results on the evolution of aggregate leverage and volume in the Online Appendix.

2.2 A bubbly asset

We now modify the example of the previous section by considering the case in which the extreme payoff ε is greater than 1. This seemingly trivial modification will reveal the differing effects of sentiment on assets with left- and right-skewed payoffs. In this case we refer to the asset as *bubbly* because—in contrast with the risky bond case—sentiment will inflate the price of the risky asset when $\varepsilon > 1$, by Result 3.

As the extreme payoff now corresponds to a good, rather than bad, outcome, we will think of the asset as paying $\varepsilon > 1$ in the “top” state, i.e. if there are T consecutive up moves, and 1 otherwise. Hence the price of the asset is 1 if ever there is a down move.

But our interest now is in the evolution of the price if there is repeated good news. For general $\alpha > 0$ and $\beta > 0$, the price at time t , following t up moves, is

$$p_{t,t} = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} \frac{\Gamma(\alpha+T)}{\Gamma(\alpha+t)} \frac{\Gamma(\alpha+\beta+t)}{\Gamma(\alpha+\beta+T)}}. \quad (17)$$

Thus the price rises with each successive piece of good news. In part, of course, these rises simply reflect good news about fundamentals, which would also cause price rises in a homogeneous-belief economy.

To isolate the influence of belief heterogeneity on pricing, we therefore define the *sentiment multiplier* as the ratio of the price (17) to the price that would prevail in a homogeneous economy in which all agents perceive $h = \alpha/(\alpha + \beta)$. Along the path on which good news keeps on coming,

$$\text{sentiment multiplier}_t = \frac{p_{t,t}}{\bar{p}_{t,t}} = \frac{1 + \frac{1-\varepsilon}{\varepsilon} \left(\frac{\alpha}{\alpha+\beta} \right)^{T-t}}{1 + \frac{1-\varepsilon}{\varepsilon} \frac{\Gamma(\alpha+T)}{\Gamma(\alpha+t)} \frac{\Gamma(\alpha+\beta+t)}{\Gamma(\alpha+\beta+T)}}. \quad (18)$$

We define the sentiment multiplier for the risky bond analogously. Heterogeneity in beliefs drives the bubbly asset’s price up but the risky bond’s price down, by Result 3, so the sentiment multiplier is initially greater than 1 for the bubbly asset and less than 1 for the risky bond. In either case it equals 1 at the terminal time T (as the price is then equal to the payoff whether or not there is heterogeneity in beliefs).

The left panel of Figure 5 shows the evolution of the sentiment multiplier over time along the path in which the extreme outcome—the “top” outcome in the case of the bubbly asset, and the “bottom” outcome in the case of the risky bond—remains possible. Beliefs are uniformly distributed between zero and one, that is, $\alpha = \beta = 1$. We set $\varepsilon = 25$ for the bubbly asset and (symmetrically, from the point of view of a log investor)

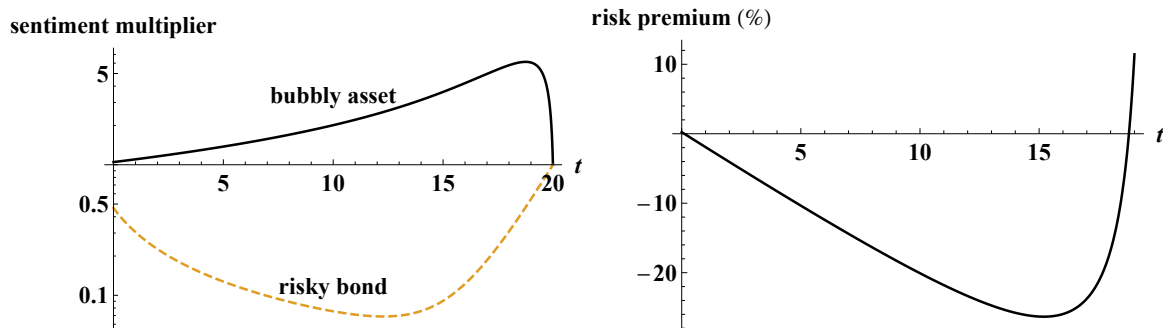


Figure 5: Left: Sentiment multipliers along the extreme paths. Log scale. Right: The risk premium on the bubbly asset, as perceived by the median investor.

$\varepsilon = 1/25$ for the risky bond, and $T = 20$.

Sentiment has little effect on the pricing of the bubbly asset early in its life, but becomes much more important following repeated good news: the multiplier is initially only slightly greater than one, but accelerates—it is convex, even on a log scale—toward a peak shortly before time T . Conversely, sentiment has a substantial effect on the price of the risky bond early in its life. We provide a formal result along these lines for arbitrary α and β below.

The risk premium perceived by the median investor is positive at first—though small because, as we will see, volatility is initially very low. As the bubble emerges, the median investor’s perceived risk premium turns negative and declines as sentiment drives the asset’s price up. But it then starts to rise, and ultimately turns positive toward the height of the bubble, as the terminal date T approaches (Figure 5, right panel).¹²

These facts have a striking implication: the median investor reverses his position *twice* over the lifetime of the bubble. He starts out long, as the representative investor at time 0. Following good news, he goes short as optimists drive the price higher than he thinks reasonable. But if good news keeps coming, he reverses position a second time to go long again at time $T - 1$.¹³

Implied volatility, as measured by the VIX index, rises as the bubbly asset experiences repeated good news (Figure 6, left panel).¹⁴ Conversely, implied volatility declines as the

¹²By contrast, in the risky bond example, the median investor perceives that the risk premium rises monotonically along the extreme price path.

¹³Somewhat more optimistic agents are long for a more extended period at both the beginning and the end. For example, investor $h = 0.80$ starts out long, goes short in period 5, and reverses position to go long again in period 15, as can be seen in Figure 6, right panel.

¹⁴The relationship between volatility and bubbles has been widely noted. For example, [Cochrane \(2003\)](#) links high volatility to the high prices of growth stocks around the turn of the millennium; more

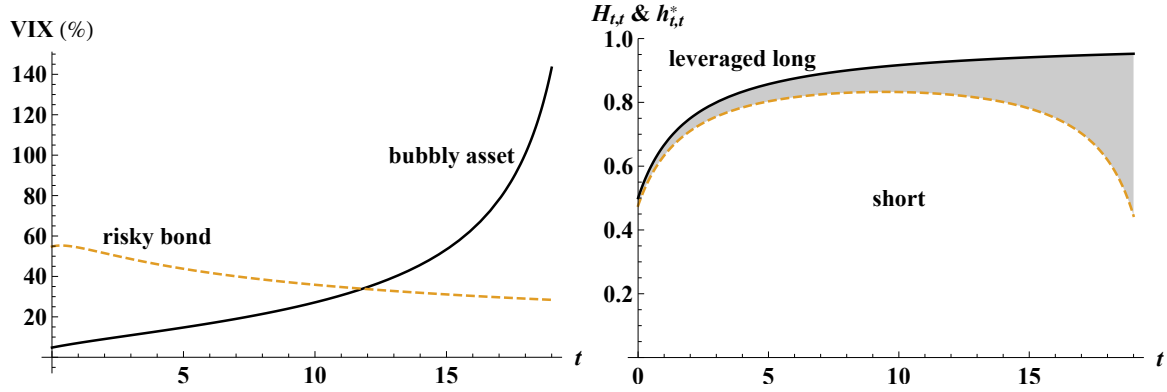


Figure 6: Left: The evolution of the one-period VIX index, following consistently good news along the same path. Right: The identity, t , following consistently good news, of the representative agent, $H_{t,t}$ (solid); and of the investor who is fully invested in the riskless bond, $h_{t,t}^*$ (dashed).

risky bond experiences repeated bad news. The central asymmetry that separates the two examples is that risk considerations drive the price down and *toward* the extreme payoff for the risky bond, but down and *away* from the extreme payoff for the bubbly asset.

We can interpret the behavior of the median investor through this lens. As shown in equation (12), movements in VIX measure the difference in beliefs between the representative agent ($h = H_{t,t}$) and the investor who is out of the market and on the boundary between the longs and shorts ($h = h_{t,t}^*$). As there is a limit to how optimistic the representative agent can become, $h_{t,t}^*$ must eventually decline to open the gap as VIX rises along the bubble path (Figure 6, right panel, which should be contrasted with Figure 3, right panel)—to the extent that the median agent ends up long, i.e., $h_{t,t}^* < 1/2$.

We close this section with a result that applies for arbitrary values of α and β . In order to formulate a clean statement, we consider the two extreme cases in which ε tends to zero or to infinity. We refer to these as the *risky bond limit* and *bubbly asset limit*, respectively. The result formalizes a sense in which sentiment has most effect early in the life of a risky bond, but late in the life of a bubbly asset.

Result 5. *Let $\alpha > 0$ and $\beta > 0$ be arbitrary.*

In the risky bond limit, the sentiment multiplier is less than 1 for $0 \leq t \leq T - 1$. Sentiment becomes less important over time along the extreme path: the multiplier increases monotonically from a minimum at time 0 to 1 at time T .

recently, [Gao and Martin \(2021\)](#) argue empirically that the high and rising level of implied volatility in the late 1990s points to bubblieness in the stock market at the time.

In the bubbly asset limit, the sentiment multiplier is greater than 1 for $0 \leq t \leq T - 1$. Sentiment becomes more important over time along the extreme path: the multiplier increases from time 0 toward a maximum at time $T - 1$, before dropping back to 1 at time T .

2.3 Speculation on sentiment

Many authors have noted that markets exhibit more volatility than seems justified by fundamental news; classic references include Shiller (1981) and Roll (1984). In our setting, investors may speculate on sentiment even if there is essentially no news arriving about fundamentals. In doing so, they generate excess volatility.

Consider an example in which the terminal payoff is up in the air until the very last period. If there have been an even number of up-moves by the terminal date, the asset pays off $\frac{1}{1+\varepsilon}$, where ε lies between 0 and 1; if there have been an odd number of up-moves, the asset pays $\frac{1}{1-\varepsilon}$. To streamline the analysis, suppose that T is odd (so that there is an even number of terminal nodes).

In a homogeneous economy with $\bar{H} = 1/2$, the asset trades at a price of 1 at every node until the terminal payoff: there is therefore *no* volatility, and the asset is totally riskless, until the very last period.

With heterogeneity in beliefs, it follows from Result 1 that the initial price is 1 if the distribution of beliefs is symmetric around $1/2$ in the sense that $f(h) = f(1 - h)$ for all h .¹⁵ But sentiment creates volatility, and this volatility reflects changing risk premia in the minds of all investors. Suppose that beliefs are uniformly distributed, as they are in Figure 7, which illustrates with $T = 3$ and $\varepsilon = 1/2$. Then, at time 1, the risky asset's price rises to $1/(1 - \varepsilon/T)$ if there is an up-move but drops to $1/(1 + \varepsilon/T)$ if there is a down-move.¹⁶

In equilibrium, our investors use the risky asset to speculate against each other. This cannot end well for all of them. If we write $\tilde{\mathbb{E}}$ to indicate a cross-sectional mean and $R_{0 \rightarrow T}^{(h)}$ to denote the return on agent h 's chosen strategy, then we have $\tilde{\mathbb{E}} R_{0 \rightarrow T}^{(h)} = R_{0 \rightarrow T}$, because (by market clearing) the cross-sectional average return on investors' strategies

¹⁵This implies that $c_m = c_{T-m}$ and hence, as T is odd, $\sum_{m=0}^T (-1)^m c_m = 0$. Together with the fact that $\sum_{m=0}^T c_m = 1$, this gives the result. (The initial price is also 1 if there is learning, as \tilde{c}_m , defined in Result 3, also has the properties $\sum_{m=0}^T (-1)^m \tilde{c}_m = 0$ and $\sum_{m=0}^T \tilde{c}_m = 1$.)

¹⁶This follows from Lemma 1 in the Appendix. There is also an equilibrium in which the asset's price is 1 until time $T - 1$, as in the homogeneous economy. Then the market is incomplete, and agents have no means of betting against one another. But this equilibrium is not robust to vanishingly small generic perturbations of the terminal payoffs, which would restore market completeness.

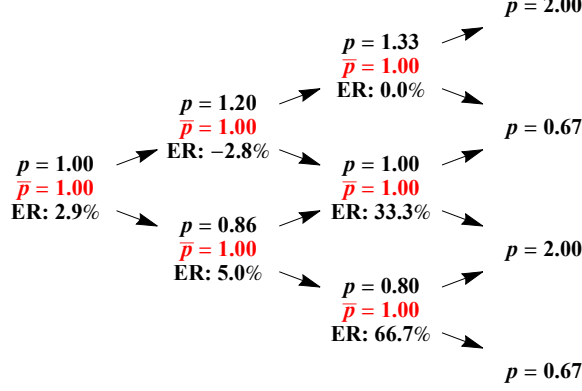


Figure 7: Speculation on sentiment. Heterogeneous-economy price (p), homogeneous-economy price (\bar{p}), and the cross-sectional average perceived excess return in the heterogeneous economy (ER).

equals the return on aggregate wealth. It follows, by Jensen's inequality, that average realized utility is lower than it would be if all agents held the risky asset statically:

$$\tilde{\mathbb{E}} \log R_{0 \rightarrow T}^{(h)} < \log \tilde{\mathbb{E}} R_{0 \rightarrow T}^{(h)} = \log R_{0 \rightarrow T}.$$

In this sense, speculation is socially costly. Note, however, that every investor believes that speculation is in his or her selfish interest: the ability to speculate raises expected utility above what is attainable by statically holding the risky asset. We return to this point in a more conventional example in Section 2.4.3.

2.4 A Brownian limit

In this section, we consider a natural continuous time limit by allowing the number of periods to tend to infinity and specifying geometrically increasing terminal payoffs. This is the setting of Cox et al. (1979), in which the option price formula of Black and Scholes (1973) emerges in the corresponding limit with homogeneous beliefs.

We divide the time interval from time 0 to time T into $2N$ periods of length $T/(2N)$. (The choice of an even number of periods is unimportant, but it simplifies the notation in some of our proofs.) Terminal payoffs are $p_{m,T} = e^{2\sigma\sqrt{\frac{T}{2N}}(m-N)}$, as in the Cox–Ross–Rubinstein model. As we will see, σ can be interpreted as the volatility of log terminal payoffs, on which all agents agree.

As the number of steps increases, the extent of disagreement over any individual step must decline to generate sensible limiting results. We achieve this by setting $\alpha = \theta N + \eta\sqrt{N}$ and $\beta = \theta N - \eta\sqrt{N}$ in (1), which makes the distribution of h —that is,

of investor beliefs about the probability of a single up-move—increasingly spiky as N increases. Small values of θ correspond to substantial belief heterogeneity, while the limit $\theta \rightarrow \infty$ represents the homogeneous case. The parameter η allows for asymmetry in the distribution of beliefs. Using tildes to denote cross-sectional means and variances, the cross-sectional mean of h satisfies $\widetilde{\mathbb{E}}[h] = \frac{1}{2} + \frac{\eta}{2\theta\sqrt{N}}$ and $\widetilde{\text{var}}[h] = \frac{1}{8\theta N} + O\left(\frac{1}{N^2}\right)$.

Given that, by design, the cross-sectional variance of h shrinks toward zero, it becomes convenient to parametrize an agent by the number of standard deviations, $z = (h - \widetilde{\mathbb{E}} h) / \sqrt{\widetilde{\text{var}} h}$, by which his or her belief deviates from the mean. Thus an agent with $z = 2$ is two standard deviations more optimistic than the mean agent. Standard results on the beta distribution imply that the cross-sectional distribution of z is asymptotically standard Normal. When we use this parametrization, we write superscripts z rather than h : for example, $\mathbb{E}^{(z)}$ rather than $\mathbb{E}^{(h)}$.

Using Result 1 to price the asset, then taking the limit as $N \rightarrow \infty$, we have the following result.

Result 6. *The price of the asset at time 0 is*

$$p_0 = \exp\left(\frac{\eta}{\theta}\sigma\sqrt{2T} - \frac{\theta+1}{2\theta}\sigma^2T\right).$$

Consistent with Result 3, the price declines as beliefs become more heterogeneous (i.e., as θ decreases with η/θ , and hence the mean level of optimism, held constant).

We now study agents' return expectations.

Result 7. *The return of the asset from time 0 to time t , from the perspective of agent $h = \widetilde{\mathbb{E}}[h] + z\sqrt{\widetilde{\text{var}}[h]}$ has a lognormal distribution with*

$$\begin{aligned}\mathbb{E}^{(z)} \log R_{0 \rightarrow t} &= \frac{\theta+1}{\theta + \frac{t}{T}} \left(\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta+1}{2\theta}\sigma^2 \right) t \\ \text{var}^{(z)} \log R_{0 \rightarrow t} &= \left(\frac{\theta+1}{\theta + \frac{t}{T}} \right)^2 \sigma^2 t.\end{aligned}$$

Thus agents agree on the second moment but disagree on the first moment of log returns. Agents also agree that log returns are negatively autocorrelated:

$$\text{corr}^{(z)}(\log R_{0 \rightarrow t_1}, \log R_{t_1 \rightarrow t_2}) = -\frac{1}{\sqrt{1 + \frac{(\theta T + t_1)^2}{t_1(t_2 - t_1)}}}.$$

The annualized expected return of the asset from 0 to t is

$$\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} = \frac{\theta + 1}{\theta + \frac{t}{T}} \left[\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta + 1}{2\theta} \frac{2\theta + \frac{t}{T}}{\theta + \frac{t}{T}} \sigma^2 \right].$$

The cross-sectional mean (or median) expected return is

$$\tilde{\mathbb{E}} \left[\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} \right] = \frac{(\theta + 1)^2 \left(\theta + \frac{t}{2T} \right)}{\theta \left(\theta + \frac{t}{T} \right)^2} \sigma^2.$$

Disagreement—that is, the cross-sectional standard deviation of $\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t}$ —is

$$\sqrt{\widetilde{\text{var}} \left[\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} \right]} = \frac{\theta + 1}{\theta + \frac{t}{T}} \frac{\sigma}{\sqrt{\theta T}}.$$

Note that if dynamic trade were shut down entirely, so that all agents had to trade once at time 0 and then hold their positions statically to time T , then equilibrium would not exist in the limit. To see this, write ψ_z for the share of wealth invested by agent z in the risky asset. Given any positive time 0 price, $R_{0 \rightarrow T}$ is lognormal from every agent's perspective by Result 7 (which applies even in the static case at horizon T , because the terminal payoff is specified exogenously). Hence we must have $\psi_z \in [0, 1]$ for all z to avoid the possibility of terminal wealth becoming negative. Market clearing requires that $\psi_z = 1$ on average across agents, so we must in fact have $\psi_z = 1$ for all z . But there is no positive price at which all agents choose to invest fully in the risky asset.

Our next result characterizes option prices. The unusual feature of the result is not that options can be quoted in terms of the Black–Scholes formula, as this is always possible, but that the associated implied volatilities $\tilde{\sigma}_t$ can be expressed in a simple yet non-trivial closed form. (We denote risk-neutral variance with an asterisk in Result 8 and throughout the paper.)

Result 8. *The time 0 price of a call option with maturity t and strike price K obeys the Black–Scholes formula with maturity-dependent implied volatility $\tilde{\sigma}_t$:*

$$C(t, K) = p_0 \Phi \left(\frac{\log \frac{p_0}{K} + \frac{1}{2} \tilde{\sigma}_t^2 t}{\tilde{\sigma}_t \sqrt{t}} \right) - K \Phi \left(\frac{\log \frac{p_0}{K} - \frac{1}{2} \tilde{\sigma}_t^2 t}{\tilde{\sigma}_t \sqrt{t}} \right), \quad \text{where} \quad \tilde{\sigma}_t = \frac{\theta + 1}{\sqrt{\theta \left(\theta + \frac{t}{T} \right)}} \sigma.$$

It follows that the level of the VIX index (at time 0, for settlement at time t) is $VIX_{0 \rightarrow t} =$

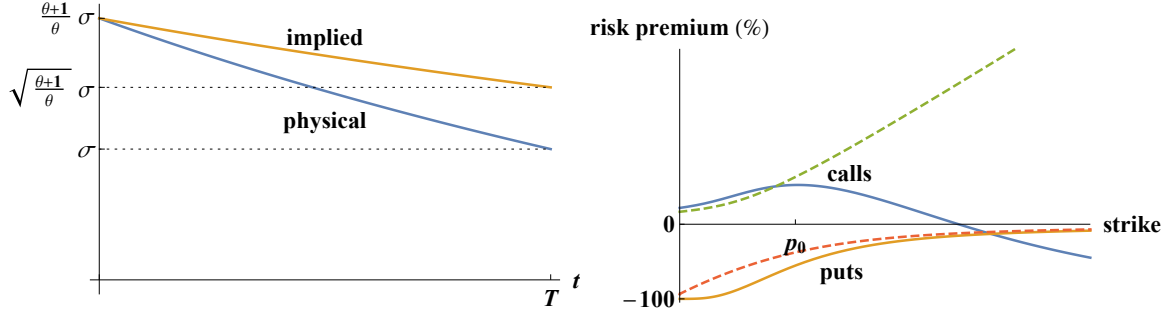


Figure 8: Left: The term structures of implied and physical volatility. Right: Expected excess returns on options of different strikes, K , as perceived by the median investor, $z = 0$. Solid/dashed lines denote heterogeneous/homogeneous beliefs.

$\tilde{\sigma}_t$, and that there is a variance risk premium, on which all agents agree:

$$\frac{1}{T} (\text{var}^* \log R_{0 \rightarrow T} - \text{var} \log R_{0 \rightarrow T}) = \frac{\sigma^2}{\theta}.$$

In the limit as $\theta \rightarrow \infty$, implied and physical volatility are each equal to σ and there is no variance risk premium, as in [Black and Scholes \(1973\)](#). But with heterogeneity, $\theta < \infty$, speculation boosts implied and physical volatility, particularly in the short run, and opens up a gap between implied and physical volatility in the long run. The existence of such a variance risk premium is a robust feature of the data; see, for example, [Bakshi and Kapadia \(2003\)](#), [Carr and Wu \(2009\)](#), and [Bollerslev et al. \(2011\)](#). (Implied volatility is constant across strikes here, but this is not a general property of our framework: the Poisson limit of Section 2.5 generates a volatility “smirk.”)

To understand intuitively *why* there is a variance risk premium, note that for any tradable payoff X and SDF M , one has the identity

$$\text{var}^* X - \text{var} X = R_f \text{cov} [M, (X - \kappa)^2], \quad (19)$$

where R_f is the gross riskless rate and $\kappa = (\mathbb{E} X + \mathbb{E}^* X)/2$ is a constant. (This identity requires only that there is no arbitrage; we are not aware of any prior references to it in the literature.) In our setting, $X = \log R_{0 \rightarrow T}$ and $R_f = 1$; different people perceive different physical probabilities and SDFs but agree on physical variance, $\text{var} \log R_{0 \rightarrow T}$,

as shown in Result 7; and $\kappa = z\sigma\sqrt{T}/(2\sqrt{\theta})$ is person-specific, so (19) specializes to

$$\text{var}^* \log R_{0 \rightarrow T} - \text{var} \log R_{0 \rightarrow T} = \text{cov}^{(z)} \left[M_{0 \rightarrow T}^{(z)}, \left(\log R_{0 \rightarrow T} - \frac{z\sigma\sqrt{T}}{2\sqrt{\theta}} \right)^2 \right].$$

From the perspective of the median agent ($z = 0$), for example, the presence of a variance risk premium indicates that the SDF is positively correlated with $(\log R_{0 \rightarrow T})^2$, i.e., that bad times are associated with extreme values of $\log R_{0 \rightarrow T}$.

To see why this is the case, we will study individual agents' trading strategies in the next section. For now, as a suggestive indication, the right panel of Figure 8 shows the risk premia on options perceived by the median investor. In a homogeneous economy, out-of-the-money call options have—as levered claims on the risky asset—high expected excess returns. With heterogeneous beliefs, the median investor perceives that deep out-of-the-money calls are so overvalued due to the presence of extremists that they earn *negative* expected excess returns.¹⁷

A calibration.—We illustrate the predictions of the model in a simple calibration. We do so with the obvious (but important) caveat that our model is highly stylized; moreover, the results above show that the parameter θ , which controls belief heterogeneity, simultaneously dictates several quantities that a priori need not be linked. The goal of the exercise is merely to point out that a single value of θ can generate predictions of broadly the right order of magnitude across multiple dimensions.

We set the horizon over which disagreement plays out to $T = 10$ years, and we set σ , which equals the volatility of log fundamentals (i.e., payoffs), to 12%. In our baseline calibration, we set $\theta = 1.8$, which implies that one-month, one-year, and two-year implied volatilities are 18.6%, 18.2%, and 17.7%, respectively, as shown in Table 1. These numbers are close to their empirically observed counterparts: in the data of Martin and Wagner (2019), mean implied volatility is 18.6%, 18.1%, and 17.9% at the one-month, one-year, and two-year horizons.

The model-implied cross-sectional mean expected returns are 3.2% and 1.8% at the one- and 10-year horizons. For comparison, in the survey data of Ben-David et al. (2013), the corresponding time-series average levels of cross-sectional average expected returns are 3.8% and 3.6%. The cross-sectional standard deviations of expected returns

¹⁷This perception is qualitatively consistent with the findings of Coval and Shumway (2001), who “find considerable evidence that both call and put contracts earn exceedingly low expected returns. A strategy of buying zero-beta straddles has an average return of around -3 percent per week.”

	Data	Model ($\theta = 1.8$)	Model ($\theta = 0.2$)
1mo implied vol	18.6%	18.6%	70.5%
1yr implied vol	18.1%	18.2%	58.8%
2yr implied vol	17.9%	17.7%	50.9%
1yr cross-sectional mean risk premium	3.8%	3.2%	28.8%
1yr disagreement	4.8%	4.2%	33.9%
10yr cross-sectional mean risk premium	3.6%	1.8%	5.0%
10yr disagreement	2.9%	2.8%	8.5%

Table 1: Observables in the model with $\theta = 1.8$ (baseline) and $\theta = 0.2$ (high disagreement) and, time-averaged, in the data.

(“disagreement”) at the one- and 10-year horizons are 4.2% and 2.8% in the model and 4.8% and 2.9%, on average, in the data of [Ben-David et al. \(2013\)](#).

An alternative interpretation of our model would interpret time 0 as a time when the market is preoccupied by some new phenomenon over which there is considerable disagreement. With 2008 in mind, one might imagine agents disagreeing about the implications of the Lehman Brothers default and the likely severity of the ensuing recession; in early 2020, the COVID-19 coronavirus is sweeping the world. On both occasions, short-term measures of implied volatility rose to extraordinarily high levels. Within our model, heightened belief heterogeneity (low θ) generates steeply downward-sloping term structures of volatility and of risk premia. To capture scenarios such as these, the table also reports results for a calibration with $\theta = 0.2$. (For comparison, the implied volatility measure SVIX, introduced in [Martin \(2017\)](#), rose to 74.1% at the one-month horizon and 45.9% at the one-year horizon in late November 2008.) We plot the term structures of physical and implied volatilities, and of the average risk premium and disagreement, in the two calibrations in the Online Appendix.

2.4.1 Speculation in equilibrium

Our investors speculate using complicated dynamic trading strategies. These determine, for each investor, an equilibrium return on wealth that is a function of the return on the underlying risky asset. To express this in a convenient form, we make two definitions. First, we refer to the investor

$$z = z_g = -\frac{\theta + 1}{\sqrt{\theta}} \sigma \sqrt{T} \quad (20)$$

as the *gloomy investor*. As we will see, the gloomy investor is an Eeyore-like figure (Milne, 1926) who has the lowest expected utility of all investors. There are, of course, more pessimistic investors ($z < z_g$), but they are less gloomy in the sense that they perceive attractive opportunities associated with short positions in the risky asset. Second, we introduce the notion of an investor-specific *target return* $K^{(z)}$ defined via¹⁸

$$\log K^{(z)} = \mathbb{E}^{(z)} \log R_{0 \rightarrow T} + (z - z_g) \sigma \sqrt{\theta T}. \quad (21)$$

The target return represents the ideal outcome for investor z : it is the realized return on the risky asset that maximizes wealth, and hence utility, ex post.

Result 9. *Agent z 's equilibrium return on wealth, $R_{0 \rightarrow T}^{(z)}$, can be expressed as a function of the return on the risky asset, $R_{0 \rightarrow T}$, as*

$$R_{0 \rightarrow T}^{(z)} = \sqrt{\frac{\theta + 1}{\theta}} \exp \left\{ \frac{1}{2} (z - z_g)^2 - \frac{1}{2(1 + \theta) \sigma^2 T} [\log (R_{0 \rightarrow T} / K^{(z)})]^2 \right\}. \quad (22)$$

Thus agent z 's terminal wealth is maximized when $R_{0 \rightarrow T} = K^{(z)}$, and as

$$\mathbb{E}^{(z)} \log R_{0 \rightarrow T}^{(z)} = \frac{1}{2} \log \frac{\theta + 1}{\theta} + \frac{(z - z_g)^2 - 1}{2(1 + \theta)},$$

the gloomy investor has the lowest expected utility of all investors.

Figure 9a shows how different investors' outcomes depend on the risky asset's realized payoff. The best-case scenario for investor z is that the target return is attained, $R_{0 \rightarrow T} = K^{(z)}$, in which case $R_{0 \rightarrow T}^{(z)} = \sqrt{\frac{\theta + 1}{\theta}} \exp \left\{ \frac{1}{2} (z - z_g)^2 \right\}$. An extremist's best-case scenario is better than that of a moderate investor, because it is cheap to purchase claims to states of the world that extremists consider likely, as few people are extremists. Furthermore, the best case scenario for an optimistic agent $z > 0$ is better than that of the symmetrically pessimistic investor—agent $-z < 0$ —because there is more aggregate wealth to go around in good states than in bad states.¹⁹

In our model, there is a useful distinction between what investors expect to happen and what they would like to happen. (The distinction also exists, but is uninteresting, in representative-agent models, as the target price is then infinity.) The gloomy investor would like to be proved right in logs: by equation (21), his target log return equals his

¹⁸Result 7 expresses the expected log return in terms of exogenous parameters.

¹⁹Note that z_g is negative, so that if $z > 0$ we have $(z - z_g)^2 > (-z - z_g)^2$.

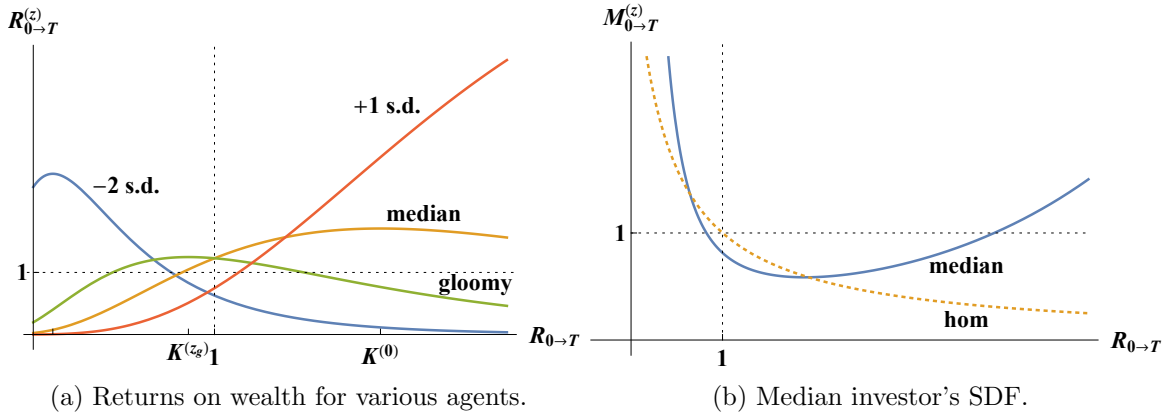


Figure 9: Left: Return on wealth, as a function of the realized return on the risky asset, for different agents. Dots indicate the expected return on the risky asset perceived by each investor. Right: All investors have U-shaped SDFs when beliefs are heterogeneous. The figure shows the SDF of the median investor, together with the SDF that would prevail in a homogeneous economy.

expected log return. Targets and expectations differ for all other investors. More optimistic investors have a target return that exceeds their expectations—i.e., they are best off if the risky asset modestly outperforms their expectations—while more pessimistic investors are best off if the risky asset modestly underperforms their expectations. But any investor does very poorly if the asset performs far better or worse than he or she anticipated, consistent with the presence of a variance risk premium and the discussion surrounding identity (19).

As our investors have log utility, their perceived SDFs satisfy $M_{0 \rightarrow T}^{(z)} = 1/R_{0 \rightarrow T}^{(z)}$. (Note that SDFs differ across investors because they—the investors—disagree on true probabilities but agree on asset prices.) As all investors' wealth returns are decreasing in the return on the risky asset once that return becomes sufficiently large, our model therefore generates a U-shaped SDF (Aït-Sahalia and Lo, 2000; Jackwerth, 2000) for every investor. Figure 9b illustrates by plotting the median investor's SDF as a function of $R_{0 \rightarrow T}$ in the heterogeneous economy and, for comparison, in a homogeneous economy.

Thus far we have thought of investors' strategies in dynamic terms. But their optimally chosen wealth return can also be implemented via a static trade, by holding long (or short) positions in options in regions in which the wealth return is convex (or concave) in the underlying risky asset return.

Result 10. *The optimal wealth return can be implemented by holding put options at strikes $K < p_0 K^{(z)}$ and call options at strikes $K > p_0 K^{(z)}$, with position size at strike K*

proportional to $R_{0 \rightarrow T}^{(z)''}(K)$; together with a riskless bond position. Thus investors are long (short) options in regions in which $R_{0 \rightarrow T}^{(z)}$ is convex (concave) as a function of $R_{0 \rightarrow T}$. In particular, moderate investors—including the gloomy investor, the median investor, and those in between—are short options with strikes close to $\exp \mathbb{E}^{(z)} \log p_T$, whereas extremists are long options with strikes in the corresponding range.

We can translate this result back to the dynamic setting to gain intuition for how investors can implement their optimal strategy by timing the market. For, as is well known, the exposure of a short option position to its underlying asset moves in the opposite direction to the asset itself. As the underlying asset appreciates, the short option position becomes less exposed to it (or more negatively exposed to it); conversely, if the underlying asset sells off, the short option position becomes more exposed to it. (In the option-trading jargon, short option positions have negative “gamma.”) Hence Result 10 formalizes a sense in which moderate investors seek negative exposure to the market in states of the world in which the market is high, and positive exposure in states in which the market is low. That is, they trade in a contrarian fashion over the region of possible outcomes that they consider likely to materialize.

2.4.2 Maximum-Sharpe-ratio strategies: a cautionary tale

As the log return on the risky asset is perceived as Normally distributed by all agents, we can use equation (22) to calculate the first and second (subjectively perceived) moments of each agent’s chosen return. These are

$$\mathbb{E}^{(z)} R_{0 \rightarrow T}^{(z)} = \frac{1 + \theta}{\sqrt{\theta(2 + \theta)}} \exp \left\{ \frac{(z - z_g)^2}{2 + \theta} \right\} \quad \text{and} \quad \mathbb{E}^{(z)} \left[R_{0 \rightarrow T}^{(z)2} \right] = \frac{1 + \theta}{\theta} \sqrt{\frac{1 + \theta}{3 + \theta}} \exp \left\{ \frac{3(z - z_g)^2}{3 + \theta} \right\},$$

and they pin down the Sharpe ratio of agent z ’s chosen investment strategy. Similarly, the Sharpe ratio of a static investment in the risky asset can be calculated using Result 7.

We can contrast these with the *maximum* Sharpe ratios that investors perceive as attainable. We use the Hansen and Jagannathan (1991) bound to compute the latter; the bound can be attained as the market is dynamically complete.

Result 11. *The maximum Sharpe ratio (MSR) perceived by investor z is $\text{MSR}_{0 \rightarrow t}^{(z)} = \sqrt{\text{var}^{(z)} M_{0 \rightarrow t}^{(z)}}$, where investor z ’s SDF variance is finite for $\theta > 1$ and equal to*

$$\text{var}^{(z)} M_{0 \rightarrow t}^{(z)} = \frac{\theta}{\sqrt{\theta^2 - (t/T)^2}} \exp \left\{ \frac{(z - z_g)^2 t/T}{\theta - t/T} \right\} - 1. \quad (23)$$

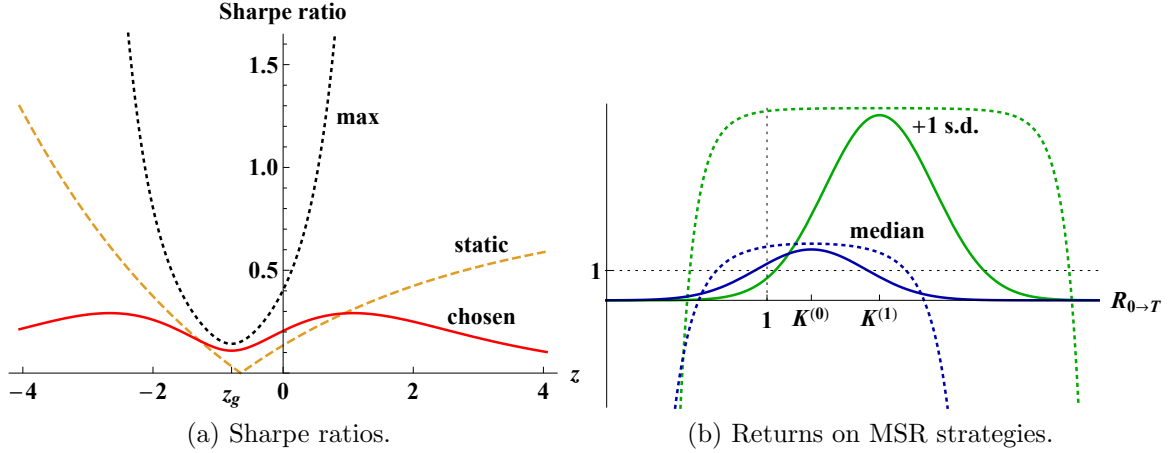


Figure 10: Left: The annualized Sharpe ratios, from 0 to T , that investors perceive on their own chosen strategies (solid) and on a static position in the risky asset (dashed); and the perceived maximum Sharpe ratio attainable through dynamic trading (dotted). Baseline calibration. Right: Realized returns on the strategies chosen by investors $z = 0$ and 1 (solid) and the realized returns on their MSR strategies (dotted) as a function of the realized return on the risky asset. Log scale on x -axis.

Hence the gloomy investor perceives the minimal maximum Sharpe ratio.

It follows that the annualized MSR perceived by agent z over very short horizons is

$$\lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} \text{MSR}_{0 \rightarrow t}^{(z)} = \frac{|z - z_g|}{\sqrt{\theta T}}. \quad (24)$$

(We annualize, here and in the figures below, by scaling the Sharpe ratio by $\frac{1}{\sqrt{t}}$.) This equals the instantaneous Sharpe ratio of the risky asset. But over longer horizons, all agents believe that there are dynamic strategies with Sharpe ratios strictly exceeding that of the risky asset.

Although the gloomy investor perceives that it is impossible to earn positive Sharpe ratios in the very short run, as is clear from equation (24), he perceives that positive Sharpe ratios are attainable at longer horizons: by Result 11,

$$\text{MSR}_{0 \rightarrow T}^{(z_g)} = \sqrt{\frac{\theta}{\sqrt{\theta^2 - 1}}} - 1.$$

Figure 10a shows that the maximum attainable Sharpe ratio exceeds the Sharpe ratio on a static position in the risky asset, indicating that all investors must trade dynamically (that, must speculate) to achieve their perceived MSR. But the figure also shows that

the Sharpe ratios that investors perceive on their own optimally chosen strategy are not in general close to the maximum Sharpe ratio or to the Sharpe ratio of the market.

More strikingly, Result 11 implies that if there is substantial disagreement, $\theta \leq 1$, all investors perceive that *arbitrarily* high Sharpe ratios are attainable at long horizons. At first sight, this might seem obviously inconsistent with equilibrium. But our investors are not mean-variance optimizers so Sharpe ratios do not adequately summarize investment opportunities.²⁰

To see why, we can study the strategies that achieve these maximal Sharpe ratios. By the work of Hansen and Richard (1987), a MSR strategy for investor z must take the form $a - bM_{0 \rightarrow T}^{(z)}$ for some constants $a > 0$ and $b > 0$, where $a = 1 + b\mathbb{E}^{(z)}[M_{0 \rightarrow T}^{(z)2}]$. As the return on wealth chosen by investor z , which we derived in Result 9, reveals the investor's SDF, $M_{0 \rightarrow T}^{(z)} = 1/R_{0 \rightarrow T}^{(z)}$, we can write an MSR return as

$$R_{\text{MSR},0 \rightarrow T}^{(z)} = 1 + b \left(\text{var}^{(z)} M_{0 \rightarrow T}^{(z)} + 1 \right) - \frac{b}{R_{0 \rightarrow T}^{(z)}}, \quad (25)$$

where $\text{var}^{(z)} M_{0 \rightarrow T}^{(z)}$ is provided in equation (23) and b can be any positive constant (the free parameter reflects the fact that any strategy can be combined with a position in the riskless asset without altering its Sharpe ratio). Figure 10b plots the realized return $R_{\text{MSR},0 \rightarrow T}^{(z)}$ as a function of the risky return $R_{0 \rightarrow T}$, for investors $z = 0$ and $z = 1$ in the baseline calibration. The MSR strategies could be implemented dynamically via a contrarian market-timing strategy that goes long if the market sells off and short if the market rallies, thereby exploiting what investors view as irrational exuberance on the upside and irrational pessimism on the downside; or statically—by the logic of Result 10—via extremely short positions in out-of-the-money call and put options.

We view this as a cautionary tale. If betas are calculated with respect to the market return, or to any investor's optimally chosen return, then MSR strategies—or factors that load up on tail risk—will earn large alphas. But our investors do not do mean-variance analysis, so alphas are not useful or interesting measures for them, and although it is possible to earn high Sharpe ratios via short option positions, these strategies are not remotely attractive to our investors, who prefer to choose strategies that lie well inside the mean-variance efficient frontier. Indeed, as MSR strategies feature the possibility of unboundedly negative gross returns, our investors would prefer to invest fully in (say) cash than to rebalance, even slightly, toward a MSR strategy.

²⁰The correct risk-adjusted measure of the attractiveness of investment opportunities is, for our investors, the expected log return. As shown in Result 9, this is finite for all investors for any $\theta > 0$.

Note, finally, that this is all true even in an ostensibly well-behaved setting in which investors have log utility and returns are universally agreed to be lognormally distributed. In Section 2.5, we will show that in a limit featuring jumps, all investors perceive that arbitrarily high Sharpe ratios are attainable in *any* calibration.

2.4.3 Ex ante attitudes to speculation

We have seen that investors believe that substantial gains in Sharpe ratio can be achieved by speculating. But Sharpe ratios do not adequately capture our investors' attitudes to speculation. A better measure is provided by agent z 's perceived gain from speculation, $\xi^{(z)}$, which satisfies the equation $\mathbb{E}^{(z)} \log R_{0 \rightarrow T}^{(z)} = \mathbb{E}^{(z)} \log [(1 + \xi^{(z)}) R_{0 \rightarrow T}]$. This is the proportional increase in wealth that would leave investor z as happy, holding the market, as he or she would have been when allowed to speculate. More generally, we can ask what investor z thinks investor x 's gain from speculation is. When we do so, we assume that investor z uses his or her own beliefs in assessing investor x 's expected utility, and we assume that other investors continue to trade, so that prices are unaffected by investor x 's absence: thus we wish to solve

$$\mathbb{E}^{(z)} \log R_{0 \rightarrow T}^{(x)} = \mathbb{E}^{(z)} \log [(1 + \xi^{(z,x)}) R_{0 \rightarrow T}] \quad (26)$$

for $\xi^{(z,x)}$. (Note that $\xi^{(z)} = \xi^{(z,z)}$.) As $\xi^{(z,x)}$ is a dollar measure of the gain from speculation, we can then aggregate over x to determine agent z 's view of the impact of speculation on social welfare. In doing so, we are committing to the utilitarian idea of cardinal utility that can be compared across people.²¹

Result 12 (Ex ante attitudes to speculation). *Investor z 's perception of investor x 's gain from speculation, $\xi^{(z,x)}$, is*

$$\xi^{(z,x)} = \sqrt{\frac{\theta + 1}{\theta}} \exp \left\{ \frac{z^2 - 1}{2(1 + \theta)} - \frac{(z - x)^2}{2\theta} \right\} - 1.$$

This is positive for investor types x that are sufficiently close to z and negative otherwise. Aggregating over x , investor z 's perception of the aggregate gain to speculation is

$$\xi = \exp \left\{ -\frac{1}{2(1 + \theta)} \right\} - 1,$$

²¹See, for example, [Harsanyi \(1955\)](#) on the use of cardinal utility in interpersonal welfare comparisons.

which is independent of z and negative for all $\theta > 0$.

Ex ante, all investors perceive that the ability to speculate is in their own interest and in the interest of investors with beliefs sufficiently similar to their own, as $\xi^{(z,z)} > 0$. But they all also think that speculation is socially costly, as $\xi < 0$. In the terminology of Brunnermeier et al. (2014), speculation is *belief-neutral inefficient*, despite every investor finding it attractive.²² Moreover, as heterogeneity increases—that is, as θ decreases—the degree of dissonance increases, in the sense that all investors perceive that speculation is increasingly beneficial for them personally but increasingly costly for the population as a whole.

2.4.4 Ex post regret and inequality

Ex post, there will always be some investors who are happy to have speculated—because their chosen return $R_{0 \rightarrow T}^{(z)}$ turned out to be higher than the static return $R_{0 \rightarrow T}$ —and others who are regretful (that is, whose realized utility is lower as a result of speculating than it would have been if they had held the risky asset statically).

On average, however, people are regretful, in the utilitarian sense that average realized utility is lower than it would have been had all agents held a static position in the risky asset. This is a direct consequence of inequality in the presence of risk aversion. To see this, we can measure inequality using the Atkinson (1970) inequality index, A , which satisfies²³

$$\log(1 - A) = \tilde{\mathbb{E}} \log R_{0 \rightarrow T}^{(z)} - \log R_{0 \rightarrow T}. \quad (27)$$

Average ex post regret, $\log R_{0 \rightarrow T} - \tilde{\mathbb{E}} \log R_{0 \rightarrow T}^{(z)}$, is therefore a function of ex post inequality, A . Equation (27) shows that the Atkinson index can be interpreted as the fraction of wealth that could be sacrificed while holding social welfare constant, if wealth were redistributed equally across the population ex post.

The extent of ex post inequality depends on how surprising the realized outcome is, in the mind of the median investor—specifically, on the number of standard deviations by

²²Brunnermeier et al. (2014) present some examples of economies with inefficient speculation in the presence of heterogeneous beliefs, but their examples have no aggregate risk.

²³Atkinson (1970) defined a family of indices indexed by an inequality aversion parameter, ε . In equation (27) we are considering the case $\varepsilon = 1$, which is widely used in practice and which has a natural interpretation in our equilibrium.

which the realized log return on the risky asset exceeds the median investor's expectation,

$$s = \frac{\log R_{0 \rightarrow T} - \mathbb{E}^{(0)} \log R_{0 \rightarrow T}}{\sqrt{\text{var}^{(0)} \log R_{0 \rightarrow T}}}.$$

Result 13 (Ex post inequality). *The Atkinson inequality index satisfies*

$$A = 1 - \sqrt{\frac{\theta + 1}{\theta}} \exp \left\{ -\frac{s^2}{2(1 + \theta)} - \frac{1}{2\theta} \right\}.$$

Thus inequality is minimized if the realized log return on the risky asset meets the expectations of the median investor, and is high if the realized log return is far from the median investor's expectations.

2.5 A Poisson limit

We now consider an alternative continuous time limit in which the risky asset is subject to jumps that arrive at times dictated (in the limit) by a Poisson process. We think of this setting as representing a stylized model of insurance or credit markets in which credit events or catastrophes arrive suddenly and cause large losses.

We divide the period from 0 to T into N steps, and we will let N tend to infinity. We want the mean agent to perceive a jump arrival rate of λ , and the cross-sectional standard deviation to be of a similar order of magnitude. These considerations dictate that the distribution of agent types h should be concentrated around a mean of $1 - \lambda dt$ (so that the probability of a decline is λdt , where we write $dt = T/N$) and should have standard deviation $\omega \lambda dt$. Exploiting the flexibility of the beta family of belief distributions (1), we therefore set

$$\alpha_N = \frac{N}{\omega^2 \lambda T} \quad \text{and} \quad \beta_N = \frac{1}{\omega^2},$$

which achieves the desired mean and standard deviation in the limit as $N \rightarrow \infty$.

If there are no jumps, the terminal payoff is one; we assume that each jump causes the same proportional loss to the terminal payoff, so that the payoffs are $p_{m,T} = e^{-(N-m)J}$ for some constant J . This setup might be viewed as a stylized model of a risky bond, for example. Our next result applies Result 1 to characterize pricing in the limit as $N \rightarrow \infty$. The price is only defined under an assumption that jumps are not too frequent

or severe, and that there is not too much disagreement:

$$\omega^2 \lambda T (e^J - 1) < 1. \quad (28)$$

(We will treat J as positive, so that jumps represent bad news, but our results go through for negative J , in which case a jump represents good news and (28) is always satisfied.) As before, we parametrize investors by z , which indexes the number of standard deviations more optimistic than the mean a given investor is; thus person z thinks that the Poisson process has jump arrival rate $\lambda(1 - z\omega)$.

Result 14. *The price at time t , if q jumps have occurred, is*

$$p_{q,t} = e^{-qJ} \left(1 - \frac{\omega^2 \lambda (T-t)}{1 + \omega^2 \lambda t} (e^J - 1) \right)^{q + \frac{1}{\omega^2}}. \quad (29)$$

Investor z 's SDF at time t is a function of q , the number of jumps that have occurred:

$$M_{0 \rightarrow t}^{(z)} = \frac{\Gamma(q + \frac{1}{\omega^2})}{\Gamma(\frac{1}{\omega^2})} [1 - \omega^2 \lambda T (e^J - 1)]^{\frac{1}{\omega^2}} [1 - \omega^2 \lambda T (e^J - 1) + \omega^2 \lambda t e^J]^{-q - \frac{1}{\omega^2}} \left[\frac{\omega^2 e^J}{1 - z\omega} \right]^q e^{\lambda(1-z\omega)t}.$$

Expected utility is well defined for all investors because $\mathbb{E}^{(z)} \log R_{0 \rightarrow T}^{(z)} = -\mathbb{E}^{(z)} \log M_{0 \rightarrow T}^{(z)}$ is finite. But all investors perceive that arbitrarily high Sharpe ratios are attainable, because $M_{0 \rightarrow t}^{(z)}$ has infinite variance.

Agent z 's return on optimally invested wealth is $R_{0 \rightarrow t}^{(z)} = 1/M_{0 \rightarrow t}^{(z)}$, so the richest agent at time t can be identified by minimizing $M_{0 \rightarrow t}^{(z)}$ with respect to z , giving $z_{\text{richest}} = (\lambda t - q)/(\omega \lambda t)$. This agent perceives arrival rate $\lambda_{\text{richest}} = \frac{q}{t}$, so has beliefs that appear correct in hindsight.

We can calculate the risky asset share of agent z by comparing the return on wealth with the return on the risky asset (which can be computed using the price (29)):

$$\text{risky share}_t^{(z)} = 1 + \frac{\omega}{e^J - 1} \underbrace{\left[1 - \frac{\omega^2 \lambda}{1 + \omega^2 \lambda T} e^J (T-t) \right]}_{> 0 \text{ by assumption (28)}} \frac{1 + \omega^2 \lambda t}{1 + \omega^2 q} \left[\frac{\omega(q - \lambda t)}{1 + \omega^2 \lambda t} + z \right].$$

The representative agent (with risky share equal to one) is therefore $z = -\frac{\omega(q - \lambda t)}{1 + \omega^2 \lambda t}$, with perceived jump arrival rate $\lambda_{\text{rep},t} = \lambda + \frac{\omega^2 \lambda t}{1 + \omega^2 \lambda t} \left(\frac{q}{t} - \lambda \right)$. Thus initially the mean investor is representative. Subsequently, the representative investor's perceived arrival rate grows if the realized jump arrival rate is higher than expected ($q/t > \lambda$) and declines

otherwise. For large t , the representative investor perceives an arrival rate close to the historically realized arrival rate q/t .

The investor who is out of the market at time t perceives arrival rate

$$\lambda_t^* = \frac{1 + q\omega^2}{1 - \omega^2\lambda T(e^J - 1) + \omega^2\lambda te^J} e^J \lambda. \quad (30)$$

Agents who are more pessimistic are short the risky asset. They lose money while nothing happens, but experience sudden gains if a jump arrives. Conversely, agents who are more optimistic are long, so do well if nothing happens but are exposed to jump risk; one can think of the pessimists as having purchased jump insurance from the optimists.

It follows from equation (30) that if jumps are sufficiently large—if $e^J - 1 \geq 1$ —then $\lambda_t^* \geq \lambda$ for all t and q . In this case, the mean investor is never short the risky asset, no matter what happens. By contrast, in any calibration of the Brownian limit there are sample paths on which the mean investor goes short the risky asset.

The risk-neutral arrival rate measures the cost of insuring against a jump. We will refer to it as the CDS rate, λ_t^* , as it equals the price (scaled by the length of contract horizon) of a very short-dated CDS contract that pays \$1 if there is a jump:

$$\lambda_t^* = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{P}_t^* (\text{at least one jump occurs in } [t, t + \varepsilon]). \quad (31)$$

We have already used λ_t^* to denote the arrival rate (30) perceived by the investor who is out of the market, but the next result shows that the two quantities coincide.

Result 15. *The risk-neutral arrival rate, or CDS rate, is λ_t^* as defined in equation (30).*

The CDS rate jumps when there is a Poisson arrival and declines smoothly as t increases during periods where there are no arrivals. (For comparison, the CDS rate is constant over time in the homogeneous case: $\lambda_{\text{hom}}^* = e^J \lambda$.) Initially, when $t = q = 0$, the CDS rate is unambiguously higher in the presence of belief heterogeneity:

$$\lambda_0^* = \frac{1}{\underbrace{1 - \omega^2\lambda T(e^J - 1)}_{\in (0,1) \text{ by assumption (28)}}} e^J \lambda > \lambda_{\text{hom}}^*.$$

By the terminal date, $t = T$, we have $\lambda_T^* = \frac{1+q\omega^2}{1+\lambda T\omega^2} \lambda_{\text{hom}}^*$. Thus λ_T^* may be larger or smaller than λ_{hom}^* , depending on whether the realized number of jumps exceeded the mean agent's expectations ($q > \lambda T$) or not.

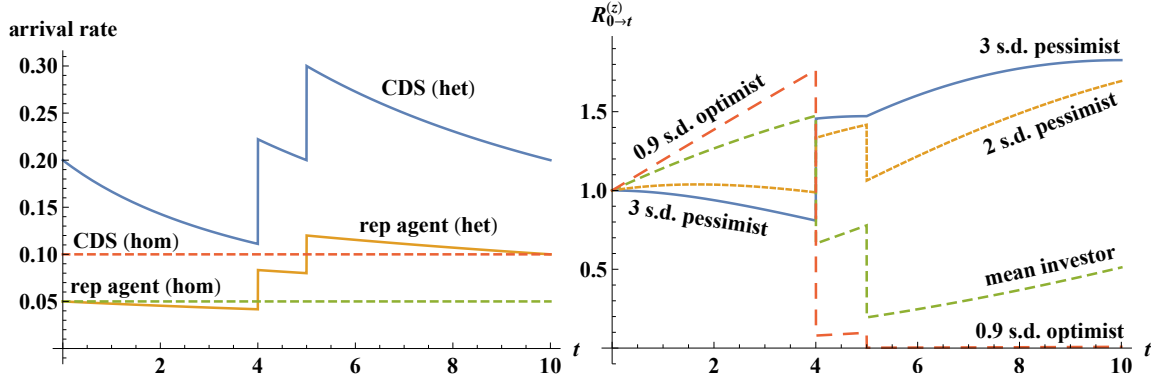


Figure 11: Left: The evolution of the representative agent’s subjectively perceived arrival rate, and of the CDS rate (i.e., risk-neutral arrival rate), in the heterogeneous and homogeneous economies, on a sample path with jumps occurring at times $t = 4$ and $t = 5$. Right: The evolution over time of the wealth of four agents ($z = -3, -2, 0, 0.9$) on the same sample path.

Figure 11 shows how the equilibrium evolves along a particular sample path on which two jumps occur in quick succession, at times $t = 4$ and $t = 5$. We set $\omega = 1$, $\lambda = 0.05$ and $T = 10$ and assume that half of the fundamental value is destroyed every time there is a jump, that is, $e^{-J} = 1/2$, or $e^J - 1 = 1$. The figure shows a relatively unlucky sample path, on which the expectations of the pessimistic agent $z = -3$ are realized; for comparison, the mean agent only expected 0.5 jumps over the ten years.

The left panel shows the evolution of the representative agent’s subjectively perceived arrival rate, and of the CDS rate. These two quantities decline smoothly during quiet periods with no jumps, but spike immediately after a jump arrives. (Similar patterns have been documented in catastrophe insurance markets by [Froot and O’Connell \(1999\)](#) and [Born and Viscusi \(2006\)](#), and have also been studied theoretically by [Duffie \(2010\)](#).) By contrast, in a homogeneous economy, each would be constant over time.

As we have seen, the CDS rate reveals the identity of the investor who is out of the market. More optimistic investors hold long positions in the risky asset, analogous to selling insurance or shorting CDS contracts. They accumulate wealth in quiet times, but experience sudden losses when bad news arrives. Pessimistic investors, who perceive higher arrival rates than the CDS rate, are short the risky asset, which is analogous to buying insurance or going long CDS. Their wealth bleeds away during quiet times, but they experience sudden windfalls if bad news arrives.

The right panel plots the cumulative return on wealth for four different agents over the same sample path. The figure shows two pessimists, who are two and three standard

deviations below the mean, and who therefore perceive arrival rates of 0.15 and 0.20, respectively; the mean investor, with perceived arrival rate 0.05; and an optimist who is 0.9 standard deviations above the mean, with perceived arrival rate 0.005. (All agents must perceive a positive arrival rate, and this imposes a limit on how optimistic an agent can be: as $\omega = 1$ in our calibration, we must have $z < 1$.)

The optimist and the mean investor are both long the asset (i.e., short jump insurance) throughout the sample path. The two pessimists buy or sell insurance depending on whether the CDS rate is above or below their subjectively perceived arrival rates. By the time of the first jump, both are short the asset—long jump insurance—so experience sudden increases in wealth at $t = 4$. In this example, the positions of the four investors in the wealth distribution are reversed as a result of the first jump. As the CDS rate then spikes, the two pessimists reverse their positions temporarily, and are short jump insurance between times 4 and 5. At the instant the jump occurs at time 5, the $z = -3$ pessimist is out of the market, so her wealth is unaffected by the jump. The $z = -2$ pessimist is still selling insurance, however, so experiences a loss.

We present an option-pricing formula for the Poisson limit in the Online Appendix. Notably, the model generates a smile with high volatility at low strikes, and a hump-shaped term structure of implied volatility.

3 Conclusion

We have presented a dynamic model in which individuals have heterogeneous beliefs. Short sales are allowed; all agents are risk-averse; and all agents are marginal. Wealth shifts toward agents whose beliefs are correct in hindsight, whether through luck or judgment, so the identity of the representative investor, “Mr. Market,” changes constantly over time, becoming more optimistic following good news and more pessimistic following bad news. These shifts in sentiment induce speculation—that is, agents take on positions they would not wish to hold to maturity.

The model can be interpreted as a stylized account of a single market episode, a period during which investors are preoccupied by some phenomenon over whose implications there is considerable disagreement: examples include the aftermath of 9/11, the subprime crisis of 2008–9, or the COVID-19 coronavirus which has spread across the world as we write. Fuller models might allow such events to occur repeatedly, perhaps at times dictated by a Poisson process; or for disagreement to be multi-dimensional. We leave such extensions for future research.

As our framework allows for general terminal payoffs, we can characterize conditions under which increasing belief heterogeneity drives prices up or down. If the risky asset is a “growth” asset with a high payoff in an extreme state, it will be overvalued relative to the homogeneous benchmark; but for a wide class of payoffs sentiment has the effect of driving prices lower, and risk premia higher. As the framework is also extremely tractable, we are able to solve the model analytically rather than relying on numerical solutions, and to go further than the prior literature in studying the interplay between concrete quantities ranging from volume and leverage, to the level of the VIX index, the variance risk premium or CDS rates, to the finer details of investors’ beliefs and behaviors.

References

- Aït-Sahalia, Y. and Lo, A. W. (2000). Nonparametric risk management and implied risk aversion. *Journal of Econometrics*, 94(1):9–51.
- Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory*, 2(3):244–263.
- Atmaz, A. and Basak, S. (2018). Belief dispersion in the stock market. *The Journal of Finance*, 73(3):1225–1279.
- Avramov, D., Chordia, T., Jostova, G., and Philipov, A. (2009). Dispersion in analysts’ earnings forecasts and credit rating. *Journal of Financial Economics*, 91(1):83–101.
- Bakshi, G. and Kapadia, N. (2003). Delta-hedged gains and the negative market volatility risk premium. *The Review of Financial Studies*, 16(2):527–566.
- Banerjee, S. (2011). Learning from prices and the dispersion in beliefs. *The Review of Financial Studies*, 24(9):3025–3068.
- Banerjee, S. and Kremer, I. (2010). Disagreement and learning: Dynamic patterns of trade. *The Journal of Finance*, 65(4):1269–1302.
- Barberis, N., Greenwood, R., Jin, L., and Shleifer, A. (2015). X-CAPM: An extrapolative capital asset pricing model. *Journal of Financial Economics*, 115(1):1–24.
- Basak, S. (2005). Asset pricing with heterogeneous beliefs. *Journal of Banking & Finance*, 29(11):2849–2881.
- Ben-David, I., Graham, J. R., and Harvey, C. R. (2013). Managerial miscalibration. *Quarterly Journal of Economics*, 128(4):1547–1584.
- Bhamra, H. S. and Uppal, R. (2014). Asset prices with heterogeneity in preferences and beliefs. *Review of Financial Studies*, 27(2):519–580.

- Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659.
- Blume, L. and Easley, D. (1993). Economic natural selection. *Economics Letters*, 42(2):281–289.
- Blume, L. and Easley, D. (2006). If you’re so smart, why aren’t you rich? Belief selection in complete and incomplete markets. *Econometrica*, 74(4):929–966.
- Blume, L. and Easley, D. (2010). Heterogeneity, selection, and wealth dynamics. *Annual Review of Economics*, 2:425–450.
- Bollerslev, T., Gibson, M., and Zhou, H. (2011). Dynamic estimation of volatility risk premia and investor risk aversion from option-implied and realized volatilities. *Journal of Econometrics*, 160(1):235–245.
- Born, P. and Viscusi, W. K. (2006). The catastrophic effects of natural disasters on insurance markets. *Journal of Risk and Uncertainty*, 33:55–72.
- Borovička, J. (2020). Survival and long-run dynamics with heterogeneous beliefs under recursive preferences. *Journal of Political Economy*, 128(1):000–000.
- Breeden, D. T. and Litzenberger, R. H. (1978). Prices of state-contingent claims implicit in option prices. *Journal of Business*, 51(4):621–651.
- Brunnermeier, M. K., Simsek, A., and Xiong, W. (2014). A welfare criterion for models with biased beliefs. *Quarterly Journal of Economics*, 129(4):1711–1752.
- Buraschi, A. and Jiltsov, A. (2006). Model uncertainty and option markets with heterogeneous beliefs. *The Journal of Finance*, 61(6):2841–2897.
- Carr, P. and Wu, L. (2009). Variance risk premiums. *Review of Financial Studies*, 22(3):1311–1341.
- Chan, Y. L. and Kogan, L. (2002). Catching up with the Joneses: Heterogeneous preferences and the dynamics of asset prices. *Journal of Political Economy*, 110(6):1255–1285.
- Chen, H., Joslin, S., and Tran, N.-K. (2012). Rare disasters and risk sharing with heterogeneous beliefs. *The Review of Financial Studies*, 25(7):2189–2224.
- Chen, J., Hong, H., and Stein, J. C. (2002). Breadth of ownership and stock returns. *Journal of Financial Economics*, 66(2-3):171–205.
- Cochrane, J. H. (2003). Stocks as money: Convenience yield and the tech-stock bubble. In Hunter, W. C., Kaufman, G. G., and Pomerleano, M., editors, *Asset Price Bubbles*. MIT Press, Cambridge.
- Coval, J. D. and Shumway, T. (2001). Expected option returns. *Journal of Finance*, 56(3):983–1009.

- Cox, J., Ross, S., and Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, 7:229–263.
- Cvitanić, J., Jouini, E., Malamud, S., and Napp, C. (2011). Financial markets equilibrium with heterogeneous agents. *Review of Finance*, 16(1):285–321.
- De Long, J. B., Shleifer, A., Summers, L. H., and Waldmann, R. J. (1990). Noise trader risk in financial markets. *Journal of Political Economy*, 98(4):703–738.
- Duffie, D. (2010). Presidential address: Asset price dynamics with slow-moving capital. *Journal of Finance*, 65(4):1237–1267.
- Dumas, B. (1989). Two-person dynamic equilibrium in the capital market. *The Review of Financial Studies*, 2(2):157–188.
- Dumas, B., Kurshev, A., and Uppal, R. (2009). Equilibrium portfolio strategies in the presence of sentiment risk and excess volatility. *Journal of Finance*, 64(2):579–629.
- Froot, K. and O’Connell, P. (1999). The pricing of U.S. catastrophe reinsurance. In Froot, K., editor, *The Financing of Catastrophe Risk*, pages 195–232. University of Chicago Press, Chicago.
- Gao, C. and Martin, I. W. R. (2021). Volatility, valuation ratios, and bubbles: An empirical measure of market sentiment. *Journal of Finance*. forthcoming.
- Geanakoplos, J. (2010). The leverage cycle. In Acemoglu, D., Rogoff, K., and Woodford, M., editors, *NBER Macroeconomic Annual 2009*, volume 24, pages 1–65. University of Chicago Press, Chicago.
- Giglio, S., Maggiori, M., Stroebel, J., and Utkus, S. (2019). Five facts about beliefs and portfolios. Working paper.
- Greenwood, R. and Shleifer, A. (2014). Expectations of returns and expected returns. *Review of Financial Studies*, 27(3):714–746.
- Hansen, L. P. and Jagannathan, R. (1991). Implications of security market data for models of dynamic economies. *Journal of Political Economy*, 99(2):225–262.
- Hansen, L. P. and Richard, S. F. (1987). The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models. *Econometrica*, 55(3):587–613.
- Harrison, J. M. and Kreps, D. M. (1978). Speculative investor behavior in a stock market with heterogeneous expectations. *Quarterly Journal of Economics*, 92(2):323–336.
- Harsanyi, J. C. (1955). Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *Journal of Political Economy*, 63(4):309–321.
- Jackwerth, J. C. (2000). Recovering risk aversion from option prices and realized returns. *Review of Financial Studies*, 13(2):433–451.

- Johnson, T. C. (2004). Forecast dispersion and the cross section of expected returns. *The Journal of Finance*, 59(5):1957–1978.
- Jouini, E. and Napp, C. (2007). Consensus consumer and intertemporal asset pricing with heterogeneous beliefs. *The Review of Economic Studies*, 74(4):1149–1174.
- Keynes, J. M. (1936). *The General Theory of Employment, Interest and Money*. Palgrave Macmillan.
- Kogan, L., Ross, S. A., Wang, J., and Westerfield, M. M. (2006). The price impact and survival of irrational traders. *Journal of Finance*, 61(1):195–229.
- Kremens, L. and Martin, I. W. R. (2019). The quanto theory of exchange rates. *American Economic Review*, 109(3):810–843.
- Martin, I. W. R. (2017). What is the expected return on the market? *Quarterly Journal of Economics*, 132(1):367–433.
- Martin, I. W. R. and Wagner, C. (2019). What is the expected return on a stock? *Journal of Finance*, 74(4):1887–1929.
- Meeuwis, M., Parker, J. A., Schoar, A., and Simester, D. I. (2019). Belief disagreement and portfolio choice. Working paper, MIT.
- Milne, A. A. (1926). *Winnie-the-Pooh*. Methuen & Co. Ltd.
- Paul, S. R. and Plackett, R. L. (1978). Inference sensitivity for Poisson mixtures. *Biometrika*, 65(3):591–602.
- Roll, R. (1984). Orange juice and weather. *American Economic Review*, 74(5):861–880.
- Rothschild, M. and Stiglitz, J. E. (1970). Increasing risk: I. A definition. *Journal of Economic Theory*, 2(3):225–243.
- Rubinstein, M. (1976). The strong case for the generalized logarithmic utility model as the premier model of financial markets. *The Journal of Finance*, 31(2):551–571.
- Sandroni, A. (2000). Do markets favor agents able to make accurate predictions? *Econometrica*, 68(6):1303–1341.
- Scheinkman, J. A. and Xiong, W. (2003). Overconfidence and speculative bubbles. *Journal of Political Economy*, 111(6):1183–1219.
- Shiller, R. J. (1981). Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review*, 71(3):421–436.
- Shiller, R. J. (1987). Investor behavior in the October 1987 stock market crash: Survey and evidence. NBER Working Paper No. 2446.
- Simsek, A. (2013). Belief disagreements and collateral constraints. *Econometrica*, 81(1):1–53.

Yu, J. (2011). Disagreement and return predictability of stock portfolios. *Journal of Financial Economics*, 99(1):162–183.

Zapatero, F. (1998). Effects of financial innovations on market volatility when beliefs are heterogeneous. *Journal of Economic Dynamics and Control*, 22(4):597–626.

A Proofs of results

Proof of Result 1. Observe from the recurrence relation (13) that $z_{0,0}$ is a linear combination of the reciprocals of the terminal payoffs,

$$z_{0,0} = \sum_{m=0}^T c_m z_{m,T}. \quad (32)$$

Each constant c_m is a sum of products of terms of the form $H_{j,s}$ and $1 - H_{j,s}$ over appropriate j and s . In order to better handle these products it will be helpful to introduce $J_{m,t}(h) = h^m(1-h)^{t-m}f(h) \propto w_h f(h)$. Then $\int_0^1 J_{m,t}(h) dh \propto \int_0^1 w_h f(h) dh = p$ and hence

$$H_{m,t} = \frac{\int_0^1 h w_h f(h) dh}{p} = \frac{\int_0^1 h J_{m,t}(h) dh}{\int_0^1 J_{m,t}(h) dh} = \frac{\int_0^1 J_{m+1,t+1}(h) dh}{\int_0^1 J_{m,t}(h) dh} \quad (33)$$

and

$$1 - H_{m,t} = \frac{\int_0^1 J_{m,t+1}(h) dh}{\int_0^1 J_{m,t}(h) dh}. \quad (34)$$

We first show that path independence holds, so that all the possible ways of getting from the initial node to node m at time T make an equal contribution to c_m . It suffices to show that, starting from any node, the risk-neutral probability of down-up equals the risk-neutral probability of up-down—that is, for any m and t ,

$$h_{m,t}^*(1 - h_{m+1,t+1}^*) = (1 - h_{m,t}^*)h_{m,t+1}^*.$$

Rewriting equation (8) to insert subscripts, we have

$$h_{m,t}^* = H_{m,t} \frac{p_{m,t}}{p_{m+1,t+1}} \quad \text{and} \quad 1 - h_{m,t}^* = (1 - H_{m,t}) \frac{p_{m,t}}{p_{m,t+1}}. \quad (35)$$

It follows that we have path independence if and only if $H_{m,t}(1 - H_{m+1,t+1}) = (1 - H_{m,t})H_{m,t+1}$. But this follows immediately from equations (33) and (34).

Path independence implies that we can evaluate c_m by considering the path that travels down for $T - m$ periods and then up for m periods, and then multiplying by the number of paths, $\binom{T}{m}$:

$$c_m = \binom{T}{m} (1 - H_{0,0}) \cdots (1 - H_{0,T-m-1}) H_{0,T-m} H_{1,T-m+1} \cdots H_{m-1,T-1}.$$

Equations (33) and (34) allow us to write c_m as a telescoping product:

$$\begin{aligned} c_m &= \binom{T}{m} \frac{\int_0^1 J_{m,T}(h) dh}{\int_0^1 J_{0,0}(h) dh} \\ &= \binom{T}{m} \int_0^1 h^m (1-h)^{T-m} f(h) dh. \end{aligned}$$

If $f(h)$ is the pdf of a $\text{Beta}(\alpha, \beta)$ distribution, we can evaluate the integral explicitly to give

$$c_m = \binom{T}{m} \frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}.$$

This quantity is the probability that a random variable with beta-binomial distribution with parameters (T, α, β) equals m . Thus the price at time zero satisfies $p_{0,0}^{-1} = \mathbb{E}[p_{\tilde{m},T}^{-1}]$, where the expectation is over $\tilde{m} \sim \text{BetaBinomial}(T, \alpha, \beta)$.

The risk-neutral probability of ending at node (m, T) , q_m^* , can be determined using (35) and path-independence:

$$\begin{aligned} q_m^* &= \binom{T}{m} (1 - h_{0,0}^*) \cdots (1 - h_{0,T-m-1}^*) \cdot h_{0,T-m}^* h_{1,T-m+1}^* \cdots h_{m-1,T-1}^* \\ &= \binom{T}{m} (1 - H_{0,0}) \frac{p_{0,0}}{p_{0,1}} \cdots (1 - H_{0,T-m-1}) \frac{p_{0,T-m-1}}{p_{0,T-m}} \cdot H_{0,T-m} \frac{p_{0,T-m}}{p_{1,T-m+1}} \cdots H_{m-1,T-1} \frac{p_{m-1,T-1}}{p_{m,T}} \\ &= c_m \frac{p_{0,0}}{p_{m,T}} \quad \square \end{aligned}$$

We also have the following generalization of Result 1, which gives the (inverse of) the price of the risky asset at node (m, t) . We omit the proof, which is essentially identical to the above.

Lemma 1. *We have $z_{m,t} = \sum_{j=0}^{T-t} c_{m,t,j} z_{m+j,T}$, where j represents the number of further up-moves after time t , and*

$$c_{m,t,j} = \binom{T-t}{j} \frac{\int_0^1 h^{m+j} (1-h)^{T-m-j} f(h) dh}{\int_0^1 h^m (1-h)^{t-m} f(h) dh}.$$

If, in particular, $f(h)$ is the pdf of a $\text{Beta}(\alpha, \beta)$ distribution, then

$$c_{m,t,j} = \binom{T-t}{j} \frac{B(\alpha + m + j, \beta + T - m - j)}{B(\alpha + m, \beta + t - m)}.$$

This is the probability that a random variable with $\text{BetaBinomial}(T - t, \alpha + m, \beta + t - m)$ distribution equals $j \in \{0, \dots, T - t\}$.

Moreover, the risk-neutral probability of ending up at node $(m + j, T)$ starting from node (m, t) is given by

$$q_{m,t,j}^* = c_{m,t,j} \frac{p_{m,t}}{p_{m+j,T}}.$$

Proof of Result 2. At time t , following m up-moves, let the investor's posterior belief about the probability of an up-move be denoted by $h_{m,t}$. In this general case we assume the belief h_0 of the representative agent has a density function $f(h)$. Then, using Bayes' rule, the posterior

density function, $f_{m,t}(\cdot)$, satisfies

$$f_{m,t}(h) = \frac{h^m(1-h)^{t-m}f(h)}{\int_0^1 h^m(1-h)^{t-m}f(h)dh}.$$

If for instance $f(h)$ is the density function of a $\text{Beta}(\alpha, \beta)$ distribution, then $f_{m,t}(h)$ is the probability density function of a $\text{Beta}(\alpha + m, \beta + t - m)$ distribution. Thus, in particular, using equation (33):

$$\mathbb{E}[h_{m,t}] = \frac{\int_0^1 h^{m+1}(1-h)^{t-m}f(h)dh}{\int_0^1 h^m(1-h)^{t-m}f(h)dh} = H_{m,t}. \quad (36)$$

That is, the expected belief of the representative agent is the same as the wealth-weighted belief in the heterogeneous economy.

The agent's portfolio problem at time t , following m up moves, is therefore

$$\max_{x_h} \mathbb{E}[h_{m,t} \log(w_h - x_h p + x_h p_u) + (1 - h_{m,t}) \log(w_h - x_h p + x_h p_d)],$$

with associated first-order condition

$$x_h = w_h \left(\frac{\mathbb{E}[h_{m,t}]}{p - p_d} - \frac{1 - \mathbb{E}[h_{m,t}]}{p_u - p} \right).$$

Market clearing dictates that $x_h = 1$ and $w_h = p$. Thus

$$p = \frac{p_u p_d}{\mathbb{E}[h_{m,t}] p_d + (1 - \mathbb{E}[h_{m,t}]) p_u}.$$

By equation (36), this is equivalent to the price (6) in the heterogeneous economy. \square

Proof of Result 3. We use the fact (noted in the proof of Result 1) that the price at time zero satisfies $p_{0,0}^{-1} = \mathbb{E}[p_{\tilde{m},T}^{-1}]$, where the expectation is over $\tilde{m} \sim \text{BetaBinomial}(T, \alpha, \beta)$. Note that an increase in belief heterogeneity corresponds to a decrease in α and β with α/β held constant. The key to the proof is then the following lemma. We presume it is well known but have not found a reference, so we include a proof in the Online Appendix.

Lemma 2. *If $\tilde{m}_1 \sim \text{BetaBinomial}(T, \bar{\alpha}, \lambda \bar{\alpha})$ and $\tilde{m}_2 \sim \text{BetaBinomial}(T, \underline{\alpha}, \lambda \underline{\alpha})$, where $\bar{\alpha} > \underline{\alpha}$ and $\lambda > 0$, then \tilde{m}_1 second order stochastically dominates \tilde{m}_2 .*

If \tilde{m}_1 second order stochastically dominates \tilde{m}_2 then $\mathbb{E}[u(\tilde{m}_1)] \geq \mathbb{E}[u(\tilde{m}_2)]$ for any concave function $u(\cdot)$ (Rothschild and Stiglitz, 1970). Therefore, if $\frac{1}{p_{m,T}}$ is convex (so that $-\frac{1}{p_{m,T}}$ is concave) then $\mathbb{E}[\frac{1}{p_{\tilde{m}_1,T}}] \leq \mathbb{E}[\frac{1}{p_{\tilde{m}_2,T}}]$, from which the first part of the result follows. If instead $\frac{1}{p_{m,T}}$ is concave, then the inequality is reversed.

Finally, log-concavity of p is equivalent to $(p')^2 \geq pp''$. This implies that $2(p')^2 \geq pp''$, which is equivalent to $1/p$ being convex. \square

Proof of Result 4. As the beta distribution is conjugate to the binomial distribution, investor h 's posterior probability of an up move at node (m, t) follows the distribution $\tilde{h}_{m,t} \sim \text{Beta}(\zeta h +$

$m, \zeta(1-h) + t - m$); thus $\mathbb{E}[\tilde{h}_{m,t}] = \frac{h+m/\zeta}{1+t/\zeta}$. The agent's first-order condition is therefore

$$x_h = w_h \left(\frac{\frac{h+m/\zeta}{1+t/\zeta}}{p-p_d} - \frac{1 - \frac{h+m/\zeta}{1+t/\zeta}}{p_u - p} \right).$$

As in the main text, we have suppressed the dependence of asset demand x_h (and, below, of p and h^*) on m and t for notational convenience.

Starting from node (m, t) , the wealth of an investor is $w_h + x_h(p_u - p) = w_h \frac{h+m/\zeta}{h^*(1+t/\zeta)}$ following an up-move (i.e., at node $(m+1, t+1)$) or $w_h + x_d(p_d - p) = w_h \frac{1 - \frac{h+m/\zeta}{1+t/\zeta}}{1-h^*}$ following a down-move (i.e., at node $(m, t+1)$). It follows, by induction, that the wealth of an investor at node (m, t) is $w_h = \tilde{\lambda}_{path} \cdot I_{m,t}(h)$, where

$$\begin{aligned} I_{m,t}(h) &= (1-h) \underbrace{\left(1 - \frac{h}{1 + \frac{1}{\zeta}}\right) \cdots \left(1 - \frac{h}{1 + \frac{t-m-1}{\zeta}}\right)}_{t-m \text{ down moves}} \underbrace{\left(\frac{h}{1 + \frac{t-m}{\zeta}}\right) \cdots \left(\frac{h + \frac{m-1}{\zeta}}{1 + \frac{t-1}{\zeta}}\right)}_{m \text{ up moves}} \\ &= \frac{B(\zeta h + m, \zeta(1-h) + t - m)}{B(\zeta h, \zeta(1-h))}. \end{aligned} \quad (37)$$

(The ordering of up- and down-moves is immaterial because $\mathbb{E}[1-\tilde{h}_{m,t}] \mathbb{E}[\tilde{h}_{m,t+1}] = \mathbb{E}[\tilde{h}_{m,t}] \mathbb{E}[1-\tilde{h}_{m+1,t+1}]$.) As initial wealth does not depend on h , we have $I_0(h) = 1$. We can find the constant $\tilde{\lambda}_{path}$ by equating aggregate wealth to the value of the risky asset:

$$p = \tilde{\lambda}_{path} \int_0^1 I_{m,t}(h) f(h) dh. \quad (38)$$

To clear the market, we must have

$$1 = \tilde{\lambda}_{path} \left[\int_0^1 I_{m,t}(h) \left(\frac{\frac{h+m/\zeta}{1+t/\zeta}}{p-p_d} - \frac{1 - \frac{h+m/\zeta}{1+t/\zeta}}{p_u - p} \right) f(h) dh \right]. \quad (39)$$

If we define

$$G_{m,t} = \frac{\int_0^1 I_{m,t}(h) (h + \frac{m}{\zeta}) f(h) dh}{(1 + \frac{t}{\zeta}) \int_0^1 I_{m,t}(h) f(h) dh} = \frac{\int_0^1 I_{m+1,t+1}(h) f(h) dh}{\int_0^1 I_{m,t}(h) f(h) dh} \quad (40)$$

then one can check that

$$1 - G_{m,t} = \frac{\int_0^1 I_{m,t+1}(h) f(h) dh}{\int_0^1 I_{m,t}(h) f(h) dh}. \quad (41)$$

In these terms, equations (38) and (39) imply that

$$\frac{1}{p} = \frac{G_{m,t}}{p-p_d} - \frac{1-G_{m,t}}{p_u-p}.$$

Defining $z_{m,t} = 1/p$, $z_{m+1,t+1} = 1/p_u$, and $z_{m,t+1} = 1/p_d$, we can rewrite this as

$$z_{m,t} = G_{m,t} z_{m+1,t+1} + (1 - G_{m,t}) z_{m,t+1}.$$

By backward induction, and using the fact that $(1 - G_{m,t}) G_{m,t+1} = G_{m,t} (1 - G_{m+1,t+1})$, we have $z_{0,0} = \sum_{m=0}^T \tilde{c}_m \cdot z_{m,T}$, where $\tilde{c}_m = \binom{T}{m} (1 - G_{0,0}) \cdots (1 - G_{0,T-m-1}) G_{0,T-m} \cdots G_{m-1,T-1}$. Using equations (40) and (41) to evaluate this as a telescoping product,

$$\tilde{c}_m = \binom{T}{m} \int_0^1 I_{m,T}(h) f(h) dh,$$

which completes the proof of the first part of the Result.

To prove the second part of the result, note from (37) that $\binom{T}{m} I_{m,T} = \mathbb{P}(\tilde{m} = m)$ where $\tilde{m} \sim \text{BetaBinomial}(T, \zeta h, \zeta(1-h))$, so $z_{0,0} = \int_0^1 \mathbb{E}[z_{\tilde{m}}] f(h) dh$. If $\tilde{m}_i \sim \text{BetaBinomial}(T, \zeta_i h, \zeta_i(1-h))$ for $i = 1, 2$, where $\zeta_1 > \zeta_2$, then \tilde{m}_1 second order stochastically dominates \tilde{m}_2 by Lemma 2. It follows that if z_m is convex, $\mathbb{E}[z_{\tilde{m}_1}] < \mathbb{E}[z_{\tilde{m}_2}]$ for all h , and hence $p_{0,0}^{(1)} > p_{0,0}^{(2)}$. Also by Lemma 2, the converse is true if z_m is concave. \square

Proof of Result 5. We will repeatedly use the fact that $\Gamma(z+1)/\Gamma(z) = z$ without comment. In the risky bond limit, $(1-\varepsilon)/\varepsilon \rightarrow \infty$, so the sentiment multiplier (18) simplifies to

$$g(t) = \left(\frac{\alpha}{\alpha + \beta} \right)^{T-t} \frac{\Gamma(\alpha + t) \Gamma(\alpha + \beta + T)}{\Gamma(\alpha + T) \Gamma(\alpha + \beta + t)}. \quad (42)$$

It follows that $g(t)$ is increasing:

$$\frac{g(t+1)}{g(t)} = \frac{\alpha + \beta}{\alpha} \frac{\Gamma(\alpha + t + 1)}{\Gamma(\alpha + t)} \frac{\Gamma(\alpha + \beta + t)}{\Gamma(\alpha + \beta + t + 1)} = \frac{\alpha + \beta}{\alpha} \frac{\alpha + t}{\alpha + \beta + t} \geq 1.$$

In the bubbly asset limit, $(1-\varepsilon)/\varepsilon \rightarrow -1$, so the sentiment multiplier (18) simplifies to

$$g(t) = \frac{1 - \left(\frac{\alpha}{\alpha + \beta} \right)^{T-t}}{1 - \frac{\Gamma(\alpha + T) \Gamma(\alpha + \beta + t)}{\Gamma(\alpha + t) \Gamma(\alpha + \beta + T)}}. \quad (43)$$

Let us write

$$x(t) = \left(\frac{\alpha}{\alpha + \beta} \right)^{T-t} \quad \text{and} \quad y(t) = \frac{\Gamma(\alpha + T) \Gamma(\alpha + \beta + t)}{\Gamma(\alpha + t) \Gamma(\alpha + \beta + T)},$$

so that

$$g(t) = \frac{1 - x(t)}{1 - y(t)} \quad \text{and} \quad g(t+1) = \frac{1 - x(t+1)}{1 - y(t+1)} = \frac{1 - x(t) \frac{\alpha + \beta}{\alpha}}{1 - y(t) \frac{\alpha + \beta + t}{\alpha + t}}.$$

It follows that $g(t+1) > g(t)$ if and only if

$$\frac{t}{\alpha + t} x(t) y(t) + \frac{\alpha}{\alpha + t} y(t) > x(t). \quad (44)$$

We can write

$$y(t) = \frac{\alpha + t}{\alpha + \beta + t} \frac{\alpha + t + 1}{\alpha + \beta + t + 1} \cdots \frac{\alpha + T - 1}{\alpha + \beta + T - 1},$$

which immediately implies that

$$y(t) > \left(\frac{\alpha + t}{\alpha + \beta + t} \right)^{T-t}. \quad (45)$$

We can use this fact to establish that inequality (44) holds, as required:

$$\begin{aligned} \frac{t}{\alpha + t} x(t) y(t) + \frac{\alpha}{\alpha + t} y(t) &> \frac{t}{\alpha + t} \left(\frac{\alpha}{\alpha + \beta} \frac{\alpha + t}{\alpha + \beta + t} \right)^{T-t} + \frac{\alpha}{\alpha + t} \left(\frac{\alpha + t}{\alpha + \beta + t} \right)^{T-t} \\ &> \left(\frac{t}{\alpha + t} \frac{\alpha}{\alpha + \beta} \frac{\alpha + t}{\alpha + \beta + t} + \frac{\alpha}{\alpha + t} \frac{\alpha + t}{\alpha + \beta + t} \right)^{T-t} \\ &= \left(\frac{\alpha}{\alpha + \beta} \right)^{T-t} \\ &= x(t). \end{aligned}$$

The first inequality uses the definition of $x(t)$ and (45); the second is Jensen's inequality. \square

Proof of Result 6. As shown in equation (32), $p_{0,0}^{-1} = \sum_{j=0}^{2N} c_m z_{m,T}$. From Result 1, as we now have $2N$ periods in total, we have $c_m = \binom{2N}{m} \frac{B(\alpha+m, \beta+2N-m)}{B(\alpha, \beta)}$. Hence we can write $p_{0,0}^{-1} = \mathbb{E}[z_{\tilde{m}, T}] = \mathbb{E} \left[e^{-\sigma \sqrt{2T} \frac{\tilde{m}-N}{\sqrt{N}}} \right]$, where $\tilde{m} \sim \text{BetaBinomial}(2N, \alpha, \beta)$ and $\alpha = \theta N + \eta \sqrt{N}$ and $\beta = \theta N - \eta \sqrt{N}$. Paul and Plackett (1978) show that \tilde{m} , appropriately shifted by mean and scaled by standard deviation, converges in distribution and in MGF to a Normal random variable: $\frac{\tilde{m}-N-\frac{\eta}{\theta} \sqrt{N}}{\sqrt{\frac{1+\theta}{2\theta} N}} \rightarrow \Psi \sim N(0, 1)$. Thus

$$p_{0,0}^{-1} \rightarrow \mathbb{E} \left[e^{-\sigma \sqrt{2T} \left(\Psi \sqrt{\frac{1+\theta}{2\theta}} + \frac{\eta}{\theta} \right)} \right] = \exp \left(-\frac{\eta}{\theta} \sigma \sqrt{2T} + \frac{\theta+1}{2\theta} \sigma^2 T \right) \quad \square$$

Proof of Result 7. We want to find the perceived expectation and variance of returns from 0 to t . To do so, we compute $p_{m,t}$ following the lines of the proof of Result 6, and then find its limiting distribution from the perspective of investor h .

We fix t , so that we can drop subscripts whenever possible, and we define $\phi = \frac{t}{T}$ and set $m = \phi N + \psi \sqrt{\phi N}$ so that ψ is a convenient parametrization of m . At time t , we are in the $2\phi N$ -th period, with $2(1-\phi)N$ periods remaining; hence (by Lemma 1) $c_{m,t,j} = \binom{2(1-\phi)N}{j} \frac{B(m+\alpha+j, 2N-m+\beta-j)}{B(m+\alpha, 2\phi N-m+\beta)}$. As $z_{m+j,T} = e^{-\sigma \sqrt{2T} \frac{m+j-N}{\sqrt{N}}}$, we have

$$p_{m,t}^{-1} = \mathbb{E} \left[e^{-\sigma \sqrt{2T} \frac{m+\tilde{j}-N}{\sqrt{N}}} \right] \quad (46)$$

where $\tilde{j} \sim \text{BetaBinomial} \left(2(1-\phi)N, (\phi+\theta)N + (\psi\sqrt{\phi} + \eta)\sqrt{N}, (\phi+\theta)N - (\psi\sqrt{\phi} + \eta)\sqrt{N} \right)$.

By the result of Paul and Plackett (1978), the standardized version of j converges in distribution and in MGF to a standard Normal random variable. Therefore we can find the limiting expectation on the right hand side of (46) by considering the expectation under a Normal distribution with the corresponding mean and variance. As N tends to infinity, we will write $p_\psi \equiv p_{m,t}$ (where $\psi = \frac{m-\phi N}{\sqrt{\phi N}}$), to emphasize that we are considering the continuous

time limit, in which ψ becomes the relevant state variable. We get:

$$p_\psi = e^{-\frac{1-\phi}{2} \frac{\theta+1}{\phi+\theta} \sigma^2 T + \frac{1-\phi}{\phi+\theta} \eta \sigma \sqrt{2T} + \frac{\theta+1}{\phi+\theta} \sigma \sqrt{2\phi T} \psi}. \quad (47)$$

We know that m has a binomial distribution with mean $2\phi Nh$ and variance $2\phi Nh(1-h)$ from the perspective of agent h . Hence, by the Central Limit Theorem or the de Moivre–Laplace theorem (as stated in the Online Appendix), a standardized version of m converges in distribution and in moment generating function to a standard Normal distribution:

$$\frac{m - 2\phi Nh}{\sqrt{2\phi Nh(1-h)}} \rightarrow N(0, 1) \quad \text{or,} \quad \sqrt{2} \left[\psi - \sqrt{\phi} \left(\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}} \right) \right] \rightarrow \xi \sim N(0, 1). \quad (48)$$

We can rewrite equation (47) from the agent z 's perspective as $p_t = b_t e^{\frac{\theta+1}{\phi+\theta} \sigma \sqrt{2\phi T} \left(\frac{\xi}{\sqrt{2}} + \sqrt{\phi} \left(\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}} \right) \right)}$, where $b_t = e^{-\frac{1-\phi}{2} \frac{\theta+1}{\phi+\theta} \sigma^2 T + \frac{1-\phi}{\phi+\theta} \eta \sigma \sqrt{2T}}$. Hence, the perceived expectation and variance of $\log p_t$ (in the limit as $N \rightarrow \infty$) are

$$\mathbb{E}^{(z)} \log p_t = \frac{t(\theta+1) \frac{z}{\sqrt{\theta}} \sigma \sqrt{T} - \frac{1}{2}(T-t)(\theta+1) \sigma^2 T}{\theta T + t} + \frac{\eta}{\theta} \sigma \sqrt{2T} \quad \text{and} \quad \text{var}^{(z)} \log p_t = \sigma^2 t \left(\frac{\theta+1}{\theta + \frac{t}{T}} \right)^2.$$

Moreover, the expected return is:

$$\mathbb{E}^{(z)} [R_{0 \rightarrow t}] = \mathbb{E}^{(z)} \left[\frac{p_t}{p_{0,0}} \right] = p_{0,0}^{-1} \cdot b_t \cdot \mathbb{E}^{(z)} \left[e^{\frac{\theta+1}{\phi+\theta} \sigma \sqrt{2\phi T} \left(\frac{\xi}{\sqrt{2}} + \sqrt{\phi} \left(\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}} \right) \right)} \right].$$

Thus, after some algebra, $\mathbb{E}^{(z)} [R_{0 \rightarrow t}] = e^{\frac{\phi(\theta+1)}{\theta+\phi} \left[\frac{z}{\sqrt{\theta}} \sigma \sqrt{T} + \frac{\theta+1}{2} \left(\frac{1}{\theta} + \frac{1}{\theta+\phi} \right) \sigma^2 T \right]}$. Setting $\phi = \frac{t}{T}$, and using the fact that z has zero cross-sectional mean and unit variance we derive the cross-sectional mean expectation and disagreement.

Finally, we find the autocorrelation of returns from the perspective of time 0. Let m and $m+j$ be the random variables representing the number of up-moves by times t_1 and t_2 respectively. Then, as in equation (48), as $N \rightarrow \infty$ we have $\frac{m-2\phi_1 Nh}{\sqrt{2\phi_1 Nh(1-h)}} \rightarrow \xi_1$ and $\frac{m+j-2\phi_2 Nh}{\sqrt{2\phi_2 Nh(1-h)}} \rightarrow \xi_2$, where ξ_1, ξ_2 are standard Normal random variables. We then have

$$\text{cov}(\log R_{0 \rightarrow t_1}, \log R_{t_1 \rightarrow t_2}) = \text{cov} \left(\frac{\theta+1}{\phi_1+\theta} \sigma \sqrt{\phi_1 T} \xi_1, \frac{\theta+1}{\phi_2+\theta} \sigma \sqrt{\phi_2 T} \xi_2 - \frac{\theta+1}{\phi_1+\theta} \sigma \sqrt{\phi_1 T} \xi_1 \right).$$

Note that $\sqrt{\phi_2} \xi_2 = \sqrt{\phi_1} \xi_1 + \sqrt{\phi_2 - \phi_1} \Xi$, where $\Xi \sim N(0, 1)$ is independent of ξ_1 . It follows that

$$\text{cov}(\log R_{0 \rightarrow t_1}, \log R_{t_1 \rightarrow t_2}) = \frac{(\theta+1)^2}{(\phi_1+\theta)^2} \frac{\phi_1 - \phi_2}{\phi_2 + \theta} \sigma^2 \phi_1 T$$

As $\text{var}[\log R_{0 \rightarrow t_1}] = \frac{(\theta+1)^2}{(\phi_1+\theta)^2} \sigma^2 \phi_1 T$, and using the fact that $\text{var}[\log R_{t_1 \rightarrow t_2}] = \text{var}[\log(p_{t_1})] + \text{var}[\log(p_{t_2})] - 2 \text{cov}(\log(p_{t_1}), \log(p_{t_2}))$, we have

$$\text{var}[\log R_{t_1 \rightarrow t_2}] = \frac{(\theta+1)^2}{(\phi_1+\theta)^2 (\theta+\phi_2)^2} \sigma^2 \phi_1 T \left[(\phi_2 + \theta)^2 + \frac{\phi_2}{\phi_1} (\phi_1 + \theta)^2 - 2(\phi_1 + \theta)(\phi_2 + \theta) \right].$$

Combining these, we find the expression given in the result. \square

Proof of Result 8. Note that $2\phi N$ is the number of periods corresponding to $t = \phi T$. Writing $q_{m,t}$ for the risk neutral probability of going from node $(0,0)$ to node (m,t) , we have (as in Lemma 1) $q_{m,t} = \frac{p_{0,0}}{p_{m,t}} c_{m,t}$, where $c_{m,t} = \binom{2\phi N}{m} \frac{B(\alpha+m, \beta+2\phi N-m)}{B(\alpha, \beta)}$. As the risk-free rate is 0, it follows that the time zero price of a call option with strike K , maturing at time t , is

$$C(0, t; K) = \sum_{m=0}^{2\phi N} q_{m,t} (p_{m,t} - K)^+ = p_0 \sum_{m=0}^{2\phi N} c_{m,t} \left(1 - \frac{K}{p_{m,t}}\right)^+ \rightarrow p_0 \mathbb{E} \left[\left(1 - \frac{K}{b_t} e^{-\frac{\theta+1}{\phi+\theta} \sigma \sqrt{2\phi T} \psi}\right)^+ \right],$$

where the expectation is taken with respect to a BetaBinomial($2\phi N, \alpha, \beta$) distribution and we have used (47), to substitute for p_ψ . By the result of Paul and Plackett (1978), m is asymptotically Normal:

$$\frac{m - \phi N - \frac{\eta}{\theta} \phi \sqrt{N}}{\sqrt{\frac{\phi+\theta}{2\theta} \phi N}} \rightarrow \Psi \sim N(0, 1), \quad \text{or} \quad \frac{1}{\sqrt{\frac{\phi+\theta}{2\theta}}} \left(\psi - \frac{\eta}{\theta} \sqrt{\phi}\right) \rightarrow \Psi \sim N(0, 1).$$

Thus

$$C(0, t; K) = p_0 \mathbb{E} \left[\left(1 - \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2\phi T} (\Psi \sqrt{\frac{\phi+\theta}{2\theta}} + \frac{\eta \sqrt{\phi}}{\theta})}\right)^+ \right].$$

(Convergence in distribution implies convergence in expectation by the Helly–Bray theorem, as the function of Ψ inside the expectation is bounded and continuous.) The expectation is standard, and we have

$$C(0, t; K) = p_0 \left[\Phi \left(-\frac{\log(X)}{\tilde{\sigma} \sqrt{t}} \right) - e^{\frac{\tilde{\sigma}^2 t}{2}} \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2T} \frac{\eta \phi}{\theta}} \Phi \left(-\frac{\log(X) + \tilde{\sigma}^2 t}{\tilde{\sigma} \sqrt{t}} \right) \right]$$

where $X = \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2T} \frac{\eta \phi}{\theta}}$ and

$$\tilde{\sigma}^2 t = \frac{(\theta+1)^2}{\theta(\theta+\phi)} \sigma^2 t = \text{var} \left[\log \left(\frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2T} (\Psi \sqrt{\frac{\phi+\theta}{2\theta}} + \frac{\eta \phi}{\theta})} \right) \right].$$

The result follows because $p_0 e^{\frac{\tilde{\sigma}^2 t}{2}} \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2T} \frac{\eta \phi}{\theta}} = K$.

Lastly, we can calculate the variance risk premium at arbitrary horizons $t < T$. We have $\text{var}^* \log R_{0 \rightarrow t} = \mathbb{E}^* \left[(\log R_{0 \rightarrow t})^2 \right] - [\mathbb{E}^* (\log R_{0 \rightarrow t})]^2$. Each of the risk-neutral expectations is determined by the prices of options expiring at time t , by the logic of Breeden and Litzenberger (1978). Hence risk-neutral variance is the same as in the Black–Scholes model with constant volatility $\tilde{\sigma}_t$. As is well known, this is $\tilde{\sigma}_t^2$ in annualized terms. Using the expression for $\text{var} \log R_t$ provided in Result 7, we have a generalization of the result given in the text:

$$\frac{1}{t} (\text{var}^* \log R_{0 \rightarrow t} - \text{var} \log R_{0 \rightarrow t}) = \frac{(\theta+1)^2 \frac{t}{T}}{\theta \left(\theta + \frac{t}{T}\right)^2} \sigma^2 \quad \square$$

Proof of Result 9. As all investors have log utility, $R_{0 \rightarrow T}^{(z)}$ is the growth optimal return from 0 to T as perceived by investor z , which equals $R_{0 \rightarrow T}^{(z)} = 1/M_T^{(z)}$, where $M^{(z)}$ is the SDF perceived by investor z . The following lemma provides a formula for this quantity:

Lemma 3. *The SDF of investor z , $M_t^{(z)}$ is given by:*

$$M_t^{(z)} = \sqrt{\frac{\phi}{\theta + \phi}} \exp \left\{ \frac{\theta + \phi}{2(1 + \theta)^2 \sigma^2 T} \left[\log \left(R_{0 \rightarrow t} / K_t^{(z)} \right) \right]^2 - \frac{1}{2} (z - z_g)^2 \right\}. \quad (49)$$

where $\log K_t^{(z)} = \mathbb{E}^{(z)} \log R_{0 \rightarrow t} + \frac{\theta+1}{\theta+\phi} (z - z_g) \sigma \sqrt{\theta T}$.

Proof. Note that $M^{(z)}$ links investor's perceived true probabilities to the objectively observed risk-neutral probabilities, which we have computed in Lemma 1. In particular, the value of agent z 's SDF if there have been m up-moves by time t is $M_t^{(z)}(m) = \frac{p_{0,0}}{p_{m,t}} \frac{c_{m,t}}{\pi_t^{(z)}(m)}$, where $\pi_t^{(z)}(m)$ is the probability that we will end up at node (m, t) , as perceived by agent z at time 0. From the proofs of Results 7, 8, we have established that $c_{m,t}$, $\pi_t^{(z)}(m)$ correspond to the probability mass functions (pmf) of a beta-binomial distribution and of a binomial distribution, and hence they both converge asymptotically to the probability density function of a Normal random variable. In particular, using ψ to parametrize m , we get that c_ψ converges to the pdf of a Normal with mean $\frac{\eta}{\theta} \sqrt{\phi}$ and variance $\frac{\phi+\theta}{2\theta}$, while $\pi^{(z)}(\psi)$ converges to the pdf of a Normal with mean $\sqrt{\phi}(\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}})$ and variance $\frac{1}{2}$. Therefore, we get the following characterization for the SDF in the limit as $N \rightarrow \infty$:

$$M_t^{(z)}(\psi) = \sqrt{\frac{\theta}{\phi + \theta}} \frac{1}{R_{0 \rightarrow t}} e^{-\frac{\theta}{(\phi+\theta)}(\psi - \frac{\eta}{\theta} \sqrt{\phi})^2 + (\psi - \sqrt{\phi}(\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}}))^2}, \quad (50)$$

where $\psi = \frac{m - \phi N}{\sqrt{\phi N}}$. Thus $M_t^{(z)}$ is asymptotically equivalent to a function of the random variable $\sqrt{2}(\psi - \sqrt{\phi}(\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}})) = \frac{\log R_{0 \rightarrow t} - \mathbb{E}^{(z)}[\log R_{0 \rightarrow t}]}{\sqrt{\log R_{0 \rightarrow t}}}$, which converges in distribution to a standard normal from the perspective of any agent, as shown in equation (48). As this function is continuous, $M_t^{(z)}$ converges in distribution to the corresponding function of a standard Normal random variable, by the continuous mapping theorem. Finally, after some algebra we can rewrite the above in the form of equation (49). \square

Equation (22) and the remaining statements of the result follow immediately. \square

Proof of Result 10. We start by proving a lemma in the spirit of Breeden and Litzenberger (1978). The lemma only relies on the absence of static arbitrage opportunities, and not on market completeness or on the particular setting of this paper.

Lemma 4. *Let $W(\cdot)$ be such that $W(0) = 0$. Then choosing terminal wealth $W(p_T)$ is equivalent to holding a portfolio that is (i) long $W'(K_0)$ units of the underlying asset, (ii) long bonds with face value $W(K_0) - K_0 W'(K_0)$, (iii) long $W''(K) dK$ put options with strike K , for every $K < K_0$, and (iv) long $W''(K) dK$ call options with strike K , for every $K > K_0$. The constant $K_0 > 0$ can be chosen arbitrarily.*

Proof of Lemma 4. Integrate $W(p_T) = \int_0^\infty W'(K) \mathbf{1}_{\{p_T > K\}} dK$ by parts. \square

The claims in the first paragraph of the result follow by applying Lemma 4 with $K_0 = K^{(z)}$, and noting that $W^{(z)'}(K^{(z)}) = 0$ from the definition (21) of $K^{(z)}$.

To establish the truth of the second paragraph, note from (22) that

$$\text{sign} \left[W^{(z)''} \left(\exp \mathbb{E}^{(z)} \log p_T \right) \right] = \text{sign} \left[z^2 - z_g z - \frac{\theta + 1}{\theta} \right],$$

which is negative for moderate investors (including all investors with z between z_g and zero), and positive if $|z|$ is sufficiently large. \square

Proof of Result 11. Note first that as the market is complete, there is a strategy that attains the maximal Sharpe ratio (MSR) implied by the Hansen and Jagannathan (1991) bound.

In order to be able to take expectations of M_t^2 —for the rest of the proof, we suppress the dependence on z in our notation—we will prove that the above sequence of random variables is uniformly integrable. To do so, rewrite equation (50) as $(M_t^2)^{(N)} := D e^{A(\psi^{(N)})^2 + B\psi^{(N)} + C}$ to denote a sequence of random variables whose limiting expectation we want to find (where we include superscripts to emphasize the dependence on N). It suffices to show that there exists an $\varepsilon > 0$ such that $\sup_N \mathbb{E}[(e^{A(\psi^{(N)})^2 + B\psi^{(N)} + C})^{1+\varepsilon}] < \infty$. Let us set $\varepsilon = 1 - A = 1 - \frac{2\phi}{\phi + \theta} > 0$.

By Hoeffding's inequality,

$$\mathbb{P} \left(\left| \frac{m - \phi N}{\sqrt{\phi N}} \right| \geq k \right) \leq 2e^{-k^2} \quad (51)$$

for any $k > 0$. For $x > 0$, the above implies that $\mathbb{P} \left(e^{\frac{1}{1+\varepsilon^2} \frac{(m - \phi N)^2}{\phi N}} \geq x \right) \leq \frac{2}{x^{1+\varepsilon^2}}$. Therefore, using that $e^{\frac{1}{1+\varepsilon^2} \frac{(m - \phi N)^2}{\phi N}} \geq 1$, we get:

$$\mathbb{E}[e^{\frac{1}{1+\varepsilon^2} \psi^2}] = \mathbb{E}[e^{\frac{1}{1+\varepsilon^2} \frac{(m - \phi N)^2}{\phi N}}] \leq \int_0^\infty \mathbb{P} \left(e^{\frac{1}{1+\varepsilon^2} \frac{(m - \phi N)^2}{\phi N}} \geq x \right) dx \leq 1 + \int_1^\infty \frac{2}{x^{1+\varepsilon^2}} dx < \infty.$$

Finally, note that $(1 + \varepsilon)A < 1/(1 + \varepsilon^2)$. Thus there is a constant, K , such that $(1 + \varepsilon)(A\psi^2 + B\psi + C) < \frac{1}{1+\varepsilon^2}\psi^2 + K$, and therefore $\mathbb{E}[(e^{A(\psi^{(N)})^2 + B\psi^{(N)} + C})^{1+\varepsilon}] < \mathbb{E}[e^{\frac{1}{1+\varepsilon^2}(\psi^{(N)})^2 + K}] < \infty$. Thus our sequence is uniformly integrable, and hence there is convergence of expectations.

We can now find the variance of M_t from the perspective of agent z . The results above imply that this problem reduces, as $N \rightarrow \infty$, to finding the MGF of a chi-squared random variable. By computing this expectation we find that

$$\mathbb{E}[M_t^2] = \frac{\theta}{\sqrt{\theta^2 - \phi^2}} \exp \left\{ \frac{[z\sqrt{\theta\phi} + (\theta + 1)\sigma\sqrt{\phi T}]^2}{\theta(\theta - \phi)} \right\} \quad \square$$

Proof of Result 12. Write $r_{0 \rightarrow T}^{(x)} = \log R_{0 \rightarrow T}^{(x)}$ and $r_{0 \rightarrow T} = \log R_{0 \rightarrow T}$. Rearranging (22), we have

$$r_{0 \rightarrow T}^{(x)} = \frac{1}{2} \log \frac{\theta + 1}{\theta} + \frac{1}{2} (x - z_g)^2 - \frac{1}{2(1 + \theta)} \left\{ \frac{r_{0 \rightarrow T} - \mathbb{E}^{(z)} r_{0 \rightarrow T}}{\sigma\sqrt{T}} + \frac{\mathbb{E}^{(z)} r_{0 \rightarrow T} - \mathbb{E}^{(x)} r_{0 \rightarrow T}}{\sigma\sqrt{T}} - \sqrt{\theta} (x - z_g) \right\}^2.$$

As $\mathbb{E}^{(z)} r_{0 \rightarrow T} - \mathbb{E}^{(x)} r_{0 \rightarrow T} = (z - x)\sigma\sqrt{T}/\sqrt{\theta}$ and $\frac{r_{0 \rightarrow T} - \mathbb{E}^{(z)} r_{0 \rightarrow T}}{\sigma\sqrt{T}}$ is a zero-mean, unit-variance

random variable in the opinion of agent z ,

$$\mathbb{E}^{(z)} r_{0 \rightarrow T}^{(x)} = \frac{1}{2} \log \frac{\theta + 1}{\theta} + \frac{1}{2} (x - z_g)^2 - \frac{1}{2(1 + \theta)} \left\{ 1 + \left[\frac{z - x}{\sqrt{\theta}} - \sqrt{\theta} (x - z_g) \right]^2 \right\}.$$

Result 7 showed that

$$\mathbb{E}^{(z)} r_{0 \rightarrow T} = \frac{z\sigma\sqrt{T}}{\sqrt{\theta}} + \frac{\theta + 1}{2\theta} \sigma^2 T = -\frac{zz_g}{\theta + 1} + \frac{1}{2} \frac{z_g^2}{\theta + 1},$$

where we use the definition of z_g in the second equality. It follows that

$$\mathbb{E}^{(z)} \left(r_{0 \rightarrow T}^{(x)} - r_{0 \rightarrow T} \right) = \frac{1}{2} \log \frac{\theta + 1}{\theta} + \frac{z^2 - 1}{2(1 + \theta)} - \frac{(z - x)^2}{2\theta},$$

which gives the first part of the result, because $\log(1 + \xi^{(z,x)}) = \mathbb{E}^{(z)} \left(r_{0 \rightarrow T}^{(x)} - r_{0 \rightarrow T} \right)$.

Lastly, as the distribution of types x is asymptotically standard Normal, we have

$$\xi = \int_{x=-\infty}^{\infty} \xi^{(z,x)} g(x) dx$$

where $g(x)$ is the standard Normal pdf, and evaluating the integral gives the result. \square

Proof of Result 13. From the definition (27), we see that $\log(1 - A) = \frac{1}{N} \sum_{i=1}^N \log y_i - \log \mu$, where μ is cross-sectional average wealth. In our setting, with a continuum of investors, this becomes $\log(1 - A) = \tilde{\mathbb{E}} \log W^{(z)} - \log \tilde{\mathbb{E}} W^{(z)}$. (As before, we use the notation $\tilde{\mathbb{E}}$ to denote a cross-sectional expectation that averages across agents.) As $\tilde{\mathbb{E}} W^{(z)} = p_0 R_{0 \rightarrow T}$ equals aggregate wealth, whereas $\log W^{(z)} = \log(p_0 R_{0 \rightarrow T}^{(z)})$ equals the log return chosen by investor z , we have $\log(1 - A) = \tilde{\mathbb{E}} \log R_{0 \rightarrow T}^{(z)} - \log R_{0 \rightarrow T}$. Henceforth, we write $r_{0 \rightarrow T}^{(z)} = \log R_{0 \rightarrow T}^{(z)}$ and $r_{0 \rightarrow T} = \log R_{0 \rightarrow T}$, so that $\log(1 - A) = \tilde{\mathbb{E}} r_{0 \rightarrow T}^{(z)} - r_{0 \rightarrow T}$.

Investor z 's log return $r_{0 \rightarrow T}^{(z)}$ is a function of the market log return, $r_{0 \rightarrow T}$: we can rewrite equation (22) as

$$r_{0 \rightarrow T}^{(z)} = \frac{1}{2} \log \frac{\theta + 1}{\theta} + \frac{1}{2} (z - z_g)^2 - \frac{1}{2(1 + \theta)} \left[\frac{r_{0 \rightarrow T} - z\sigma\sqrt{\frac{T}{\theta}} - \frac{\theta+1}{2\theta} \sigma^2 T}{\sigma\sqrt{T}} - \sqrt{\theta} (z - z_g) \right]^2.$$

This expression is quadratic in z . As z has zero mean and unit variance, so that $\tilde{\mathbb{E}} z = 0$ and $\tilde{\mathbb{E}} z^2 = 1$, we have (after some algebra)

$$\tilde{\mathbb{E}} r_{0 \rightarrow T}^{(z)} = \frac{1}{2} \log \frac{\theta + 1}{\theta} + \frac{1}{2} (1 + z_g^2) - \frac{1}{2(1 + \theta)} \left\{ \left[\frac{r_{0 \rightarrow T}}{\sigma\sqrt{T}} - \frac{\theta + 1}{2\theta} \sigma\sqrt{T} - (\theta + 1)\sigma\sqrt{T} \right]^2 + \left(\sqrt{\theta} + \frac{1}{\sqrt{\theta}} \right)^2 \right\}.$$

Using the expression for z_g given in equation (20) and simplifying, we find that

$$\log(1 - A) = \frac{1}{2} \left(\log \frac{\theta + 1}{\theta} - \frac{1}{\theta} \right) - \frac{1}{2(1 + \theta)\sigma^2 T} \left(r_{0 \rightarrow T} - \frac{1 + \theta}{2\theta} \sigma^2 T \right)^2.$$

This is equivalent to the expression given in the text. \square

Proof of Result 14. There are N periods of length T/N . Let us write $t = \phi T$. Suppose there have been $n = q$ down-moves (jumps) and $m = \phi N - q$ up-moves by time t . If \tilde{q} of the remaining $(1 - \phi)N$ periods are down-moves and j are up-moves, then we must have $\tilde{q} + j = (1 - \phi)N$. From Lemma 1, the price at time t is $\{\mathbb{E}[e^{(q+\tilde{q})J}]\}^{-1}$, where the expectation is over $\tilde{q} \sim \text{BetaBinomial}((1 - \phi)N, q + 1/\omega^2, N/(\omega^2 \lambda T) + \phi N - q)$. We now use the fact that as $n \rightarrow \infty$, a beta binomial distribution with parameters n, α, Cn approaches a negative binomial distribution with $r = \alpha$ and $p = 1/(1 + C)$. Therefore, as $N \rightarrow \infty$, \tilde{q} is asymptotically distributed as a negative binomial distribution with parameters $q + 1/\omega^2$ and $\omega^2 \lambda T(1 - \phi)/(1 + \omega^2 \lambda T)$. Using the formula for the MGF of a negative binomial distribution, the price equals

$$e^{-qJ} \left[\frac{1 - \omega^2 \lambda T (e^J - 1) + \omega^2 \lambda T e^J}{1 + \omega^2 \lambda T} \right]^{q + \frac{1}{\omega^2}}.$$

Simplifying this expression gives the price (29).

As the riskless rate equals zero, agent z 's SDF equals the ratio of the risk-neutral probability of q jumps occurring by time t to the corresponding true probability (which is $(\lambda(1 - z\omega)t)^q e^{-\lambda(1 - z\omega)t}/q!$). As in the proof of Result 8, the risk-neutral probability of $m = \phi N - q$ up-moves having occurred during the first ϕN moves is $(p_{0,0}/p_{m,\phi N})x_N$, where x_N is the probability of m realizations in a beta-binomial distribution with parameters $\phi N, N/(\omega^2 \lambda T)$, and $1/\omega^2$ or, equivalently, the probability of $\phi N - m = q$ realizations in a beta-binomial distribution with parameters $(\phi N, 1/\omega^2, N/(\omega^2 \lambda T))$. In the limit as $N \rightarrow \infty$, using the convergence of this beta-binomial to a negative binomial distribution with parameters $1/\omega^2, \frac{\omega^2 \lambda \phi T}{1 + \omega^2 \lambda \phi T}$, we find that the probability x_N is therefore equal to

$$\frac{\Gamma(q + \frac{1}{\omega^2})}{q! \Gamma(\frac{1}{\omega^2})} \left(\frac{1}{1 + \omega^2 \lambda T} \right)^{\frac{1}{\omega^2}} \left(\frac{\omega^2 \lambda T}{1 + \omega^2 \lambda T} \right)^q.$$

Similarly, as N tends to infinity, $p_{0,0}/p_{m,\phi N}$ tends to the reciprocal of the return from 0 to t conditional on q jumps having occurred, as provided in Result 14. The SDF follows as stated. To calculate $\mathbb{E}^{(z)} \log M_{0 \rightarrow T}^{(z)}$, note that investor z perceives the number of jumps, q , that occur by time T as distributed according to a Poisson distribution with parameter $\lambda(1 - z\omega)T$. As $\Gamma(z) = O(z^{z-1/2}e^{-z})$, it follows that $\mathbb{E}^{(z)} \log M_{0 \rightarrow T}^{(z)}$ is finite but $\mathbb{E}^{(z)} \left[\left(M_{0 \rightarrow T}^{(z)} \right)^2 \right]$ is infinite. \square

Proof of Result 15. The risk-neutral probability inside the limit in (31) is the price of a security with unit payoff if there is at least one jump in $[t, t + \varepsilon]$. As the interest rate is zero, this price equals $1 - x_\varepsilon$, where x_ε is the price of a security with unit payoff if there are no jumps between t and $t + \varepsilon$. A straightforward calculation gives

$$x_\varepsilon = \left[1 + \frac{\varepsilon \omega^2 \lambda e^J}{1 - \omega^2 \lambda T (e^J - 1) + \omega^2 \lambda T e^J} \right]^{-q - 1/\omega^2}.$$

As $\lambda_t^* = \lim_{\varepsilon \rightarrow 0} \frac{1 - x_\varepsilon}{\varepsilon}$, the result follows by the binomial theorem. \square